MECHANICS OF FLUIDS

W. J. DUNCAN
C.B.E., D.Sc., F.R.S., M.I.Mech.E., Hon. F.R.Ae.S.,
formerly Professor of Aeronautics and Fluid Mechanics in the University of Glasgow
Fellow of University College London

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PREFACE TO FIRST EDITION

This book is intended primarily for the use of undergraduates and students in Technical Colleges but it is hoped that it may be of service to young professional engineers and physicists, also to independent and unassisted readers. The authors have had the constant aim of providing a systematic and easily understood account of the basic principles of the science of the mechanics of fluids and to cover adequately some of the more elementary technical applications. In general, they have sought to give a lucid and concise exposition, but they have not hesitated to sacrifice conciseness in passages dealing with matters which many students find difficult. The subject matter covers the motion of compressible and of viscous fluids, with an adequate discussion of the 'boundary layer', but the study of the motion of a 'perfect fluid' provides the essential avenue of approach to all the higher developments. The early chapters cover hydrostatics, a thorough treatment of the kinematics of fluids, the basic dynamics of fluids and the application of the principles of similarity and dimensional analysis.

The mechanics of fluids may be regarded as a department of 'classical' physics and it is unquestionably of the greatest importance as a basic science in most of the branches of engineering. For example, it is a central subject for hydraulic engineering, naval architecture, aeronautics, nuclear engineering and heat engines. Further, the mechanics of fluids, in its numerous branches, is now and has for a long time been one of the most vigorous and rapidly developing parts of physics, making a very great contribution to the substance and volume of physical research. It is hoped that this book will provide a useful and helpful introduction to these recent developments.

Unquestionably, the mechanics of fluids is an intensely interesting subject; it presents many surprises and some apparent paradoxes to the student. The authors have attempted to foster the interest of the reader and not to bore him with unnecessary detail while, at the same time, taking care to avoid the snares of superficiality. No excessive demands on the reader's mathematical knowledge have been made and any mathematical techniques which seemed likely to be unfamiliar have been carefully explained. No attempt has been made to cover the discussion of constructional detail in those chapters which deal with technical applications.

The Table of Contents will show that this book covers much of the subject matter of traditional hydraulics, but the scope of the book is so much wider and the outlook so much less empirical that the word 'hydraulics' has been deliberately kept out of the title. 'Hydraulics' is, in fact, a defunct subject although hydraulic engineering is very much alive.

W. J. D. A. S. T. A. D. Y.

CONTENTS

CHAPTER 1. GENERAL PROPERTIES OF FLUIDS AND HYDROSTATICS.

1.1 Definition and Types of Fluid . . . 1
1.2 Stress and Pressure . . . 2
1.3 Physical Properties of Fluids . . . 4
1.4 Viscosity . . . 7
1.5 Surface Tension . . . 8
1.6 Equilibrium of Fluids . . . 10
1.7 Steady Rotation of an Inviscid Fluid about a Vertical Axis . . . 15
1.8 Thrusts on Plane Surfaces . . . 18
1.9 Buoyancy and Thrusts on Curved Surfaces . . . 24
1.10 Units and Constants . . . 26

CHAPTER 2. KINEMATICS OF FLUIDS.

2.1 Introductory Remarks . . . 36
2.2 Description and Specification of Fluid Motion . . . 36
2.3 Continuity . . . 41
2.4 The Stream Function . . . 44
2.5 The Combination of Motions . . . 48
2.6 Conditions at the Boundary of the Fluid . . . 49
2.7 Sources, Sinks and Doublets . . . 52
2.8 Streamline Bodies . . . 60
2.9 Acceleration of a Fluid Particle . . . 66
2.10 Flow Along a Curve. Circulation . . . 69
2.11 Vorticity and Vortices . . . 74
2.12 Irrotational Flow and the Velocity Potential . . . 81
2.13 The Velocity Field of an Isolated Vortex . . . 85
2.14 The Complex Potential . . . 94
2.15 Conformal Transformation and Orthogonal Coordinates . . . 100
2.16 The Method of Images . . . 109
2.17 The Status of Kinematic Theory . . . 113
Appendix 1. Analysis of the Motion of a Fluid near a Point . . . 114
Appendix 2. Stokes' Stream Function for Symmetric Flow in Planes through a Fixed Axis . . . 115

CHAPTER 3. ELEMENTS OF THE DYNAMICS OF INCOMPRESSIBLE FLUIDS.

3.1 Introduction . . . 124
3.2 The Basic Dynamical Principles . . . 124
3.3 Pressure at a Point in a Moving Fluid . . . 127
3.4 Bernoulli's Theorem . . . 128
3.5 The Dynamical Equations . . . 132
3.6 The Equations of Momentum . . . 138
3.7 Kelvin's Circulation Theorem . . . 142

Uttarpara

3.8 Method of Solution of Problems on the Motion of Perfect Fluids . 146
3.9 Kinetic Energy and Virtual Mass 151
3.10 Elementary Consideration of Viscous Fluids 155
3.11 Flow in a Rotating Channel 164
3.12 Cavitation 168
Appendix. The Acceleration Potential 171

CHAPTER 4. SIMILARITY AND THE APPLICATION OF DIMENSIONAL ANALYSIS.

4.1 Introduction 178
4.2 Similar Physical Systems 179
4.3 Similarity of Behaviour 180
4.4 Dynamical Similarity 180
4.5 Similarity of Flow 185
4.6 Units and Measures 188
4.7 Measure or Dimensional Formulae 189
4.8 Dimensional Analysis 191
4.9 Perfect Fluids 192
4.10 Viscous Incompressible Fluid 195
4.11 Compressible Inviscid Fluid 199
4.12 Compressible Viscous Fluid 200
4.13 Convection 201
4.14 Similar Fluids 203
4.15 Periodic Phenomena in Fluids 204
4.16 Non-dimensional Force and Moment Coefficients 207

CHAPTER 5. EXPERIMENTAL TECHNIQUES.

5.1 Introduction 212
5.2 Towing, Free Flight and Tank Tests 213
5.3 Flumes 215
5.4 Whirling Arms. 215
5.5 Low Speed Wind Tunnels 216
5.6 High Speed Wind Tunnels. 218
5.7 Water and Cavitation Tunnels 219
5.8 Tunnel Corrections 220
5.9 The Measurement of Velocity 222
5.10 The Measurement of Discharge 226
5.11 The Measurement of Pressure and Head 230
5.12 Devices for Making Flow Visible 234
5.13 The Hot Wire Anemometer 235
5.14 The Measurement of Temperature 236

CHAPTER 6. BOUNDARY LAYERS, WAKES AND TURBULENCE

6.1 Introduction 237
6.2 General Outline of the Physics of Boundary Layer Flow . . . 239
6.3 Laminar Boundary Layer Equations for Steady Incompressible Two-Dimensional Flow 246

6.4 Boundary Conditions. Boundary Layer Thicknesses . . . 250
6.5 Flow in Wakes and Jets 252
6.6 The Momentum Integral Equation of the Boundary Layer (Two Dimensional Steady Flow) 254
6.7 The Exact Solution for the Laminar Boundary Layer on a Flat Plate at Zero Incidence (Blasius Solution) 259
6.8 Flow in a Pipe or Channel of Constant Cross-Section . . . 262
6.9 Similar Solutions of the Laminar Boundary Layer Equations . 263
6.10 The General Problem and the Series Solution 266
6.11 The Laminar Wake 267
6.12 Approximate Methods 270
    6.12.1 Introductory Remarks 270
    6.12.2 Pohlhausen's Method 271
    6.12.3 Thwaites' Method 273
    6.12.4 Young's Method 278
    6.12.5 Some Remarks on Other Approximate Methods. The Energy Integral Equation 280
6.13 Stability of Laminar Flow. Reynolds Stresses. Transition . . 283
    6.13.1 Introductory Remarks 283
    6.13.2 Reynolds Stresses 284
    6.13.3 Energy Balance in Disturbed Laminar Flow . . . 285
    6.13.4 Classical Stability Analysis 286
    6.13.5 The Transition Process and Turbulence 291
6.14 Turbulence and the Structure of the Turbulent Boundary Layer . 294
6.15 Mixing Length Theories. Velocity Distributions . . . 299
    6.15.1 General Remarks 299
    6.15.2 Momentum Transport Theory 300
    6.15.3 Von Kármán's Similarity Theory 303
    6.15.4 Squire's Dimensional Theory and the Inner Velocity Distribution 304
    6.15.5 The Outer Velocity Law 307
    6.15.6 Power Laws 310
6.16 Skin Friction Laws for Turbulent Flow in a Smooth Pipe of Circular Cross-Section 311
    6.16.1 Power Law Relations 311
    6.16.2 'Log' Law Relations 313
6.17 Skin Friction and Other Relations for Turbulent Flow in the Boundary Layer on a Flat Plate at Zero Incidence . . . 314
    6.17.1 Experimental Determination of Skin Friction . . . 314
    6.17.2 Power Law Relations 317
    6.17.3 'Log' Law Relations 320
6.18 Effects of Roughness 325
    6.18.1 Flow in Rough Pipes 325
    6.18.2 Flow Past a Rough Flat Plate 328
6.19 Drag of a Smooth Flat Plate at Zero Incidence with partly Laminar and partly Turbulent Boundary Layer 330
6.20 The Turbulent Boundary Layer with an External Non-Zero Pressure Gradient 333
    6.20.1 Cases where the External Pressure Gradient is Small . 333
    6.20.2 Cases of Large Pressure Gradient 337
6.21 The Measurement and Calculation of Boundary Layer Drag . 350
    6.21.1 Effect of Boundary Layer on External Flow. Boundary Layer Pressure Drag 350

6.21.2 The Calculation of the Boundary Layer or Profile Drag . 353
6.21.3 The Pitot-Traverse Method of Measuring Boundary Layer Drag 359
6.22 Free Turbulence. Wakes and Jets 365
6.23 The Calculation of Lift of a Wing Section 368
6.24 Boundary Layer Control 369
6.24.1 General 369
6.24.2 Suction or Blowing through Slots 369
6.24.3 Preserving Laminar Flow. Suction through a Porous Surface and Slots 371
6.24.4 Boundary Layer Control to Delay or Suppress Separation 378
6.24.5 Power Required for Suction and the Equivalent Drag . 386

CHAPTER 7. FLOW IN PIPES.

7.1 Introduction 393
7.2 Applications of Dimensional Analysis 393
7.3 Transition from Laminar to Turbulent Flow 396
7.4 Phenomena near the Entry to a Pipe 397
7.5 Uniform and Steady Regimes 398
7.6 Laminar Flow in Pipes 400
7.7 Turbulent Flow in Pipes 403
7.8 Bends 410
7.9 Sudden Changes of Section 411
7.10 Diffusers 413
7.11 Nozzles and Orifices 414
7.12 Branched Pipes and Systems 415

CHAPTER 8. FLOW IN OPEN CHANNELS.

8.1 Introduction 424
8.2 Steady and Uniform Regimes 424
8.3 General Theoretical Considerations 427
8.4 The Hydraulic Jump. 434
8.5 Steady Flow of Variable Depth in Channels of Arbitrary Sectional Form 438
8.6 Notches, Weirs and Orifices 446

CHAPTER 9. DYNAMICS OF COMPRESSIBLE FLUIDS.

9.1 Introduction. Importance of Mach Number 459
9.2 Some Elementary Thermodynamic Concepts 463
9.2.1 The Equation of State of a Gas. Functions of State . 463
9.2.2 The First Law of Thermodynamics 464
9.2.3 Reversible and Irreversible Processes 465
9.2.4 Internal Energy and Enthalpy. Specific Quantities . . 468
9.2.5 Entropy and the Second Law of Thermodynamics . . 469
9.3 The Dynamical and Energy Equations 471
9.4 Some Simple Relations for the Steady Adiabatic Flow of a Perfect Gas. Choking 476

9.5 Variation of Entropy with Local Reservoir Pressure in Adiabatic Flow . . . 480
9.6 Potential and Stream Functions. The Potential Function Equation. 481
9.7 Mach Waves. Simple Wave Flow . . . 482
9.8 Shock Waves . . . 489
9.9 The Shock-Expansion Method . . . 496
9.10 The Total Pressure Registered by a Pitot Tube in Compressible Flow 499
9.11 Linearised Perturbation Potential Function Theory . . . 501
  9.11.1 General Relations . . . 501
  9.11.2 Subsonic Flow . . . 502
  9.11.3 Supersonic Flow . . . 505
9.12 The Method of Characteristics . . . 509
9.13 Transonic Flow . . . 513
9.14 Hypersonic Flow. Rarefied Gas Flow . . . 521

CHAPTER 10. OSCILLATIONS AND WAVES.

10.1 Introduction . . . 527
10.2 Oscillations of Liquids in Tubes . . . 530
10.3 Sound Waves in a Fluid . . . 538
10.4 Dispersive Wave Propagation . . . 542
10.5 Waves in a Sea of Uniform Depth . . . 544
10.6 Water Waves in General . . . 549
10.7 Ship Waves and Ship Resistance . . . 553
10.8 Water Hammer . . . 556

CHAPTER 11. FORCES AND MOMENTS ON BODIES IMMERSED IN A STREAM OF FLUID.

11.1 Introduction . . . 568
11.2 The Generation of Lift and Drag . . . 571
11.3 Relation between Lift and Circulation . . . 574
11.4 Lift, Drag and Pitching Moment Characteristics of Aerofoil Sections 579
11.5 Lift and Pitching Moment in Steady Two-Dimensional Potential Flow . . . 584
  11.5.1 Introductory Remarks . . . 584
  11.5.2 Flow About a Flat Plate Set at Incidence . . . 584
  11.5.3 Flow Past a Slightly Cambered Plate. Thin Aerofoil Theory 587
  11.5.4 The Blasius Theorems . . . 592
  11.5.5 Aerodynamic Properties of General Aerofoil Sections . 594
  11.5.6 First Order Theory for Velocity Distribution on a Thin Symmetrical Wing Section at Zero Incidence . . . 597
11.6 Wings of Finite Span in Steady Flow . . . 599
  11.6.1 Introductory Remarks . . . 599
  11.6.2 Unswept Wings of Large Aspect Ratio. The Lanchester-Prandtl Lifting Line Theory . . . 604
  11.6.3 Slender Wing Theory . . . 609
11.7 Aircraft Stability Derivatives . . . 615
11.8 Cascades . . . 618
  11.8.1 Introduction . . . 618
  11.8.2 Basic Relations for Cascade Flow . . . 618
  11.8.3 Further Theoretical and Experimental Features of Cascade Flow . . . 621

CHAPTER 12. PUMPS AND TURBINES

12.1 Introduction 631
12.2 Types of Rotodynamic Machines 632
12.3 Performance Coefficients 636
12.4 Dynamical Principles 642
12.5 Velocity Diagrams 646
12.6 General Discussion of Losses 647
12.7 Pelton Wheel 649
12.8 Barker's Mill or Scotch Turbine. 651
12.9 Inward Flow Turbines 655
12.10 Kaplan Turbine 656
12.11 Centrifugal Pumps 657
12.12 Cavitation 659
12.13 Methods for the Measurement of Efficiency. 660

CHAPTER 13. PROPELLERS, FANS AND WINDMILLS.

13.1 Introduction 666
13.2 Geometry of the Screw Propeller 666
13.3 Slip and Advance Ratio 671
13.4 Propeller Coefficients 671
13.5 Theory of the Actuator Disk 674
13.6 Blade Element Theory of the Propeller 677
13.7 Vortex Theory of the Propeller 679
13.8 Blade Tip Effects and Broad-Bladed Propellers 687
13.9 The Experimental Mean Pitch of a Propeller 688
13.10 Propeller Design 689
13.11 Ship Propulsion 692
13.12 Airscrews 696
13.13 Fans 698
13.14 Windmills 700

APPENDIX 705

INDEX 715

LIST OF PLATES

Plate 9.1 Schlieren Pictures of Shock Formations on a Wing Section . . . . . . *facing p.* 518
Plate 12.1 Types of Turbine Runner . . . *facing p.* 636

CHAPTER I

GENERAL PROPERTIES OF FLUIDS AND HYDROSTATICS

1.1 Definition and Types of Fluid

This chapter includes, besides fundamental definitions and the principles of hydrostatics, a brief discussion of some basic concepts which are treated in greater detail later.

A fluid is defined to be a material substance which cannot sustain a shearing stress when it is at rest. The nature of shearing stress is discussed in § 1.2 but we shall here assume that the reader is already familiar with this concept. It is a consequence of the definition that relative motions of the elements of a mass of fluid can be set up by very small (theoretically infinitesimal) forces and continuous distortion occurs. In more popular language, fluids possess the property of flowing freely.

The definition calls for two comments. First, nothing is said about the rate of movement under the action of shearing stresses. Hence substances which respond very sluggishly to applied shearing stresses may yet be fluids. Second, substances other than fluids may flow but they do not flow under the action of exceedingly small forces. The study of such substances falls within the province of rheology, and lies beyond the scope of this book.

Fluids are conventionally classified as (a) liquids or (b) gases and vapours. These are two possible phases of a fluid which, when they coexist, form an interface (or free surface), and the liquid is the denser of the two†. From the point of view of mechanics the important distinction is between compressible and incompressible fluids. Of course, no fluid is strictly incompressible but in ordinary circumstances the compressibility of most liquids is so small that its influence on the flow is negligible. Throughout this book it will be assumed, unless the contrary is stated, that liquids are incompressible. Gases and vapours, on the other hand, are highly compressible, although if the distorting forces are small a gas may behave as if incompressible.

An inviscid or non-viscous fluid is defined to be such that, even when in motion, it cannot sustain a shearing stress. No real fluid is inviscid, but in many circumstances the influence of viscosity is negligible for fluids of small viscosity, such as air and water. Moreover, the discussion of fluid motion is greatly simplified when the influence of viscosity is neglected. We can go further and assert with truth that the study of the dynamics of inviscid fluids is a necessary preliminary to the theory of the motion of real fluids. A fluid which is both incompressible and inviscid is called *perfect*. This is an abstraction, but a very useful one.

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† These properties relate to the relative magnitudes of the pressure contributions of the random molecular movements and of the inter-molecular forces. In a typical gas the latter are relatively unimportant; in liquids the two contributions are of the same order of magnitude.

A fluid is treated in the theory as a continuum, which implies that an element of the fluid, however small, retains the properties of the fluid in bulk. This again is an abstraction from the facts since all fluids consist of molecules. However, except for gases at very low densities, no appreciable error arises from treating a fluid as continuous. The molecular structure of fluids makes itself manifest in the phenomenon of viscosity. In conformity with the fluid being treated as a continuum, the measure ρ of the *density* of the fluid at a point P is regarded as being the limit of the ratio of the measure of the mass of fluid within a small closed region containing P to the measure of its volume, when *all* the linear dimensions of the region tend to zero. The *specific volume* v is defined as the volume occupied by unit mass of the fluid, the density being constant throughout the mass. Accordingly

$$v = \frac{1}{\rho}. \tag{1.1,1}$$

A fluid is said to be *uniform* when its properties are the same at all points. Compressible fluids cannot be uniform except when the pressure and temperature are constant throughout the fluid.

1.2 Stress and Pressure

The concept of stress is, as we have seen, fundamental in the mechanics of fluids and we shall now explain it. Let P in Fig. 1.2,1 be a point within the fluid and take a plane PKL through P whose normal is PN. Let A be the area of a closed region of the plane containing P. The plane through P divides the fluid into two parts which it will be convenient to call the upper (on the same side as N) and the lower. Then the upper fluid exerts a certain force $\mathbf{F}$ on the lower fluid *through the area* A and, since action and reaction are equal and opposite, the lower fluid exerts a force $-\mathbf{F}$ on the upper fluid through the area A. The average stress on the area A is defined to be

$$\mathbf{f}_A = \frac{\mathbf{F}}{A} \tag{1.2,1}$$

and the stress at P on the plane whose normal is PN is

$$\mathbf{f} = \lim_{A \to 0} \left(\frac{\mathbf{F}}{A}\right). \tag{1.2,2}$$

In reckoning the limit the linear dimensions of the region containing P whose area is A must *all* tend to zero. Both $\mathbf{f}_A$ and $\mathbf{f}$ are vector quantities and it is convenient to resolve them along and perpendicular to the normal PN. The component of $\mathbf{f}$ along PN is the *normal stress* f_n at P on the plane PKL. This is said to be *tensile* when f_n acts upwards so that the upper fluid pulls the lower fluid toward itself and *compressive* when f_n acts downwards. The other component of $\mathbf{f}$ lies in the plane PKL; it is the *shearing stress* f_s. The shearing stress may now be resolved into two components lying in the plane PKL in such directions as may be convenient. Hence we see that the stress at a given point and in a given plane through it has three distinct components, one being the normal stress and the other two constituting the shearing stress. As we alter the orientation of the plane and the direction of the normal PN so, in general, shall we alter the values of the stress components. In the mechanics of fluids we nearly always have to deal with a compressive normal stress and this is called the pressure at the point and on the plane considered; it is given the symbol p. In accordance with the definition of a fluid, the stress in a fluid at rest is always purely normal.

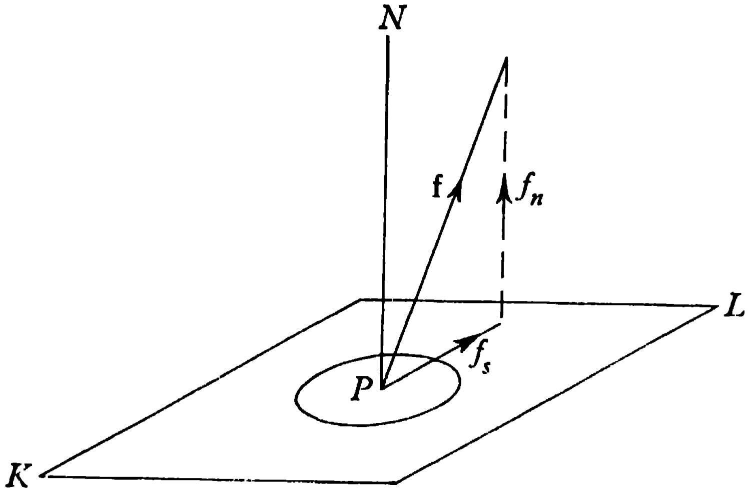

Fig. 1.2,1. Stress on the plane KL at P.

We shall now prove the following fundamental proposition:—

The pressure at a point in a fluid at rest is the same for all planes through the point at the instant considered.

In Fig. 1.2,2 $ABCD$ is a square of side l lying in any plane through the point P and $AB'C'D$ is its orthogonal projection on the reference plane. Let the pressure on the reference plane at P be p_0 and let p be the pressure on $ABCD$. We shall suppose that there is a "body force" such as gravity acting directly on the fluid and that the component of the body force per unit mass of fluid, in the direction perpendicular to the reference plane, is g. The volume of the wedge-shaped element of fluid $ABB'C'D$ is

$$\tfrac{1}{2}l^3 \cos\theta \sin\theta,$$

where θ is the angle between the planes. Hence the component of the total body force on the element, in the direction normal to the reference plane, is $\frac{1}{2}g\rho l^3 \cos\theta \sin\theta$, where ρ is the density of the fluid at P. The normal force on the face $ABCD$ is pl^2 and the component of this perpendicular to the reference plane is $pl^2 \cos\theta$ acting downwards, while the upward force on the base $AB'C'D$ is $p_0 l^2 \cos\theta$ since the area of the base is $l^2 \cos\theta$. Moreover, the sides of the element are all perpendicular to the reference plane and the pressures on them have no component normal to the reference plane. Hence the condition for equilibrium of the element is

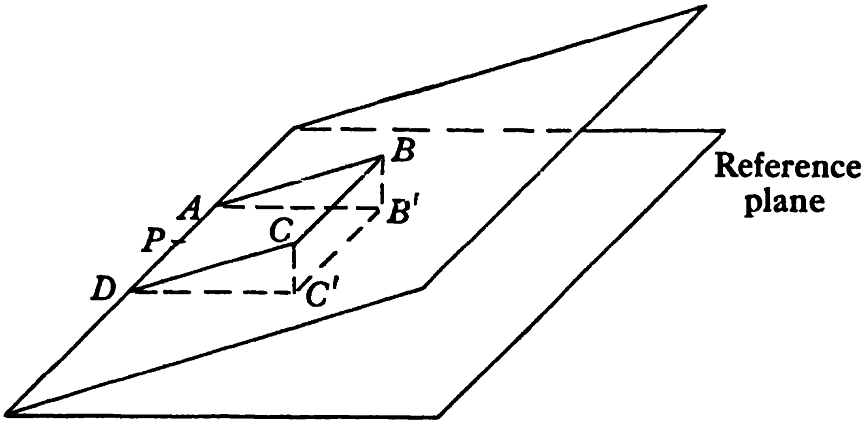

Fig. 1.2,2. Pressure on planes through P.

$$p_0 l^2 \cos\theta - pl^2 \cos\theta = \tfrac{1}{2}g\rho l^3 \cos\theta \sin\theta$$

or

$$p_0 - p = \tfrac{1}{2}g\rho l \sin\theta.$$

In the limit when l tends to zero this becomes

$$p = p_0.$$

Since $ABCD$ is any plane through P, we see that the pressure is the same on all planes through P.

When a fluid of any kind is at rest it is thus legitimate to speak of *the* pressure at a point. We shall show later, by a slight extension of the proof already given, that the same is true for an inviscid fluid in motion. However, the normal stress on a plane is not in general independent of its orientation for a viscous fluid in motion.†

1.3 Physical Properties of Fluids

We have already discussed pressure and density, while viscosity and surface tension are treated separately in §§ 1.4 and 1.5 respectively. Here we shall consider briefly other general properties of fluids which are relevant to mechanics and point out at once that it is not possible to treat the dynamics of compressible fluids without taking account of their thermal properties. Even for liquids the thermal properties are sometimes of importance in problems of mechanics.

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† The quantity often called the pressure and given the symbol p in the dynamical equations of a viscous fluid is the mean of the normal compressive stresses on three mutually perpendicular planes.

The elasticity of volume or *bulk modulus* of a fluid is

$$K = \frac{\text{pressure increment}}{\text{corresponding compressive volume strain}}.$$

Let v be the volume of unit mass of fluid when the pressure throughout the mass is p and let the volume become $v + dv$ when the pressure is $p + dp$. Then the compressive volume strain is $-dv/v$ and we accordingly have by the definition

$$K = -v\frac{dp}{dv} = \rho\frac{dp}{d\rho} \tag{1.3,1}$$

since ρv is constant. The value of the bulk modulus depends on the thermal conditions during the change of volume and there are two important cases, namely constant temperature and constant entropy.† For an isothermal change the modulus is K_T while for an isentropic change it is K_S. It is proved in treatises on thermodynamics that for all fluids

$$\frac{K_S}{K_T} = \frac{c_p}{c_v} \tag{1.3,2}$$

where c_p and c_v are the specific heats of the fluid at constant pressure and at constant volume respectively (see below). The isentropic modulus is always greater than the isothermal one but for liquids the difference is very small. The reciprocal of the bulk modulus is called the *compressibility*.

For a uniform fluid of definite chemical composition the specific volume (or density) is a function only of the pressure p and of the temperature T. The relation connecting the three variables p, v and T is called the *equation of state* or *characteristic equation* and it can be written in various ways, such as

$$F(p, v, T) = 0 \tag{1.3,3}$$

or

$$v = f(p, T). \tag{1.3,4}$$

For a perfect gas the equation of state is

$$pv = RT \tag{1.3,5}$$

where T is now the absolute temperature and where R is the *gas constant* for the gas considered.‡ For a pure gas R is inversely proportional to its molecular weight. The *coefficient of thermal expansion at constant pressure* is

$$\alpha_p = \left(\frac{\partial v}{\partial T}\right)_p \Big/ v \tag{1.3,6}$$

---

† See § 9.2.5

‡ It must be noted that v is the volume of *unit mass* of gas.

while the coefficient of pressure increase at constant volume is

$$\beta_v = \left(\frac{\partial p}{\partial T}\right)_v \Big/ p. \tag{1.3,7}$$

As an example, for a perfect gas

$$\alpha_p = \beta_v = \frac{1}{T}. \tag{1.3,8}$$

The *specific heat* of a substance is defined as the quantity of heat required to raise the temperature of unit mass of it by one unit (degree) and the value depends on the conditions as regards volume and pressure during the heating. The specific heat c_v at constant volume is measured with v constant, so p increases during the heating, whereas the specific heat c_p at constant pressure is measured with p constant and v increases during the heating. For all fluids c_p is greater than c_v but the difference is very small for liquids. In fluid mechanics it is the *heat capacity per unit volume* or product of the density and specific heat which is of importance. The *internal energy* and *enthalpy* of fluids are discussed in chapter 9, while their *thermal conductivity* is considered in § 4.12.

When a liquid of given chemical constitution is in equilibrium with its own vapour at a plane surface of separation the pressure of the vapour at the surface is called the *vapour pressure* of the liquid and this is a function of temperature only. The boiling point is the temperature at which the vapour pressure of the liquid is equal to the total pressure at the free surface. It was shown by Lord Kelvin† that the vapour pressure is increased when the surface of the liquid is convex towards the vapour and reduced when it is concave. The effective vapour pressure is increased when the liquid holds gases in solution. All these facts have a bearing on the phenomena of cavitation (see §§ 3.12 and 12.12).

Sound is propagated in a fluid with velocity a given by Newton's formula, namely

$$a = \sqrt{\left(\frac{K}{\rho}\right)}, \tag{1.3,9}$$

and the bulk modulus is that appropriate to isentropic changes of volume (see § 10.3 for the proof). When the fluid flows with speed V the non-dimensional ratio

$$M = \frac{V}{a} \tag{1.3,10}$$

is called the *Mach number* and this is a convenient variable for indicating the importance of the compressibility of the fluid in the circumstances of the flow (see §§. 45, 4.11 and Chap. 9).

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† "On the equilibrium of vapour at a curved surface of liquid", *Phil. Mag.* Ser. IV, **42,** 448 (1871).

We shall close this catalogue by pointing out that the refractive index of a fluid depends in general on its temperature and pressure; for a gas it depends on the density alone. This variability of the refractive index is used in some techniques for the experimental study of flow.

1.4 Viscosity

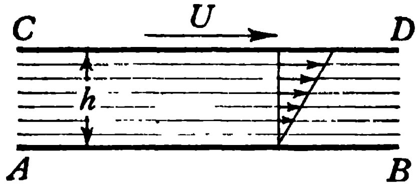

Fig. 1.4,1. Shearing motion in a fluid.

All real fluids resist motion involving internal shearing and for a very important class of fluids the shearing stress which accompanies the shearing is proportional to the rate of shearing. In Fig. 1.4,1 *AB* represents a fixed plane and *CD* a parallel plane at a distance *h* from it. The space between the planes is filled with a uniform fluid and the upper plane moves uniformly with velocity *U* in its own plane. Provided that a steady state has been reached it is found that, for the class of fluids mentioned, there is a shearing stress on the plane *AB* in the direction of *U* and of magnitude

$$\tau = \frac{\mu U}{h} \tag{1.4,1}$$

where μ is constant for the fluid considered, in one definite condition as regards temperature and pressure. The characteristic constant μ is called the *coefficient of viscosity* or simply *the viscosity.*† Fluids whose behaviour conforms with equation (1.4,1) are called Newtonian fluids in contrast with others where τ is not proportional to U. Unless the contrary is distinctly stated, all viscous fluids considered in this book have constant viscosity in this sense. All gases belong to this class, except when the density is so low that the mean free path of the molecules becomes comparable with the distance *h*, while water and other pure liquids and true solutions of substances of small or moderate molecular weight are also included.

We must now consider more closely the state of affairs in the experiment illustrated in Fig. 1.4,1. It is established by many and thorough investigations, particularly of the flow of liquids in pipes, that there is no relative velocity of fluid and rigid boundary at their common surface.‡ Therefore the fluid in contact with *AB* is at rest while that in contact with *CD* moves with *CD*. The velocity *u* of the fluid at a distance *y* from *AB* varies linearly in such a manner as to satisfy these two conditions. Thus *u* is parallel to *U* and

$$u = \frac{yU}{h}. \tag{1.4,2}$$

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† This is Clerk Maxwell's definition.

‡ Except for gases at very low pressures.

The rate of change of u in the direction y perpendicular to u, i.e. the *rate of shear*, is

$$\frac{du}{dy} = \frac{U}{h} \tag{1.4,3}$$

and we may now write (1.4,1) as

$$\tau = \mu \frac{du}{dy}. \tag{1.4,4}$$

It is evident that, since the fluid at any fixed distance y from AB moves with constant speed, there must be the same constant shearing stress on every plane parallel to AB within the fluid. Hence equation (1.4,4) applies throughout the fluid and not merely at the boundaries. Whenever the shearing motion is of the simple type here considered the shearing stress will be given by equation (1.4,4).

Experiment shows that the viscosity of a given fluid depends, in general, on its temperature and pressure. The viscosity of liquids usually decreases with rise of temperature and is nearly independent of pressure for pressures not exceeding a few atmospheres. The viscosity of gases increases with rise of temperature and is independent of pressure (Maxwell's law), except when it is very high, so that the behaviour of the gas deviates sensibly from that of a perfect gas.

When the fluid between the planes is not in a steady state of motion the acceleration caused by the tangential stresses will be proportional to the viscosity but inversely proportional to the density. Hence the acceleration will be proportional to the *kinematic viscosity*,

$$\nu = \frac{\mu}{\rho}. \tag{1.4,5}$$

We shall see later that this quantity is of great importance in relation to fluid motion in general.

1.5 Surface Tension

Liquids behave as if their free surfaces were perfectly flexible membranes having a constant tension σ per unit width; this is called the *surface tension* and it is important to observe that this is neither a force nor a stress but a force per unit length. The value of the surface tension depends on the following factors:

(a) The nature of the liquid.

(b) The nature of the substance with which it is in contact at the surface. This substance may be solid, liquid or gaseous.

(c) The temperature and pressure.

Here we shall only consider some aspects of surface tension which are important in relation to the mechanics of fluids and for fuller information the reader should consult books on physics or physical chemistry.

Suppose that we have a plane material membrane possessing the property that the tension per unit width is constant and equal to σ. Let the membrane have a straight edge of length l. Then the force required to hold the edge fixed is

$$P = \sigma l. \qquad (1.5,1)$$

Now let the edge be pulled so that it is displaced normal to itself by the distance x in the plane of the membrane. The work done in stretching the membrane is then

$$W = Px = \sigma lx. \qquad (1.5,2)$$

The stretching is evidently a reversible process when performed slowly and at constant temperature; hence the work done on the membrane is exactly equal to the additional energy F stored in the membrane. Thus we have

$$F = \sigma lx = \sigma A \qquad (1.5,3)$$

where A is the area added to the membrane and we see that σ is the free energy of the membrane per unit area.† The point of importance is that, if the energy of a surface is proportional to its area, then it will behave exactly as if it were a membrane with a constant tension per unit width and this is altogether independent of the mechanism by which the energy is stored. We thus recognise that the existence of a surface tension at the boundary between two substances is a manifestation of the fact that the stored energy contains a term proportional to the area of the surface. The energy is, in fact, attributable to molecular attractions.

If we consider geometrically similar bodies, of which l is a typical linear dimension, the surface energy will be proportional to σl^2 whereas the gravitational potential energy will be proportional to $g\rho l^3$. Hence, when l is very small, the surface energy will be dominant. This can be illustrated by the way in which a small drop of water will remain at rest on a vertical sheet of glass in spite of gravity. We must guard ourselves from the error of supposing that surface tension is never important just because it is of little importance for liquids in large bulk.

A liquid may or may not wet a solid with which it is in contact; for instance, water and alcohol wet glass but mercury does not. When wetting occurs, the free surface of the liquid (the surface of separation of liquid and atmosphere) makes an acute angle with the solid† and the level of the free surface inside a tube will be higher than that outside (capillary attraction). However, when wetting does not occur, the angle of contact is obtuse and the level of the free surface inside a tube is depressed. Liquid in contact with a granular or porous material which it wets will rise within the material to a level higher than that of the external free surface.

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† The free energy $F = E - TS$, where E is the internal energy and S is the entropy (see Chapter 9).

An important effect of surface tension is that a long cylinder of liquid, at rest or in motion, with a free surface is unstable and breaks up into parts which then assume an approximately spherical shape; this is the mechanism of the breakup of jets into drops. If the jet remained intact the resistance of the atmosphere to the motion would be very small, the jet would have almost exactly the parabolic form of a free trajectory under gravity in vacuo and there would be scarcely any dissipation of the energy and momentum of the jet. However, when drops are formed the resistance of the air becomes large and a rapid dissipation of energy and momentum results.

1.6 Equilibrium of Fluids

We shall begin by considering the equilibrium of a fluid at rest in a uniform gravitational field of intensity g (downward) and shall take the axis OZ vertical with the positive sense upward. Since the fluid is at rest there is a unique pressure p at a point P on all planes through P (see § 1.2). In Fig. 1.6,1 we have a horizontal cylinder of small cross-sectional area dA with plane ends which are normal to the axis of the cylinder and therefore vertical. The pressures on the curved surface of the cylinder are normal and therefore have no component parallel to its axis. The weight of the fluid within the cylinder is a vertical force and has no component parallel to the axis. Hence the condition for the balance of the forces on the cylinder acting parallel to its axis is equality of thrust on the ends. Since the areas of the ends are equal this requires that the pressures on the ends shall be equal. Now if P and Q are any two points on the same horizontal plane and such that the line PQ lies wholly in the fluid we can construct a small cylinder with PQ as axis and the foregoing argument shows that the pressure is the same at P and Q. We can extend this in an obvious manner and show that the pressure is the same at all points in a horizontal plane which can be connected by any continuous polygon or curve lying in the plane and wholly within the fluid (see Fig. 1.6,2).

Fig. 1.6,1. Slender horizontal cylinder lying wholly within the fluid.

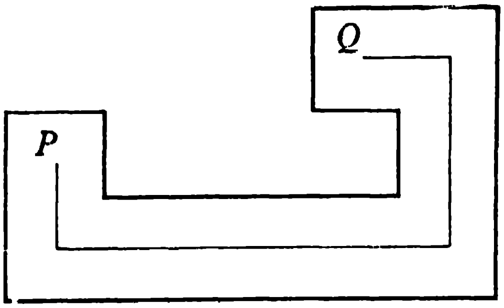

Fig. 1.6,2. Points in fluid and on same horizontal plane connected by broken line lying wholly within the fluid.

Next take a cylinder whose axis is vertical, of sectional area dA and height dz. The curved surface is vertical and the pressures on it have no vertical component. Hence the weight of the fluid contained within the cylinder is balanced by the difference in the thrusts on the ends. Let the pressure at the bottom be p, then the pressure at the top is $p + (dp/dz)\,dz$ when small quantities of the second order are neglected. Thus the condition of equilibrium is

$$p\,dA - \left(p + \frac{dp}{dz}dz\right) dA = g\rho\,dz\,dA$$

since the mass of the contained fluid is $\rho\,dz\,dA$. The last equation yields

$$\frac{dp}{dz} = -g\rho. \qquad (1.6,1)$$

First, take a fluid of constant density. The quantity on the right-hand side of equation (1.6,1) is then constant and we now obtain at once by integration

$$p = p_0 - g\rho z \qquad (1.6,2)$$

where p_0 is the pressure on the plane $z = 0$. This equation is valid throughout the whole of any continuous region filled with fluid as can be seen by supposing any two points in the region to be joined by a continuous line consisting of horizontal and vertical segments. When g is zero or ρ negligibly small equation (1.6,2) shows that the pressure is constant. It usually suffices to take the pressure to be constant when p_0 is large and z always small (positive or negative). Suppose next that the fluid is a liquid with a free surface where the pressure is equal to the atmospheric pressure p_a, assumed constant, and let p be the pressure at a vertical distance h below the free surface, which will be taken as the plane $z = 0$. Then $z = -h$ and $p_0 = p_a$. Accordingly equation (1.6,2) becomes

$$p = p_a + g\rho h. \qquad (1.6,3)$$

This equation yields the *absolute pressure* and the pressure in excess of atmospheric† is obtained by omitting the first term. Equation (1.6,3) can be written alternatively

$$p = p_a + wh \qquad (1.6,4)$$

where $w = g\rho$ is the weight of unit volume of the fluid or its *specific weight.* When the specific weight is constant the gauge pressure is determined by h, which is called the *head.* Subject to the constancy of w, all pressures may be expressed as heads and this is common practice in hydraulic engineering.

We shall next consider the equilibrium of compressible fluids. A fluid is said to be *barotropic* when its density is a function of the pressure alone. No fluid is inherently baratropic but it may behave as such in particular circumstances depending on the nature of the force field acting on it. We shall now investigate the pressure in a compressible fluid at rest in a uniform gravitational field, and we shall show that the fluid is then barotropic. Suppose that

$$1/\rho = v = f(p,. \tag{1.6,5}$$

Equation (1.6,1) then yields

$$f(p)\,dp = -g\,dz$$

and this becomes when integrated

$$\int f(p)\,dp = C - gz. \tag{1.6,6}$$

We shall work out in detail the case where pv^n is constant, with n not unity. We then have

$$v = cp^{-\frac{1}{n}} \tag{1.6,7}$$

where

$$c = p_0^{\frac{1}{n}}/\rho_0 \tag{1.6,8}$$

and p_0, ρ_0, are the pressure and density, respectively, at the level $z = 0$. Here

$$\int f(p)\,dp = c\int p^{-\frac{1}{n}}\,dp = \frac{nc}{n-1}\,p^{\frac{n-1}{n}}$$

and (1.6,6) becomes, in view of the definition of p_0,

$$\frac{nc}{n-1}\left[p_0^{\frac{n-1}{n}} - p^{\frac{n-1}{n}}\right] = gz.$$

Substitute for c from (1.6,8) and rearrange

$$z = \frac{np_0}{(n-1)g\rho_0}\left[1 - \left(\frac{p}{p_0}\right)^{\frac{n-1}{n}}\right]. \tag{1.6,9}$$

The explicit expression for the pressure as a function of height is therefore

$$p = p_0\left[1 - \frac{(n-1)g\rho_0 z}{np_0}\right]^{\frac{n}{n-1}}. \tag{1.6,10}$$

An important special problem is that of a perfect gas with a constant temperature lapse rate, for this is a good approximation to average conditions in the *troposphere*, i.e. that part of the earth's atmosphere lying between the earth's surface and the stratosphere, in which the temperature is nearly constant. Let the absolute temperature at height z above sea level be

$$T = T_0 - az \tag{1.6,11}$$

where a is the constant lapse rate. Then by the equation of state (1.3,5)

$$\rho = \frac{p}{RT} = \frac{p}{R(T_0 - az)}$$

and equation (1.6,1) yields

$$\frac{dp}{p} = -\frac{g\,dz}{R(T_0 - az)}\,.$$

Since p_0 is the pressure when z is zero we get at once by integration†

$$\ln\left(\frac{p}{p_0}\right) = \frac{g}{Ra}\ln\left(1 - \frac{az}{T_0}\right)$$

or

$$p = p_0\left(1 - \frac{az}{T_0}\right)^{\frac{g}{Ra}} \tag{1.6,12}$$

$$= p_0\left(\frac{T}{T_0}\right)^{\frac{g}{Ra}}. \tag{1.6,13}$$

Hence also by the equation of state

$$\rho = \rho_0\left(\frac{p}{p_0}\right)\left(\frac{T_0}{T}\right) = \rho_0\left(\frac{T}{T_0}\right)^{\frac{g}{Ra}-1} \quad \text{by (1.6,13)}$$

$$= \rho_0\left(1 - \frac{az}{T_0}\right)^{\frac{g}{Ra}-1}. \tag{1.6,14}$$

In the "standard atmosphere" adopted in aeronautical calculations the lapse rate is taken as 0·0065 degree C per metre or 1·9812 degree C per 1,000 ft. The index g/Ra in (1.6,13) is then 5·256 while the index in (1.6,14) is 4·256.

We shall now consider the equilibrium of a fluid in a non-uniform field of force. Take a line in any direction through a point P and let s be the distance from P measured along the line in the positive sense. Let the component of the force per unit mass in the same direction and sense be f_s. In Fig. 1.6,1 let the axis PQ of the cylinder be of length ds and lie in the chosen direction and consider the equilibrium of the components of force along PQ acting on the fluid within the cylinder. The pressures on the curved surface are normal to PQ and therefore have no component in the direction PQ. The thrust on the end at P is $p\,dA$ while that at Q is $[p + (\partial p/\partial s)\,ds]\,dA$ and the component of body force in the direction PQ is $\rho F_s ds dA$. Hence the condition for equilibrium is†

---

† The symbol ln stands for the logarithm to the base e, i.e. the "natural logarithm"

$$\frac{\partial p}{\partial s}\,ds\,dA = \rho f_s\,ds\,dA$$

or

$$\frac{\partial p}{\partial s} = \rho f_s. \tag{1.6,15}$$

Equation (1.6,1) is a special case of this since $-g$ is the component of the force per unit mass in the sense of z increasing. Suppose next that the body force is conservative i.e. can be equated to minus the gradient of a potential χ. Then we can write

$$f_s = -\frac{\partial \chi}{\partial s} \tag{1.6,16}$$

and (1.6,15) can therefore be written

$$\frac{1}{\rho}\frac{\partial p}{\partial s} + \frac{\partial \chi}{\partial s} = 0. \tag{1.6,17}$$

This equation shows that the pressure is constant on an equipotential surface, for $\partial\chi/\partial s$ is zero when the element of length ds lies in the surface and $\partial p/\partial s$ must therefore be zero; in a uniform vertical gravitational field the equipotentials are horizontal planes and the surfaces of constant pressure are therefore also horizontal planes, as we already know.

When the body forces are conservative and the fluid in equilibrium the density is necessarily constant on an equipotential surface. Let us identify the element of length ds with dn, the length of the segment of the normal to the equipotential cut off between it and the equipotential $\chi + d\chi$; these equipotentials are also the surfaces for which the pressure has the constant values p and $p + dp$ respectively. Then equation (1.6,17) becomes

$$\frac{1}{\rho}\frac{\partial p}{\partial n} = -\frac{\partial \chi}{\partial n}$$

or

$$dp = -\rho\,d\chi.$$

Since both $d\chi$ and dp are constant all over the equipotential it follows that ρ is constant on the equipotential. Hence any fluid in equilibrium in a conservative field of force is effectively barotropic, for the surfaces of constant pressure coincide with the surfaces of constant density, so the density is some function of the pressure. We are now able to integrate equation (1.6,17). Let the relation between pressure and density be given by equation (1.6,5) and let

$$\varpi = \int \frac{dp}{\rho} = \int f(p)\,dp. \tag{1.6,18}$$

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† See p. 43 for the meaning of partial differential coefficients like $\partial p/\partial s$.

Then equation (1.6,17) becomes

$$\frac{\partial \varpi}{\partial s} + \frac{\partial \chi}{\partial s} = 0.$$

Since this is true for all possible directions of the element of length ds it follows that

$$\varpi + \chi = C \tag{1.6,19}$$

where C is a constant (see also equation 3.5,21). In the case of a uniform fluid this becomes

$$\frac{p}{\rho} + \chi = C. \tag{1.6,20}$$

For a uniform gravitational field

$$\chi = gz \tag{1.6,21}$$

and (1.6,20) becomes

$$\frac{p}{\rho} + gz = C$$

which is equivalent to (1.6,2). In order to determine the function $f(p)$ which appears in equation (1.6,18) we must be given some supplementary information. For instance, we may be told that the temperature or the entropy is constant throughout the fluid or we may be given the spatial distribution of temperature, as in the case of the "standard atmosphere" worked out above.

A fluid cannot be in equilibrium in a non-conservative field of force. In such a field it is possible to move a particle in a closed circuit in such a manner that a net positive amount of work is done upon it by the field. The pressures and body forces cannot then form a balanced system and acceleration must ensue.

1.7 Steady Rotation of an Inviscid Fluid about a Vertical Axis

Although the steady rotation of a fluid about a vertical axis in a uniform gravitational field is a dynamical problem it can be treated by the methods of hydrostatics and we shall discuss it here. Whenever the motion of the fluid is known, which implies that the acceleration of every fluid element is known, we may find the distribution of pressure by taking the fluid to be at rest with an additional body force (force per unit mass) vectorially equal to the acceleration but reversed in direction.

We shall suppose that the motion is steady and that the angular velocity ω about the fixed vertical axis is a function only of r, the radial distance of the point from the axis. Hence a particle of fluid describes a circular path in a horizontal plane with constant speed $r\omega$; its acceleration is radial and inward and of magnitude $\omega^2 r$. Hence the pressure is the same as in a stationary fluid with a body force having the constant component g vertically downward and a radial outward component $\omega^2 r$. Since ω is a function of r, the radial component of the body force is a function of r alone.

In Fig. 1.6,1 let the cylinder PQ be radial (and therefore horizontal) and of length dr. The resultant inward thrust on the ends is $(\partial p/\partial r)\,dr\,dA$ and this must balance the "centrifugal force" $\omega^2 r\rho\,dr\,dA$. Hence the pressure satisfies the equations

$$\frac{\partial p}{\partial r} = \rho\omega^2 r \tag{1.7,1}$$

$$\frac{\partial p}{\partial z} = -g\rho \tag{1.7,2}$$

where the last equation is identical with (1.6,1) except that the differential coefficient is now partial. Let ρ be constant and put

$$P(r) = \int \omega^2 r\,dr \tag{1.7,3}$$

which can be determined when ω is a given function of r. Equation (1.7,1) can be integrated with respect to r, with z treated as constant, and yields

$$p = \rho P(r) + F_1(z) \tag{1.7,4}$$

where the "constant of integration" is some function of z; this last equation is the most general one which yields (1.7,1) when differentiated partially with respect to r. Similarly equation (1.7,2) yields when integrated with respect to z, with r treated as constant,

$$p = -g\rho z + F_2(r) \tag{1.7,5}$$

where the "constant of integration" is now some function of r. The right-hand sides of equations (1.7,4) and (1.7,5) must be equal and we accordingly deduce that

$$p = \rho P(r) - g\rho z + C \tag{1.7,6}$$

where C is independent of both r and z. This is the most general equation which yields (1.7,1) and (1.7,2) when differentiated partially with respect to r and z respectively.

As a simple example let us take the case where the fluid rotates about the vertical axis like a rigid body, i.e. with ω independent of r; this motion was called the *forced vortex* by Rankine. From (1.7,3) we may write

$$P(r) = \tfrac{1}{2}\omega^2 r^2$$

and (1.7,6) becomes

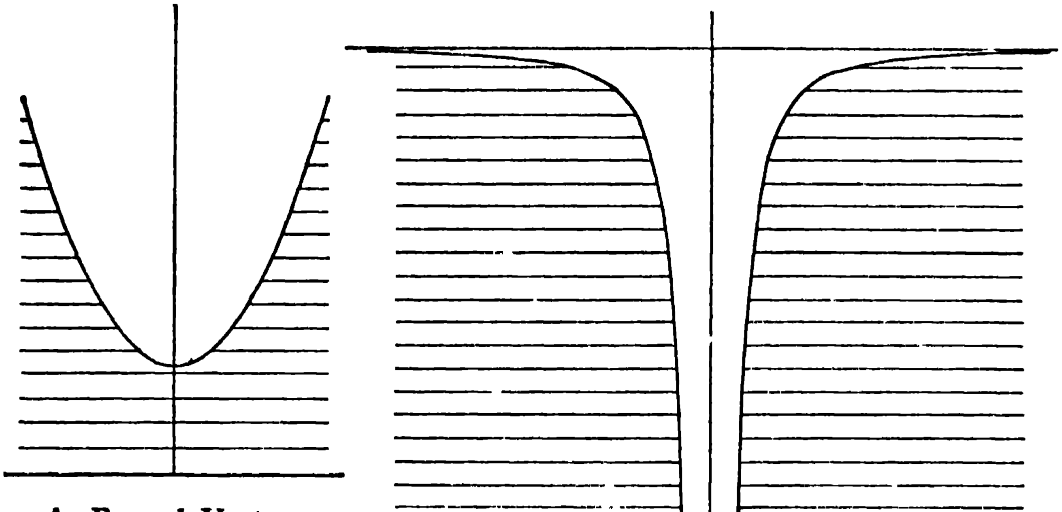
$$p = \tfrac{1}{2}\rho\omega^2 r^2 - g\rho z + p_0 \tag{1.7,7}$$

where p_0 is the pressure at the origin of coordinates. The surfaces of constant pressure are therefore the paraboloids of revolution

$$z = \frac{\omega^2 r^2}{2g} + \frac{p_0 - p}{g\rho}\,. \tag{1.7,8}$$

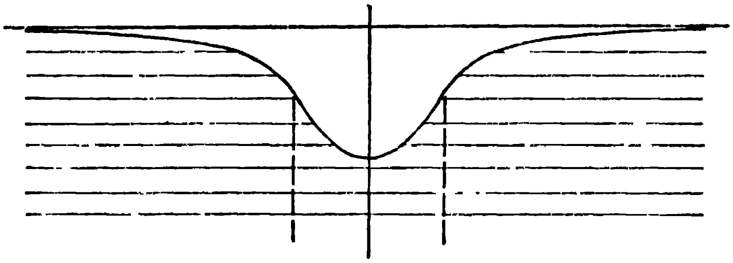
In particular, the free surface of a liquid rotating as a rigid body about a vertical axis is a paraboloid of revolution (see Fig. 1.7,1 A).

Next let us take the *free vortex* in Rankine's terminology. Here the motion of the fluid is *irrotational* (see § 2.12) which means that, although each small element of fluid describes a fixed circular path, it has zero rate of spin about its own axis. It can then be shown that the *circulation* (see § 2.10) in the circular path is independent of the radius. This requires that the product of the velocity of flow and the circumference of the circle shall be constant. Hence the velocity varies inversely as the radius and we may write



A. Forced Vortex

B. Free Vortex



C. Combined Vortex

Fig. 1.7,1.

$$\omega = \frac{c}{r^2} \tag{1.7,9}$$

where c is a constant. Accordingly (1.7,3) yields

$$P(r) = -\frac{c^2}{2r^2}$$

and (1.7,6) becomes

$$p = C - \frac{\rho c^2}{2r^2} - g\rho z. \tag{1.7,10}$$

The surfaces of constant pressure, including the free surface, are given by

$$z = \frac{(C - p)}{g\rho} - \frac{c^2}{2gr^2}. \tag{1.7,11}$$

When r is large the surface approaches a horizontal plane but when r is very small z is large and negative. The surface may be likened to a trumpet with its mouth upward and blending into a horizontal plane (see Fig. 1.7,1 B)

As a final example we shall take Rankine's *combined vortex*. This consists of an inner cylindrical core of radius a rotating as a solid with angular velocity ω, surrounded by a free vortex, the velocity being continuous at the radius a. By equation (1.7,9) the condition for continuity is

$$c = \omega a^2. \tag{1.7,12}$$

The pressure in the core ($a \geqslant r \geqslant 0$) is given by equation (1.7,7) and the pressure at radius a is

$$p(a) = \tfrac{1}{2}\rho\omega^2 a^2 - g\rho z + p_0. \tag{1.7,13}$$

But the pressure is continuous at this radius and by (1.7,10) we have

$$\begin{aligned} C &= p(a) + \frac{\rho c^2}{2a^2} + g\rho z \\ &= \rho\omega^2 a^2 + p_0 \end{aligned} \tag{1.7,14}$$

by (1.7,12) and (1.7,13). Hence by (1.7,10) and (1.7,12) the pressure when $r > a$ is given by

$$p = p_0 + \rho\omega^2 a^2\left(1 - \frac{a^2}{2r^2}\right) - g\rho z. \tag{1.7,15}$$

The free surface has a dimpled form as shown in Fig. 1.7,1 C.

1.8 Thrusts on Plane Surfaces

We shall consider the thrust on a plane area immersed in a liquid at rest and having a free surface, situated in a uniform gravitational field. The plane in which the area lies is inclined at the angle α to the vertical and the horizontal breadth of the area is b at the perpendicular distance y from the line of intersection of the plane and the free surface (see Fig. 1.8,1). Usually we are interested in the thrust caused by the presence of the fluid and not in the contribution due to the atmospheric pressure. For example, in considering the thrust on a dam we neglect the atmospheric pressure which acts on both faces of the dam and causes no stresses in the dam other than a uniform compressive stress equal to the atmospheric pressure. Unless the contrary is stated, we shall accordingly base our calculations of thrust on the *gauge pressure* as defined in § 1.6. If the total thrust is required it can be obtained at once by adding the atmospheric thrust which is the product of the area and the atmospheric pressure.

Consider a horizontal strip of length b and width dy (see Fig. 1.8,1). This lies at a perpendicular distance $y \cos \alpha$ below the free surface and the gauge pressure is accordingly $g\rho y \cos \alpha$. Hence the thrust on the strip is $g\rho by \cos \alpha \, dy$ and the total thrust is

$$P = g\rho \cos \alpha \int by \, dy, \tag{1.8,1}$$

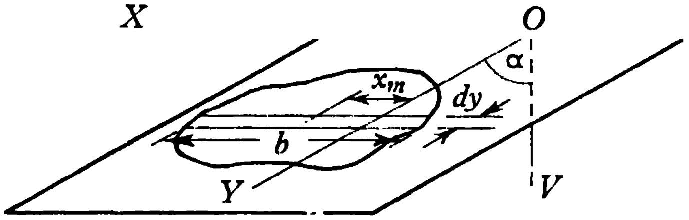


Fig. 1.8,1. Plane area immersed in liquid with free surface. OX lies in free surface and OY is perpendicular to OX.

where the integral covers the area. Let $\bar{y}$ be the ordinate of the centroid of the area whose measure is A. By definition

$$A\bar{y} = \int by \, dy \tag{1.8,2}$$

and equation (1.8,1) can therefore be written

$$P = g\rho \bar{y} A \cos \alpha. \tag{1.8,3}$$

Now the pressure at the centroid is

$$\bar{p} = g\rho \bar{y} \cos \alpha \tag{1.8,4}$$

and accordingly we have

$$P = \bar{p} A. \tag{1.8,5}$$

This important result can be put in words thus: *the thrust on any plane area is equal to the product of the area and the pressure at its centroid.* If the absolute thrust is required $\bar{p}$ must be taken as the absolute pressure at the centroid.

The thrusts on small elements of the area considered are all normal to the plane of the area and directed from the fluid towards the plane. These parallel forces have a unique resultant which has the same direction and sense as the thrusts on the elements of area and is equal to their sum. The line of action of the resultant thrust intersects the plane of the area in a point called the *centre of pressure*, whose coordinates are (x_p, y_p). By taking moments about OX we obtain

$$P y_p = g\rho \cos \alpha \int by^2 \, dy$$

and from (1.8,1) it follows that

$$y_p = \frac{\int by^2 \, dy}{\int by \, dy}. \tag{1.8,6}$$

This formula can be put in words thus: *the perpendicular distance of the centre of pressure from the line of intersection of the plane of the area with the free surface is equal to the second moment of the area about the line of intersection divided by the first moment of the area about this line.* If it be required to find the centre of pressure for the absolute thrusts we may use the same formulae but take OX to be the line of intersection of the plane of the area with a fictitious free surface situated at the vertical height $p_a/g\rho$ above the actual free surface. Next, let x_m be the abscissa of the mid-point of the horizontal chord of length b. Then we find by taking moments about OY that

$$Px_p = g\rho \cos\alpha \int bx_m y \, dy$$

or

$$x_p = \frac{\int bx_m y \, dy}{\int by \, dy}. \qquad (1.8,7)$$

The numerator of the fraction appearing in the last equation can be written alternatively

$$\iint xy \, dx \, dy = I_{xy} \qquad (1.8,8)$$

where I_{xy} is the *product moment* of the area for the axes OX, OY. This vanishes when OY is a principal axis of the area. Accordingly, the abscissa x_p of the centre of pressure is zero when OY is a principal axis. In particular, this is true when OY is an axis of symmetry of the area. Equations (1.8,6) and (1.8,7) show that the position of the centre of pressure in the area is independent of the inclination of its plane to the vertical, i.e. rotation of the plane about its line of intersection with the free surface† does not shift the centre of pressure in the plane.

Thrusts and centres of pressure are most conveniently computed in a systematic manner by supposing the area to be divided into strips perpendicular to the line of intersection of the plane of the area with the free surface. For a rectangle of width b and ordinate y with its base in OX the thrust is

$$P = \tfrac{1}{2} g\rho b y^2 \cos\alpha \qquad (1.8,9)$$

while the moment of the thrust about OX is

$$M = \tfrac{1}{3} g\rho b y^3 \cos\alpha. \qquad (1.8,10)$$

Hence the total thrust on the area shown in Fig. 1.8,2 is

$$P = \tfrac{1}{2} g\rho \cos\alpha \int (y_l^2 - y_u^2) \, dx. \qquad (1.8,11)$$

This can be written alternatively

$$P = \tfrac{1}{2} g\rho \cos\alpha \oint y^2 \, dx \qquad (1.8,12)$$

---

† When the fluid is water this is called the *water line.*

where the circuit integral is taken round the boundary of the area in the sense of the arrows in Fig. 1.8,2. The integral can be found from the graph of the squared ordinates or directly by means of a suitable integrator. Also we have

$$M = \tfrac{1}{3} g\rho \cos\alpha \int (y_l^{\,3} - y_u^{\,3})\, dx \qquad (1.8,13)$$

$$= \tfrac{1}{3} g\rho \cos\alpha \oint y^3\, dx \qquad (1.8,14)$$

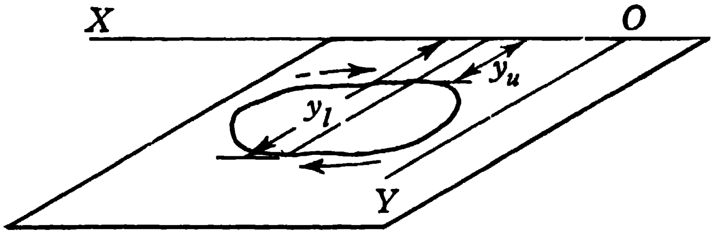


Fig. 1.8,2. Diagram to illustrate the systematic computation of thrust and centre of pressure.

and this integral can be found in a manner similar to that used for the former. The ordinate of the centre of pressure is accordingly

$$y_p = \frac{M}{P} = \frac{2}{3}\frac{\oint y^3\, dx}{\oint y^2\, dx} \qquad (1.8,15)$$

and by taking moments about OY we obtain

$$x_p = \frac{\oint xy^2\, dx}{\oint y^2\, dx}\,. \qquad (1.8,16)$$

We shall now give some useful illustrative examples.

ILLUSTRATIVE EXAMPLES

Example 1. *Trapezium with base in OX and sides normal to OX*

Fig. 1.8,3 shows the trapezium where we have taken OY coincident with one of its sides; the lengths of the sides are y_1, y_2 and the base is b. The equation to the line AB is

$$y = l + kx$$

and $\int_0^b y^2\, dx = l^2 b + lkb^2 + \tfrac{1}{3}k^2 b^3.$

But $l = y_1$ and $bk = y_2 - y_1$

so the integral is

$$\tfrac{1}{3}b[3y_1^{\,2} + 3y_1(y_2 - y_1) + (y_2 - y_1)^2]$$
$$= \tfrac{1}{3}b[y_1^{\,2} + y_1y_2 + y_2^{\,2}].$$

Hence from (1.8,11)

$$P = \tfrac{1}{6} g\rho b(y_1^{\,2} + y_1y_2 + y_2^{\,2}) \cos\alpha. \qquad (1.8,17)$$

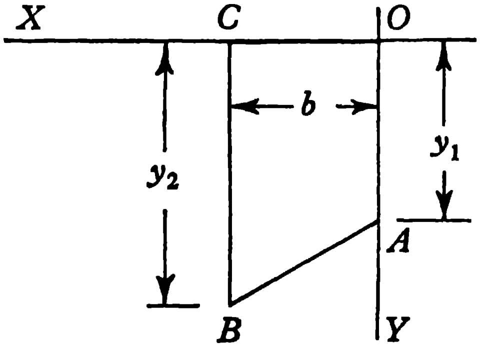


Fig. 1.8,3. Trapezium with base OC lying in the free surface. The sides OA and CB are normal to OX.

Next

$$\begin{aligned}\int_0^b y^3\,dx &= l^3b + \tfrac{3}{2}l^2kb^2 + lk^2b^3 + \tfrac{1}{4}k^3b^4\\ &= \tfrac{1}{4}b[4y_1^3 + 6y_1^2(y_2 - y_1) + 4y_1(y_2 - y_1)^2 + (y_2 - y_1)^3]\\ &= \tfrac{1}{4}b[y_1^3 + y_1^2y_2 + y_1y_2^2 + y_2^3].\end{aligned}$$

Hence from (1.8,13)

$$M = \tfrac{1}{12}g\rho b(y_1^3 + y_1^2y_2 + y_1y_2^2 + y_2^3)\cos\alpha \tag{1.8,18}$$

and

$$y_p = \frac{y_1^3 + y_1^2y_2 + y_1y_2^2 + y_2^3}{2(y_1^2 + y_1y_2 + y_2^2)} = \frac{(y_2^4 - y_1^4)}{2(y_2^3 - y_1^3)}. \tag{1.8,19}$$

The integrals of the powers of y can be obtained very neatly as follows:—

$$\int_0^b y^n\,dx = \int_{y_1}^{y_2} y^n \frac{dx}{dy}\,dy = \frac{1}{k}\int_{y_1}^{y_2} y^n\,dy = \frac{1}{(n+1)k}(y_2^{n+1} - y_1^{n+1})$$

and

$$y_2^{n+1} \quad y_1^{n+1} = (y_2 - y_1)(y_2^n + y_2^{n-1}y_1 + y_2^{n-2}y_1^2 + \dots + y_1^n).$$

Finally

$$\begin{aligned}\int_0^b xy^2\,dx &= \tfrac{1}{2}l^2b^2 + \tfrac{2}{3}lkb^3 + \tfrac{1}{4}k^2b^4\\ &= \frac{b^2}{12}[y_1^2 + 2y_1y_2 + 3y_2^2]\end{aligned}$$

and by (1.8,16)

$$x_p = \frac{b}{4}\left(\frac{y_1^2 + 2y_1y_2 + 3y_2^2}{y_1^2 + y_1y_2 + y_2^2}\right) \tag{1.8,20}$$

where x_p is the distance of the centre of pressure from the ordinate y_1.

With the help of these results the thrust and centre of pressure can be found for any polygonal area. This is illustrated in the next example.

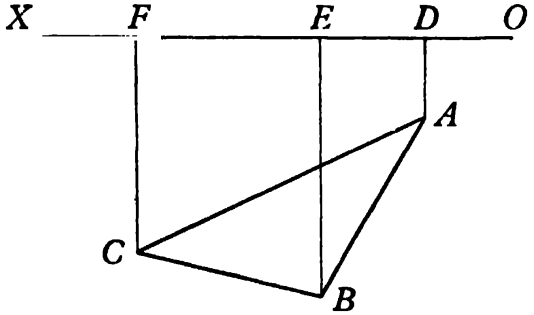


Fig. 1.8,4. General triangle.

Example 2. General triangular area

In Fig. 1.8,4 the thrust P on the triangle ABC

= thrust on trapezium $ABED$
+ thrust on trapezium $BCFE$
− thrust on trapezium $ACFD$.

Let the coordinates of A, B and C be (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively. Then by equation (1.8,17) the thrust on $ABED$

$$= \tfrac{1}{6}g\rho(x_2 - x_1)(y_1^2 + y_1y_2 + y_2^2)\cos\alpha.$$

Hence

$$\frac{P}{\frac{1}{6}g\rho\cos\alpha} = (x_2 - x_1)(y_1^2 + y_1y_2 + y_2^2) + (x_3 - x_2)(y_2^2 + y_2y_3 + y_3^2) + (x_1 - x_3)(y_3^2 + y_3y_1 + y_1^2). \qquad (1.8,21)$$

The reader's attention is drawn to the fact that the second and third terms in the expression on the right hand side of equation (1.8,21) are obtained from the first term merely by cyclic interchange of the suffixes.† Now the coefficient of x_1 in this expression can be written

$$(y_3^2 - y_2^2) + y_1(y_3 - y_2) = (y_3 - y_2)(y_1 + y_2 + y_3)$$

and the coefficients of x_2 and x_3 can be similarly factorised. Hence

$$\frac{P}{\frac{1}{6}g\rho\cos\alpha} = (y_1 + y_2 + y_3)[x_1(y_3 - y_2) + x_2(y_1 - y_3) + x_3(y_2 - y_1)].$$

But the expression in the square bracket

$$= (x_2 - x_1)(y_1 + y_2) + (x_3 - x_2)(y_2 + y_3) + (x_1 - x_3)(y_3 + y_1) = 2A \qquad (1.8,22)$$

where A is the area of the triangle ABC. Hence (1.8,21) becomes

$$P = \tfrac{1}{3}g\rho A(y_1 + y_2 + y_3)\cos\alpha. \qquad (1.8,23)$$

This shows that the depth of immersion of the centroid of the triangle is $\frac{1}{3}(y_1 + y_2 + y_3)\cos\alpha$, which can easily be proved directly.

From (1.8,18) the moment of the thrust on the triangle about OX is given by

$$\frac{M}{\frac{1}{12}g\rho\cos\alpha} = (x_2 - x_1)(y_1^3 + y_1^2y_2 + y_1y_2^2 + y_2^3) + (x_3 - x_2)(y_2^3 + y_2^2y_3 + y_2y_3^2 + y_3^3) + (x_1 - x_3)(y_3^3 + y_3^2y_1 + y_3y_1^2 + y_1^3). \qquad (1.8,24)$$

The coefficient of x_1 in the expression on the right-hand side of (1.8,24) can be written

$$(y_3^3 - y_2^3) + y_1(y_3^2 - y_2^2) + y_1^2(y_3 - y_2) = (y_3 - y_2)[y_1^2 + y_2^2 + y_3^2 + y_1y_2 + y_2y_3 + y_3y_1]$$

after a little reduction, and the coefficients of x_2 and x_3 can be similarly factorised. Therefore the expression in (1.8,24) becomes

$$[y_1^2 + y_2^2 + y_3^2 + y_1y_2 + y_2y_3 + y_3y_1] \times [x_1(y_3 - y_2) + x_2(y_1 - y_3) + x_3(y_2 - y_1)] = 2A[y_1^2 + y_2^2 + y_3^2 + y_1y_2 + y_2y_3 + y_3y_1]$$

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† See second footnote on p. 75.

and we obtain

$$M = \tfrac{1}{6}g\rho A[y_1^2 + y_2^2 + y_3^2 + y_1y_2 + y_2y_3 + y_3y_1]\cos\alpha. \quad (1.8,25)$$

Hence from (1.8,23)

$$y_p = \frac{y_1^2 + y_2^2 + y_3^2 + y_1y_2 + y_2y_3 + y_3y_1}{2(y_1 + y_2 + y_3)}. \quad (1.8,26)$$

The position of the centre of pressure can be found by the following rule which accords with (1.8,26): it is the centre of mass of three masses placed at the mid-points of the sides of the triangle and each mass is equal to the depth of the point at which it is situated. The correctness of the rule is more apparent when (1.8,26) is written in the form

$$y_p = \frac{[\frac{1}{2}(y_1 + y_2)]^2 + [\frac{1}{2}(y_2 + y_3)]^2 + [\frac{1}{2}(y_3 + y_1)]^2}{\frac{1}{2}(y_1 + y_2) + \frac{1}{2}(y_2 + y_3) + \frac{1}{2}(y_3 + y_1)} \quad (1.8,27)$$

which is also convenient for computation. The corresponding expression for the abscissa of the centre of pressure is

$$x_p = \frac{\frac{1}{2}(x_1 + x_2)\frac{1}{2}(y_1 + y_2) + \frac{1}{2}(x_2 + x_3)\frac{1}{2}(y_2 + y_3) + \frac{1}{2}(x_3 + x_1)\frac{1}{2}(y_3 + y_1)}{\frac{1}{2}(y_1 + y_2) + \frac{1}{2}(y_2 + y_3) + \frac{1}{2}(y_3 + y_1)} \quad (1.8,28)$$

which can be deduced from (1.8,20).

The position of the centre of pressure can be found very easily by application of the reciprocal property of the moment of the hydrostatic thrust enunciated in Exercise No. 16 at the end of this chapter.

1.9 Buoyancy and Thrusts on Curved Surfaces

The buoyancy of a body wholly or partly immersed in a fluid at rest, situated in a gravitational or other field of force, is defined as the resultant thrust of the fluid upon the body. All questions on buoyancy can be anwsered by application of the *Principle of Archimedes*:

The buoyancy of any body is vectorially equal and opposite to the weight of the fluid displaced by the body and has the same line of action.

In this enunciation the word *weight* must be interpreted as the resultant body force, irrespective of the cause of the force. The principle, as stated, is true of uniform and non-uniform fields of force and for all types of true fluids. It is to be understood that the fluid considered as displaced by the body is reckoned by supposing the surfaces of constant density in the fluid surrounding the body to be continued through the region occupied by the body. The principle is of special utility in the very important case of the uniform gravitational field, since the weight is then the resultant of parallel forces, whereas the thrusts on the elements of the surface of the body are not all parallel.

The proof of the principle is very simple. Let us imagine the body to be removed and let the space occupied by it be filled with fluid whose density is continuous with that of the surrounding fluid, in the manner already explained. Then the whole body of fluid will be in equilibrium while the thrust on the replaced fluid will be equal to the buoyancy of the body. But this thrust must balance the weight of the replaced fluid; therefore the buoyancy is equal and opposite to the weight of the replaced fluid.

Let us now consider in more detail the case where the fluid and the field of force are both uniform. Then the weight of the displaced fluid is $g\rho V$, where V is the immersed volume of the body. The line of action of the buoyancy is vertical and passes through the centre of gravity of the displaced fluid, i.e. the centroid of the displaced volume. This centroid is called the *centre of buoyancy*. For further information on the buoyancy and stability of floating bodies the reader should consult treatises on hydrostatics or naval architecture.

We shall now briefly discuss the stability of a totally immersed body in a uniform gravitational field. First, let the fluid be of constant density whereas the body is compressible. In the initial position the body is in equilibrium, so its constant weight just balances the buoyancy. Suppose now that the body receives a small upward displacement. The pressure is therefore reduced and the body, being elastic, expands. The buoyancy is increased and it now exceeds the weight of the body. Hence there is a resultant upward force which tends to increase the upward displacement. If the initial displacement had been downward the resultant force would be downward and would again tend to increase the initial displacement. Thus the equilibrium is unstable for vertical displacements and this can be illustrated by the behaviour of the toy called the *Cartesian diver*. Second, let the immersed body be rigid and let the density of the fluid decrease with height, as will always be true when the fluid is compressible. Here the volume of the body is constant and the buoyancy will be reduced by an upward displacement but increased by a downward one. Hence the equilibrium is stable for vertical displacements. When both the body and the fluid are compressible the stability will depend on the ratio of the compressibilities. In all cases the stability is neutral for horizontal displacements.

The foregoing has an application to the stability of the atmosphere, for we may take the "body" to be an element of the atmosphere itself. It will be assumed that when the element moves its entropy remains constant. Then in an upward displacement the pressure of the element will be that of its surroundings and the temperature of the element will fall with the pressure at the isentropic rate. If the rate of fall of temperature with pressure in the surrounding atmosphere falls short of the isentropic rate, the density in the surrounding atmosphere will be less than that of the element which will therefore tend to fall back to its original level. In such circumstances the atmosphere is stable. But if the temperature lapse rate in the atmosphere exceeds the isentropic rate the equilibrium will be unstable. As another example, a stratified liquid will be in stable equilibrium only when the density decreases upwards. This can also be deduced from the fact that the potential energy of a conservative system is a minimum in a state of stable equilibrium. Clearly the potential energy is not at a minimum if the density at any level exceeds that at any lower level.

We shall now consider the resultant thrust on a curved surface exposed to the pressure of a fluid at rest in equilibrium. Let it be required to find the component P_s of the thrust in a chosen direction OS. Let dA be an element of area of the surface and let θ be the angle between the direction OS (taken in its positive sense) and the normal to the surface at the element, drawn in the sense from the fluid towards the surface. Then the element of thrust in the chosen direction is $p\,dA\cos\theta$ and the total for the surface is to be obtained by integration. In special cases and by the use of special devices, results of great simplicity are obtainable, as exemplified below.

First, let p be constant (uniform hydrostatic pressure). This will be true when body forces are absent and will be a working approximation to the truth when the range of pressure variation due to the body forces is a small fraction of the minimum pressure, e.g. in a vessel containing compressed gas. The integral of $dA\cos\theta$ taken over the surface considered is then the net area A_s of the normal projection of the surface on a plane perpendicular to OS. Hence we obtain

$$P_s = pA_s. \tag{1.9,1}$$

When the surface is closed, the net projected area is zero and the thrust is therefore also zero. This also follows at once from the Principle of Archimedes, for the resultant thrust on a closed surface is the buoyancy and this vanishes in the absence of body forces.

Next let us take the important case of a liquid of constant density in a uniform gravitational field. Suppose that the curved surface considered has an edge which is a closed plane curve lying in a vertical plane. Imagine the edge to be covered by a plane face; we now have a closed surface which may be supposed to be totally immersed. By the Principle of Archimedes the total external thrust is a vertical force equal and opposite to the weight of the enclosed liquid. Hence the vertical component of the resultant internal thrust on the curved surface is equal to the weight of this liquid and has the same line of action. The horizontal component of the internal thrust is vectorially equal to the external thrust on the plane face; this thrust and its line of action can be found as explained in § 1.8. If the plane of the edge is not vertical, the vertical downward component of the external thrust on the plane face must be added to the weight of the enclosed fluid in calculating the vertical component of the thrust on the internal curved surface.

1.10 Units and Constants

Consistent units (see Chapter 4) are used throughout this book unless the contrary is expressly stated. As a consequence, it is not necessary to present the units for each quantity when writing the equations connecting physical quantities and the equations can then be regarded as equations between the measures of these quantities. For example, the relation between the measures of thrust, pressure and area of plane surface is then

$$P = pA \tag{1 10,1}$$

irrespective of the choice of fundamental units. The relation between the measures of force, mass and acceleration is

$$F = ma \tag{1.10,2}$$

where m and a are the measures of the mass and acceleration, respectively. The reader is warned that adherence to this rule conflicts with some conventions that are still current. For example, it is usual in some English-speaking countries to use the foot as the unit of length but to quote pressures in pounds weight per square inch (p.s.i.); this requires the introduction of the conversion factor 144, i.e. the square of the number of inches in 1 foot.

In pure science it has been usual to adopt the C.G.S. system of units, in which the unit of length is the centimetre, the unit of mass is the gram and the unit of time is the second.† There are two systems of consistent units in which the unit of length is the foot and the unit of time is the second. In the first the unit of mass is the pound and the unit of force, called the *poundal*, is such that it imparts an acceleration of 1 ft/sec^2 to a mass of 1 pound. This unit of force is small and is scarcely ever used by engineers. In the second system the unit of mass is the *slug*, consisting of 32·17 pounds. The unit force then imparts unit acceleration (1 ft/sec^2) to a mass of 32·17 pounds or an acceleration of 32·17 ft/sec^2 to a mass of 1 pound. Hence the unit of force is the same as the weight of 1 pound at any place where g has the value 32·17 ft/sec^2. This unit of force will be called thc *standard pound weight*. The variations of g over the surface of the earth are small enough to be neglected in almost all engineering calculations and it is therefore usual and convenient to assume that the unit of force is equal to the local value of the weight of 1 pound. The reader should, however, understand clearly that the weight of a mass of 1 pound is *not* constant and therefore cannot be adopted as a unit of force. In this book we shall mainly use SI units (Système International) in numerical work unless the contrary is stated. In this system the units of mass, length and time are the kilogramme, metre and second, respectively, and the unit of force, called the Newton, is such that it imparts an acceleration of 1 metre/sec^2 to a mass of 1 kilogramme. This system is now being widely adopted in both engineering and science but it may be some time before other systems cease to be used. The units of some important physical quantities are shown in Table 1.10,1.

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† The C.G.S. unit of force is the dyne and of pressure and stress is the barye = 1 dyne per sq cm. The *bar* is 10^6 barye and an atmospheric pressure of 76 cm of mercury at 0°C at London is 1·014 bar.

TABLE 1.10,1
Units of Some Important Physical Quantities

Physical quantity	Unit in system	
	Slug-foot-second	Kilogramme-metre-second
Mass	Slug = 32·17 pounds mass	Kilogramme
Length (Displacement)	Foot	Metre
Time	Second	Second
Velocity	Foot per second	Metre per second
Acceleration	Foot per second per second	Metre per second per second
Force (Weight, tension, thrust)	Slug foot per second per second = weight of 1 pound mass at a locality where g has the measure 32·17 feet per second per second, i.e. the standard pound weight	Newton = force to accelerate 1 kilogramme at 1 metre per second per second
Pressure (Stress)	Slug foot per second per second per square foot or standard pound weight per square foot	Newton per square metre
Density	Slug per cubic foot	Kilogramme per cubic metre
Momentum	Slug foot per second	Kilogramme metre per second
Viscosity	Slug per foot per second	Kilogramme per metre per second
Kinematic viscosity	Square foot per second	Square metre per second

The density of fresh water depends on its temperature and pressure but in most engineering calculations the density may be taken as 62·4 pounds per cubic foot, 1·94 slugs per cubic foot, or 1000 kg/m^3. One Imperial gallon of fresh water at 60°F has a mass of 10 pounds; the United States gallon measures approximately 0·833 Imperial gallon and is exactly 231 cubic inches. For sea water the density depends, in addition, on the salinity but it is usually sufficiently accurate to adopt the figure of 64 pounds per cubic foot or 1·99 slugs per cubic foot. In aeronautics the standard density of atmospheric air is taken to be 0·002377 slug per cubic foot, or 1·22506 kg/m^2; this is the density at 15°C or 288·2°K with a barometric pressure of 760 mm of mercury, i.e. 14·69 pounds weight per square inch or $1{\cdot}013 \times 10^5$ Newtons per square metre. The density for other temperatures and pressures can be obtained by use of the characteristic equation (see § 1.3).

TABLE 1.10,2

Viscosity and Kinematic Viscosity of Water

Pressure atmospheric or nearly so

Temperature degrees C	Viscosity		Kinematic viscosity	
	$10^5\mu$ slug foot second units $(\text{lbf s/ft})^2$	$10^3\mu$ kilogramme metre second units (Ns/m^2)	$10^5\nu$ foot second units (ft^2/s)	$10^6\nu$ metre second units (m^2/s)
0	3·743	1·792	1·929	1·792
5	3·173	1·519	1·635	1·519
10	2·732	1·308	1·408	1·308
15	2·381	1·140	1·228	1·141
20	2·099	1·005	1·084	1·007
30	1·673	0·801	0·866	0·804
40	1·370	0·656	0·712	0·661
50	1·147	0·549	0·598	0·556
60	0·980	0·469	0·513	0·477
70	0·848	0·406	0·447	0·415
80	0·746	0·357	0·395	0·367
90	0·662	0·317	0·353	0·328
100	0·593	0·284	0·319	0·296

This Table is based on Table 1 of *Modern Developments in Fluid Dynamics* (Oxford, 1938).

In the British system (whether or not the unit of mass is the slug) the unit for rate of discharge (volume/time) is the *cusec.*, equal to 1 cubic foot per second; this is approximately equal to 6·23 Imperial gallons or 7·48 U.S. gallons per second (374 Imperial gallons or 448·8 U.S. gallons per minute). The metric equivalent of the cusec. is 0·02832 cubic metre per second. Velocities are sometimes given in statute miles per hour or in knots:

1 statute mile per hour = 1·467 feet per second
= 1·60934 km/h
1 knot = 1·689 feet per second
= 1·85318 km/h.

The C.G.S. unit of viscosity is called the *poise* (after Poiseuille) and is 1 gram per centimetre per second, while the unit of kinematic viscosity is called the *stokes* (after Sir George Stokes) and is 1 square centimetre per second. In SI units one P (poise) = 10^{-1} Ns/m^2 and one St (stokes) = 10^{-4} m^2/s. For consistency in SI nomenclature, to keep the names of the units in steps of 10^3, use of the terms centipoise (cP) and centistokes (cSt) is recommended. Thus one cP = 10^{-3} Ns/m^2 and one cSt = 10^{-6} m^2/s.

Tables 1.10,2 and 1.10,3 give information about the viscosities of water

TABLE 1.10,3
Viscosity of Air

Temperature degrees C	Viscosity	
	10^7 μ slug foot second units (lbf s/ft^2)	10^5 μ kilogramme metre second units (Ns/m^2)
0	3·57	1·709
20	3·78	1·808
40	3·98	1·904
60	4·17	1·997
100	4·54	2·175
150	4·98	2·385
200	5·39	2·582
300	6·15	2·946
400	6·84	3·277
500	7·48	3·583

Note. The kinematic viscosity of air depends on the pressure as well as the temperature. At 15°C and under normal atmospheric pressure

$$\nu = 1{\cdot}57 \times 10^{-4}\ \text{ft}^2/\text{sec}$$

or

$$1{\cdot}46 \times 10^{-5}\ \text{m}^2/\text{s}.$$

and air, respectively, for a range of temperatures. The conversion factors are

Viscosity

$$1 \text{ poise} = 1 \text{ gm/(cm sec)} = 2{\cdot}089 \times 10^{-3} \text{ slug/(ft sec)}.$$

Kinematic viscosity

$$1 \text{ stokes} = 1 \text{ cm}^2\text{/sec} = 1{\cdot}0764 \times 10^{-3} \text{ ft}^2\text{/sec}.$$

The value of the gas constant R for air (see equation 1.3,5) in ft sec-degree K units is $3{\cdot}09 \times 10^3$. In SI units, R has the value 287 joules/kg.°K. R has the dimensions $L^2T^{-2}\Theta^{-1}$, where Θ stands for temperature, so it is independent of the unit of mass, provided that consistent units are adopted. The ratio $c_p/c_v = \gamma$ for air may be taken as exactly 1·4.

Unless otherwise stated, angles are measured in *radians*.

The consistent unit of power in the SI system is the watt or Newton metre per second and in the slug-foot-second system it is the foot pound-weight per second. Then the horse power (H.P.), as defined by James Watt, is 550 ft lbf per second and the 'force de cheval' defined as 75 kilogrammetres per second, is 0·9863 H.P. The kilowatt is 1·34 H.P., or 1 H.P. is 0·746 kilowatt. One watt is 1 joule per second and 1 joule is 10^7 ergs, where the erg is the work done by a force of 1 dyne in moving 1 cm parallel to itself. One joule is one Newton metre. The heat required to raise 1 kilogramme of water by 1 degree C is equivalent to 4186·8 joules.

EXERCISES. CHAPTER 1

1. According to the experiments of Regnault the density of air at 0°C is given by

$$\rho = 1{\cdot}2759p \times 10^{-9}$$

where p is measured in dynes/cm² and ρ in gram/cm³. Given that 0°C is 273·2°K, find the value of the gas constant R for air when the units are slug, ft, sec, degree K and kg, metre, sec, degree K. (Answer 3088, 287·1)

2. Find the isothermal value of the bulk modulus for a gas which obeys van der Waals' law

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT.$$

(Answer $K_T = (pv^3 - av + 2ab)/v^2(v - b)$)

3. Find the value of the pressure coefficient β_v (see equation 1.3,7) for a gas obeying van der Waals' law (see Exercise 2).

$$\left(\text{Answer } \beta_v = \left(1 + \frac{a}{pv^2}\right)\Big/T\right)$$

4. A perfect gas at constant temperature is at rest in equilibrium in a uniform gravitational field. Find the pressure as a function of height z, given that its value is p_0 where $z = 0$.

$$\left(\text{Answer } p = p_0 \exp\left(\frac{-gz}{RT}\right)\right)$$

5. Show that when $n \to 1$ the expression for the pressure in equation (1.6,10) tends to equality with that given in Exercise 4.

6. A circular cylindrical pot is of internal radius a and height b while the axis is vertical. Liquid is poured in to a depth h and the pot is set spinning with angular velocity ω about its axis. Find ω such that, when a steady state is reached, the fluid just fails to spill. (Assume that $2h > b$.)

$$\left(\text{Answer } \omega = \frac{2}{a}\sqrt{[g(b-h)]}\right)$$

7. The problem is the same as the last but $2h < b$.

$$\left(\text{Answer } \omega = \frac{b}{a}\sqrt{[g/(b-h)]}\right)$$

8. A stationary sphere of uniform liquid of density ρ and radius c is in equilibrium under the action of its own gravitational attraction (Newton's law, gravitational constant γ). Find the pressure at a distance r from the centre given that the pressure is zero at the surface. [Use the facts that a uniform spherical shell has no attraction at an internal point and attracts at an external point as if all its mass were at the centre.]

$$(\text{Answer } p = \tfrac{2}{3}\pi\rho^2\gamma(c^2 - r^2))$$

9. A stationary and uniform gravitating sphere of radius a and density ρ_1 is surrounded by a layer of liquid of density ρ_2 and constant depth h. Find the pressure at height z above the surface of the sphere, neglecting the self attraction of the liquid layer, if the pressure at the free surface is zero. [See notes on last exercise.]

$$\left(\text{Answer } p = \frac{4}{3}\frac{\pi\rho_1\rho_2\gamma a^3(h-z)}{(a+h)(a+z)}\right)$$

10. A stationary and uniform gravitating sphere of radius a and density ρ_1 is surrounded by a layer of liquid of density ρ_2 and constant depth h. If the pressure at the free surface is zero, find the pressure in the liquid at height z above the surface of the sphere. [Assume Newton's law of gravitation with gravitational constant γ and take account of the gravitational attraction of the liquid itself.] [*Hint*. Use the results of the last two exercises.]

$$\left(\text{Answer } p = \tfrac{4}{3}\pi\rho_2\gamma\left\{\frac{(\rho_1-\rho_2)a^3(h-z)}{(a+h)(a+z)} + \tfrac{1}{2}\rho_2[(a+h)^2 - (a+z)^2]\right\}\right)$$

11. Show that the pressure in a gravitating liquid of density ρ satisfies $\nabla^2 p = -4\pi\rho^2\gamma$ where γ is the gravitational constant. [*Note*. The potential satisfies Poisson's equation $\nabla^2\chi = 4\pi\rho\gamma$.]

12. Verify that the pressure in Exercise 10 satisfies $\nabla^2 p = -4\pi\rho_2{}^2\gamma$.

[*Note.* Since p depends only on z and $(a + z)$ is the distance from the centre of the sphere

$$\nabla^2 p = \frac{1}{(a+z)^2} \frac{\partial}{\partial z} \left((a+z)^2 \frac{\partial p}{\partial z}\right)\Bigg]$$

13. A liquid in a tank is carried by a truck moving on horizontal straight rails with constant acceleration a. If the liquid is at rest relative to the truck, find the inclination of its free surface to the horizontal. (Answer $\tan^{-1}(a/g)$)
14. The vertices A, B, C of a triangle are at depths of 1, 1·5 and 3·5 m below the free surface of water at rest in a standard gravitational field. If AB = 7 m, BC = 5 and CA = 6, find the thrust on the triangle. (Answer 288·6 kN)
15. Show that if a plane area be rotated about the water line the position of the centre of pressure in the plane is unaltered.
16. *A Reciprocal Theorem.* XX' is the water line of a totally immersed plane area inclined at θ to the horizontal. The moment of the hydrostatic thrust on the area is M about an axis YY' lying in the plane of the area but not intersecting it. Prove that the moment is unaltered when the axes XX' and YY' are interchanged while θ is unaltered.
17. Show that the thrust on a given totally immersed triangular area is unaltered when the depths of its vertices are interchanged in any manner.
18. The lengths (in m) of the sides of a quadrilateral $ABCD$ are AB = 2, BC = 3, CD = 1·5 and DA = 4·5 while the diagonal AC = 4. Find the depth of immersion of the vertex D, given that the depths of A, B and C are 1, 2 and 3 m respectively. [*Hint.* Consider the point in AC which is at the same depth as B.] (Answer 2·81 m)
19. Use the result of the last exercise to find the thrust on the quadrilateral, supposed immersed in water and with standard gravity. (Answer 123 kN)
20. A hemispherical cavity in a vertical wall is of radius a and its centre is at a distance $h > a$ below the free surface of a liquid of density ρ situated in a uniform gravitational field of strength g. Find the magnitude and sense of the resultant vertical thrust on the wall of the cavity, also the magnitude and direction of the resultant horizontal thrust.

    (Answer Vertical thrust $\frac{2}{3}\pi a^3\rho g$ downwards. Horizontal thrust $\pi a^2 h\rho g$ normal to plane of wall and away from liquid.)
21. *Equilibrium of a Floating Body.* The immersion must be such that the weight of the fluid displaced is equal to that of the body. Balance of couples requires that the centre of mass of the body is on the same vertical line as the centre of buoyancy. A catamaran consists of two equal circular cylinders of small diameter fixed together with their axes distant apart $2d$ and the weight is such that forces are balanced when the cylinders have three-quarters of their volume immersed. Find the greatest distance that the C.G. can be laterally off-centre for equilibrium. (Assume C.G. to be on the level of the axes.) (Answer $\frac{1}{3}d$)
22. The cross-section of a uniform prism is an isosceles triangle of height h. The material is of density ρ_1 and the prism floats with its base submerged and horizontal in a liquid of density $\rho_2 > \rho_1$. Find the height of the apical edge above the liquid. (Answer $h\sqrt{[(\rho_2 - \rho_1)/\rho_2]}$)
23. A slab of constant thickness is symmetrical about an axis midway between its parallel faces and its breadth b is a given function of distance z from the

plane base. A body of revolution is of radius r at distance z from its plane base and is of the same total height as the slab. If $r^2 = kb$ where k is constant and if the bodies are of the same density, show that, when their axes are vertical, they will float with their bases at the same depth. (Questions of stability may be ignored.)

24. *Surface of Flotation* (*or Floatation*). A given rigid body of fixed weight floats in a liquid of given specific weight in such a manner that the plane of flotation (plane of the free surface) cuts off a constant volume from the body; the orientation of the plane in the body in the state of equilibrium depends on the distribution of its weight. The envelope of the plane of flotation in the body is called the *surface of flotation.* When the body has vertical sides and a flat bottom with two perpendicular axes of symmetry, show that the surface of flotation reduces to a mere point.

25. Find the surface of flotation when the floating body is the paraboloid of revolution
$$az = x^2 + y^2.$$
Note. It may be assumed that the plane $z = mx + c$ cuts a slice from the paraboloid of volume
$$V = \frac{\pi}{32}a(m^2a + 4c)^2.$$
(Answer The surface of flotation is the paraboloid of revolution obtained by rotating
$$z = 4ak + \frac{x^2}{a} \text{ about } OZ, \text{ with } k = \sqrt{\frac{32}{\pi}\left(\frac{V}{a^3}\right)}.\Bigg)$$

26. *Surface of Buoyancy.* A plane cuts off from a rigid body a constant volume V. The locus of the centroid of V as the orientation of the plane changes is called the *surface of buoyancy.* Find the locus of the centre of buoyancy for a rectangular body in the restricted case where the plane of flotation is always parallel to one edge of the body (displacement in pure roll or pure pitch). Assume that the plane bottom is always submerged and that the upper edges are never submerged. Take the breadth of the body as a and the origin at the centroid of V when the sides are vertical; in this situation the depth of immersion of the bottom is b.

(Answer The parabola $z = \dfrac{6by^2}{a^2}$)

27. *Efficiency of a canal lock.* A ship of displacement (immersed volume) D is raised through height H in a canal lock whose plan area is A. Find the efficiency η of the lifting process, assuming that there is no leakage of water, in terms of D and $V = AH$, the volume of water required to fill the lock. (*Hint.* Consider the total volume of water passing in a complete cycle of operations.)

(Answer $\eta = \dfrac{D}{V + D}$)

28. A 5 m square lamina *KLMN* makes an angle of 30° with the horizontal. It is hinged along its upper horizontal side *KL* at a depth of 4·5 m, and subjected to water pressure on one side only. Find the vertical force required at the middle of side *MN* to balance the hydrostatic moment.

(Answer 87·3 MN)

29. A tank with vertical sides, 2 m wide, contains mercury to a depth of 1 m and water to a depth of 6 m from the bottom. Taking the specific gravity of mercury to be 13·6, determine the total force on one side of the tank and the position of the centre of pressure.
(Answer 47·7 MN, 1·54 m from bottom)

30. A cylinder of radius r, lying with axis horizontal, plugs a rectangular hole of width r in the bottom of a tank. Neglecting the weight of the cylinder, show that, when there is no force between the edges of the hole and the plug, the fluid depth will be $r[5\pi/6 + \sqrt{3}/4]$.

31. A rectangular notch of depth $2r$ in the end of a tank full of water is to be blocked by means of a D-shaped shell or half drum whose axis is horizontal and whose radius is r. Show that, when the axis is in the plane of the orifice at depth r below the surface, and the convexity of the shell is towards the inside of the tank, the resultant thrust passes through the axis of the shell, sloping upwards at $\tan^{-1} \pi/4$.

32. A cylindrical vessel with closed ends, radius 0·3 m, length 0·6 m, is full of water and rotates with its axis vertical. The vessel is connected through a centrally placed gland in its base through a pipe to a reservoir in which the water level is 3·0 m above the base of the cylinder. The cylinder, complete with water, is forced to rotate at a steady speed of 240 rev/m. Calculate the static pressure in kN/m^2 at the top of the cylinder at a radius of 0·3 m and the total upthrust on the lid. (Answer 54·86 kN/m^2, 11·5 kN)

33. The still water level behind a dam is 100 m above the level of the stream below the dam. Excess water is discharged through an energy "disperser" which "destroys" the kinetic and potential energy of the water. Calculate the rise in temperature of the water after passing through the disperser if it be assumed that half the heat generated is lost by conduction to the surroundings. (Answer 0·117 deg C)

34. Oil pressure in a reservoir is maintained at 35 MN/m^2. Calculate the energy available per kg of oil due to this pressure and the strain energy contained per kg in the reservoir. Take $K = 1·9$ GN/m^2 and $\rho = 850$ kg/m^3.
(Answer 41,200 and 380 m^2/s^2)

35. A lock gate is 3 m wide. Determine the initial turning moment required to force the gate to open when the water levels on the upstream and downstream sides are 3·06 m and 3·0 m respectively above the sill.
(Answer 8·02 kNm)

36. A condenser shell 4 m diameter is tested by water under pressure. The axis of the shell is horizontal and is 3 m above floor level. A pressure gauge connected by a pipe containing water to the lowest point in the shell reads 137 kN/m^2 when held at floor level. Calculate the hydrostatic pressure at the bottom and top of the shell. If the pressure connection were made at any other point in the shell, would the gauge reading be different?
(Answer 127·2, 88·0 kN/m^2; no)

37. A metal container has an internal volume of 1,000 m^3, and is filled with water at atmospheric pressure. How much more water must be pumped in to raise the pressure by 700 kN/m^2? Assume that the equivalent bulk modulus allowing for expansion of the container is 1·4 GN/m^2.
(Answer 0·5 m^3)

CHAPTER 2

KINEMATICS OF FLUIDS

2.1 Introductory Remarks

Kinematics is the geometry of motion. Thus, the kinematics of fluids is the science of the motion of fluids considered in abstraction from the forces which cause or accompany the motion. The subject has two main aspects:—

(a) The development of methods and techniques for describing and specifying the motions of fluids.

(b) The determination of the conditions for the kinematic possibility of fluid motions, i.e. the exploration of the consequences of continuity in the motion.

A thorough study of the kinematics of fluids is a necessary preliminary to the study of the dynamics of fluids; in the latter subject we are concerned with the manner in which fluids do in fact move in given circumstances and with the forces accompanying the motion. It is found that the actual motions of fluids in many important cases conform to simple kinematic types, so purely kinematic investigations carry us very far into the general theory of fluid motion.

The kinematics of fluids presents problems of much greater complexity than does the kinematics of rigid bodies and requires quite different theoretical methods for its treatment. For a rigid body moving in a plane, the velocities of *all* its points at a given instant become definite and calculable when the two components of the velocity of one point and the angular velocity are given. For plane motion the rigid body thus has just three degrees of freedom. However, the fluid occupying a given region has infinitely many degrees of freedom. In spite of this, the motion is not completely arbitrary for there remains the condition that two fluid elements cannot occupy the same position at the same time.

Unless the contrary is stated, we shall suppose for a liquid that cavitation does not occur, i.e. adjacent elements do not become separated, leaving gaseous regions within the liquid (see § 3.12).

2.2 Description and Specification of Fluid Motion

We consider a very small element of fluid which we call a fluid particle; this is so small that the relative velocities of its parts are negligible and, so far as position is concerned, it can be regarded as a geometric point. However, we must not forget that our fluid particle has a definite mass and a volume, which, in the case of an incompressible fluid, is constant. At any given instant the fluid particle at a point P has a definite velocity which can conveniently be specified by its components referred to fixed rectangular axes. In the most general case the velocity depends on the time t and on the coordinates of P but there are two important and simple cases:—

(a) The motion is said to be *steady* when the velocity at any point with fixed coordinates is the same at all instants.

(b) The motion is said to be *uniform* at the instant considered when the velocity is the same for all fluid particles. A uniform motion may also be steady.

With regard to the definition (a), it should be noted that a motion may be steady for one set of reference axes and unsteady for another. For example, let us consider an aircraft flying with constant speed in a horizontal straight line and without rotation. Then, if we take reference axes fixed in the aircraft, the motion of the air will be steady because the velocity at a point with fixed coordinates is independent of time. But if we take axes fixed to the earth the motion of the air will be unsteady. We shall always suppose that the reference axes are fixed in direction and that the origin is at rest or in unaccelerated motion.

There are two important and commonly used methods of specifying fluid motion. The first, known technically as the *Lagrangean method*, may be described as *historical* since we follow the history of each individual fluid particle. The particle may be identified by its spatial coordinates at the instant $t = 0$; its coordinates at any later time t are then functions of t and of these initial coordinates. The *path of a particle* is the curve of fundamental importance when the motion is considered from this point of view. The second, known technically as the *Eulerian method*, may be described as the *cinematographic* method since the basic conception is the state of velocity throughout the whole region occupied by the fluid *at one instant*; the complete state of motion is given by a succession of such instantaneous pictures of the state of flow. In accordance with the definition, the flow picture is constant when the motion is steady. In the Eulerian method the curve of fundamental importance is the *streamline*, which is next considered.

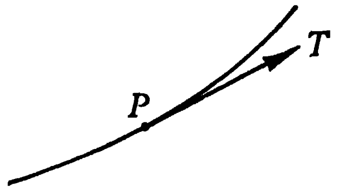

Fig. 2.2,1. Diagram to illustrate the definition of a streamline.

A *streamline* is a curve such that its tangent PT at any point P is parallel to the velocity of the fluid at P at the instant considered (see Fig. 2.2,1). The whole set of streamlines thus gives a representation of the direction of the flow at all points at one particular instant. When the motion is steady the streamlines are the same at all times. It will be seen that a streamline is defined in relation to the instantaneous field of velocity in exactly the same way as a line of force is defined in relation to a field of force (gravitational, electric or magnetic).

A method often adopted in the experimental exploration of a field of flow consists in allowing a fluid of strong and distinctive colour to exude slowly into the moving fluid from a small fixed orifice; when the flowing substance is liquid a highly coloured solution is used while a dense smoke is commonly used when the fluid is gaseous. At any given instant the exuded fluid lies on a curve which is called a *filament line* or *streak line*. The formal definition is as follows:—a *filament line* is the locus at a given instant of all the fluid particles which have passed through or will pass through a fixed point P within the fluid. We now have the important proposition:—*When the motion is steady the streamlines and filament lines are identical with the paths of the particles.* In order to see that this is true, consider a particle at P on a streamline (Fig. 2.2,1). Since the motion is steady the flow at P is at all times in the direction of the tangent PT to the streamline. Let the particle move to P' in the short interval of time δt. Then in the limit when δt tends to zero P' will be a point on the streamline, which is a fixed curve. The particle at P' moves in the direction of the tangent to the streamline at P' and is carried in a further short interval δt to a third point P'' on the streamline. Hence the particle continues to move along the fixed streamline, which is thus the path of the particle. Clearly *all* the particles which pass through P have the streamline as their path and, in accordance with the definition, the streamline is also a filament line. The filament technique can thus be used to find the streamline experimentally whenever the motion is steady. Alternatively, let us suppose that the filament lines are the same at all times, i.e. fixed curves. Then each filament line is also the path of every particle upon it and it is also a streamline since the motion of a particle at any instant is along the tangent to its path. Thus the filament lines, paths of particles and streamlines are identical.

Let us now consider the determination of the equation to a streamline and first let us take the case where the flow occurs in the plane OXY. Let u and v be the components of the velocity of the fluid in the directions OX, OY respectively at the time t and at the point whose coordinates are x, y. Since the tangent to the streamline is parallel to the velocity we have at time t

$$\frac{dy}{dx} = \frac{v}{u} \tag{2.2,1}$$

which may also be written

$$\frac{dx}{u} = \frac{dy}{v}. \tag{2.2,2}$$

Suppose, for example, that

$$u = \frac{kx}{x^2 + y^2} \tag{2.2,3}$$

and

$$v = \frac{ky}{x^2 + y^2}$$

where k is a constant. Then (2.2,2) becomes after removal of the common factor

$$dx \quad \frac{dy}{y}$$

and this can be integrated at once to yield

$$\ln x = \ln y + \text{const.}$$

which is equivalent to

$$y = Cx \tag{2.2,4}$$

where C is a constant of integration. Hence the streamlines are straight lines radiating from the origin (see Fig. 2.2,2). The motion represented by (2.2,3) is that of a two-dimensional source situated at O (see § 2.7).

As another simple example we may take the case where

$$\begin{aligned} u &= kx \\ v &= -ky \end{aligned} \tag{2.2,5}$$

with k constant. Equation (2.2,2) becomes

$$\frac{dx}{x} = -\frac{dy}{y}$$

or

$$\ln x = -\ln y + \text{const.}$$

which is equivalent to

$$xy = C \tag{2.2,6}$$

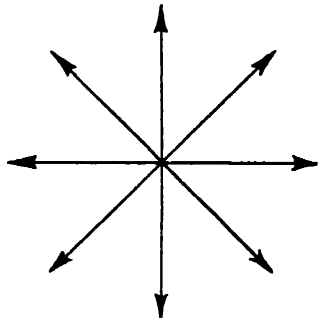

Fig. 2.2,2. Streamlines of two-dimensional source.

where C is a constant of integration. Hence the streamlines are rectangular hyperbolas. This motion can be interpreted as that of a perfect fluid in the neighbourhood of a rectangular corner (see Fig. 2.2,3). In these examples, and generally, the constant of integration has a distinct value for each streamline and the value of the constant serves to identify the streamline.

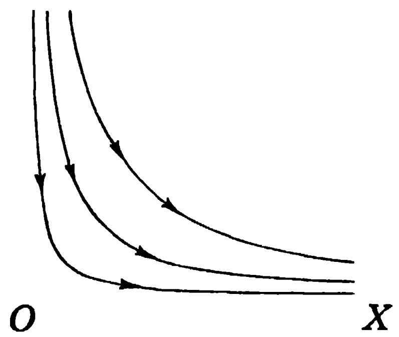

Fig. 2.2,3. Streamlines of two-dimensional flow in a right-angled corner.

For flow in three dimensions we take fixed rectangular axes OX, OY, OZ and the corresponding components of the velocity are u, v, w respectively. Let an element of the streamline of length ds have projections dx, dy and dz on the coordinate axes. Then, since the element ds is parallel to the velocity, we have

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{ds}{q} \tag{2.2,7}$$

where q is the magnitude of the resultant velocity and given by

$$q^2 = u^2 + v^2 + w^2. \tag{2.2,8}$$

A streamline now has *two equations* which are to be obtained by integrating a pair of the equations contained in (2.2,7). As a simple example, suppose that

$$u = ax, \qquad v = ay, \qquad w = -2az \tag{2.2,9}$$

where a is a constant. From (2.2,7) we have

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad \frac{dx}{x} = \frac{dy}{y}$$

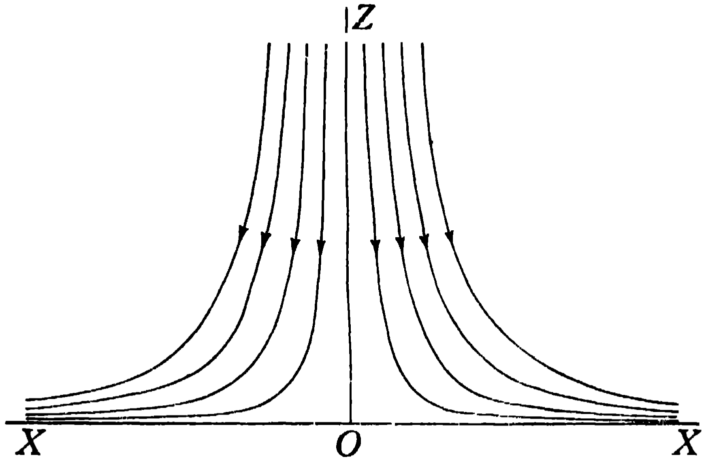

Fig. 2.2,4. Streamlines in plane OXZ for a symmetric three-dimensional flow in contact with a fixed plane.

which yields [compare equation (2.2,4)]

$$y = Cx. \qquad \text{(2.2,10}$$

We also have from (2.2,7)

$$\frac{dx}{u} = \frac{dz}{w} \quad \text{or} \quad \frac{dx}{x} \qquad \frac{dz}{2z}$$

which yields on integration

$$x^2z = D \tag{2.2,11}$$

where D is a second constant of integration. Equation (2.2,10) represents a family of planes through the axis OZ while (2.2,11) represents a family of cylindrical surfaces. The streamlines are the curves of intersection of the planes (2.2,10) with the cylindrical surfaces (2.2,11). Since u is here a function of x alone and v is the same function of y it is evident that the flow is symmetrical about OZ, i.e. it occurs in planes containing OZ and the pattern of streamlines is the same in all such planes. If we take the plane OZX the equations to the streamlines are given by (2.2,11). The flow (see Fig. 2.2,4) can be interpreted as a stream of perfect fluid impinging normally on the fixed plane OXY. Here, and generally for three-dimensional flow, the pair of equations which determine the streamlines each contains a constant of integration and the values of these two constants, such as C and D in the present example, serve to identify the streamline.

A *streamtube* is defined as follows (see Fig. 2.2,5): take a simple closed curve (usually supposed to be very small) lying in the fluid. Then the streamlines through the points on the curve generate a tubular surface which is called a streamtube.

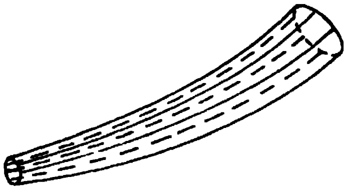

Fig. 2.2,5. Diagram of a streamtube.

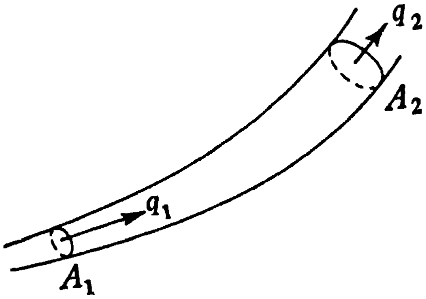

Fig. 2.3,1. Continuity of flow.

2.3 Continuity

The principle of continuity is based on two fundamental propositions:

(a) The mass of the fluid is conserved, i.e. the fluid is neither created nor destroyed in the field of flow.

(b) The flow is continuous, i.e. empty spaces do not occur between particles which were in contact.

The status of these propositions is different, for the first is always true, whereas the second is an assumption about the nature of the flow which, in fact, is sometimes violated. In this section we shall work out the consequences of these two propositions.

Let the flow be *steady* and consider the flow in a *streamtube* (see Fig. 2.3,1). Since the motion is steady the streamtube will be a fixed surface as the streamlines which generate it are all fixed curves. We shall now assume that the cross-section of the tube is so small that the velocity of the fluid can be treated as constant over a normal section of the tube. Take two normal sections of areas, A_1, A_2 at fixed positions and let the velocities at these sections be q_1, q_2 respectively while the densities are ρ_1, ρ_2 respectively. Under steady conditions the mass of fluid contained in the part of the tube between A_1 and A_2 is constant. Hence the mass entering at A_1 in unit time must equal the mass leaving at A_2 in unit time. The volume of fluid entering at A_1 in unit time is q_1A_1 and its mass is $\rho_1q_1A_1$. Hence we derive the equation of continuity in the form

$$\rho_1q_1A_1 = \rho_2q_2A_2. \tag{2.3,1}$$

For a homogeneous (incompressible) fluid the density is constant and the factor ρ may be divided out of the last equation. Thus the equation of continuity takes the simpler form

$$q_1A_1 = q_2A_2. \tag{2.3,2}$$

When the flow is *two-dimensional*, we take the streamtube to be of unit depth perpendicular to the plane of the flow and of normal width t in this plane. Hence the area of section becomes t and equation (2.3,2) becomes

$$q_1 t_1 = q_2 t_2 \tag{2.3,3}$$

which implies that

$$q_2 = \frac{q_1 t_1}{t_2} = \frac{\text{const.}}{t_2}. \tag{2.3,4}$$

Thus the velocity at any point of a given streamtube varies inversely as the width of the tube at the point. If we draw adjacent streamlines so that qt has the same value k for all the streamtubes of the system we shall have the general result

$$q = \frac{k}{t}. \tag{2.3,5}$$

Thus the direction of the fluid velocity is tangential to the streamlines and its magnitude is proportional to the inverse of their normal spacing. The pattern of streamlines gives complete information about the field of flow.

When the fluid is incompressible its volume is conserved. Hence, in the absence of cavitation, if we take any closed surface of constant volume lying within the fluid the instantaneous rate at which the fluid flows outwards across the surface must be equal to the instantaneous rate at which it flows inwards across the surface, or the net rate of outflow is zero. The instantaneous rate at which a fluid flows across a surface (volume per unit time) is called the flux across the surface. For an incompressible fluid the total flux across any closed surface of constant volume is zero, as we have just seen, and this is true whether the flow is steady or unsteady and whether the surface is fixed or moving, rigid or deformable, so long as the enclosed volume is constant. In order to calculate the flux Q across any fixed surface we take a small element of the surface of area dS where the resultant velocity is q making the angle θ with the normal to the surface at the element (see Fig. 2.3,2); this normal is drawn in a particular sense which is regarded as positive, e.g. from the inside of a closed surface towards the outside. The flux across the element is then equal to the area multiplied by the normal component of the velocity and the whole flux is

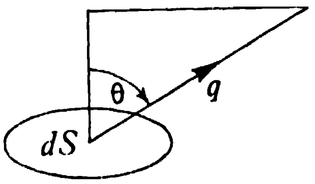

Fig. 2.3,2. Diagram to illustrate the calculation of flux across a surface.

$$Q = \iint q \cos\theta \, dS \tag{2.3,6}$$

where the surface integral covers the whole of the surface considered.

Let us now revert to Fig. 2.3,1 and suppose that the fluid is incompressible and the flow *unsteady*. Consider the flux outward across a *fixed* tubular surface whose ends are A_1, A_2 and whose curved surface contains the streamlines as they exist at the instant considered. Everywhere on the curved surface the resultant velocity is, by the definition of a streamline, tangential; hence the angle θ is 90° and the normal component of velocity is zero. Consequently the instantaneous flux across the curved surface is zero. The net outward flux is therefore

$$q_2A_2 - q_1A_1$$

and this must be zero since the enclosed volume is constant. Hence equation (2.3,2) is universally valid for an incompressible fluid although it was first proved for steady motion only. Likewise equation (2.3.3) is always true for the two-dimensional flow of an incompressible fluid.

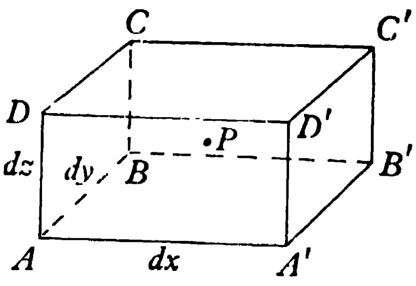

Fig. 2.3,3. Rectangular element with centre at P.

The general form of the *equation of continuity* can be obtained as follows. Take a small rectangular element $ABCDA'B'C'D'$ whose edges are parallel to the coordinate axes and whose centre P has the coordinates x, y, z; let the lengths of the edges be dx, dy, dz (see Fig. 2.3,3). The velocity at P has the components u, v, w. Now the face $ABCD$ has the abscissa $x - \frac{1}{2}\,dx$ so the average rate of transfer of mass per unit area inwards across this face is

$$\rho u - \frac{1}{2}\frac{\partial(\rho u)}{\partial x}\,dx$$

while the average rate of transfer of mass per unit area outwards across the face $A'B'C'D'$ is

$$\rho u + \frac{1}{2}\frac{\partial(\rho u)}{\partial x}\,dx.$$

Hence the net rate of loss of mass across this pair of faces is $\{\partial(\rho u)/\partial x\}$ $dx\,dy\,dz$ since the area of each face is $dy\,dz$. There are similar expressions for the rate of loss of mass across the other two pairs of faces and the total rate of loss is

$$\left\{\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right\} dx\,dy\,dz.$$

But the mass contained is $\rho\,dx\,dy\,dz$ and its rate of decrease is $-(\partial\rho/\partial t)$ $dx\,dy\,dz$. When we equate these expressions and divide by the volume of the element we obtain the *equation of continuity* in its general form†

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} + \frac{\partial\rho}{\partial t} = 0. \qquad (2.3,7)$$

---

† It is important to be quite clear about the meanings of the partial differential coefficients which appear in (2.3,7). The term $\frac{\partial(\rho u)}{\partial x}$, for example, is the rate of change of the product ρu with x when y, z and t are all constant. The term $\frac{\partial\rho}{\partial t}$ is the rate of change of ρ with t when x, y and z are all constant.

Whenever the motion is steady the density at a fixed point is independent of time and the equation becomes

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0. \tag{2.3,8}$$

Next, when the fluid is incompressible (ρ constant) the last term in (2.3,7) vanishes and the factor ρ may be removed from the remainder; hence the equation of continuity becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \tag{2.3,9}$$

It is important to note that this equation is true for both steady and unsteady motions. The expression on the left of (2.3,9) is called the *expansion* and it is the *divergence of the velocity vector* i.e. div **V**.

We see from equation (2.3,9) that we cannot have three completely arbitrary functions of position as components of the velocity. Thus the preservation of continuity imposes a restriction on the nature of the flow. We can immediately verify that when $u = ax$, $v = ay$ and $w = -2az$ equation (2.3,9) is satisfied (cp. equation 2.2,9) so these expressions for the components of velocity are possible for an incompressible fluid. But if, say, we reverse the sign of w the velocity field is not a possible one for an incompressible fluid.

2.4 The Stream Function

We shall now consider *the motion of an incompressible fluid in two dimensions*. The motion in all planes parallel to the plane OXY is the same as in this plane and there is no component of velocity perpendicular to this plane anywhere. Hence the velocity is completely specified by its components u, v in the directions OX, OY respectively while both these components are functions of x, y and t only. For steady motion u and v are independent of t. We shall find it convenient to consider a layer of fluid lying between the plane OXY and a parallel plane at unit distance from it. When we speak of the *flux across a curve* in the plane OXY it is to be understood that we mean the flux in this layer of unit depth across a cylindrical surface whose base is the given curve.

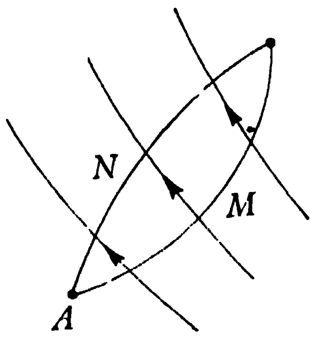

Fig. 2.4,1. Diagram illustrating the definition of the stream function.

In Fig. 2.4,1 A is a fixed point while P is any point in the plane. These points are joined by a pair of curves AMP, ANP which together form a closed region. Since the fluid is incompressible the total flux out of this region is zero. Hence we have

$$\text{flux across } ANP = \text{flux across } AMP$$

where we adopt the convention that the flux is positive in the sense from right to left across the curve. Now regard *AMP* as a fixed curve while the curve *ANP* is allowed to vary its situation. Then we see from the last equation that the flux across the curve *ANP* is fixed so long as its end points *A* and *P* are given. Since *A* is a fixed point the flux across *ANP* is thus merely a function of the coordinates x,y of the point *P*. This function is called the ***stream function*** and is represented by the symbol ψ or $\psi(x, y)$ when it is desired to emphasise the coordinates of the point to which it refers. The fixed point *A* may be taken to coincide with the origin *O* but this is not necessary. We shall show that when we know ψ as a function of x and y the components u, v of the velocity can easily be derived and the equations of the streamlines written down at once. In order to find ψ when the flow is given it is only necessary to find the flux across any curve connecting the point *P* to the chosen fixed point *A*. It will usually be possible to simplify the calculation of ψ by a suitable choice of the curve. The function ψ refers to the flow at one particular instant. When the flow is steady ψ is independent of time but for unsteady flow it becomes a function of the three variables x, y and t.

Let us calculate the increment $d\psi$ of the stream function in passing from *P* to *P'* where *PP'* is parallel to *OX* and equal to the infinitesimal dx. We may pass from the fixed point *A* to *P'* by any path so we may choose the path *AMPP'*. The value of the stream function at *P'* is thus

$$\begin{aligned}\psi(x + dx, y) &= \text{flux across } AMP + \text{flux across } PP' \\ &= \psi(x, y) + v\,dx\end{aligned}$$

for the component of velocity normal to *PP'* is v. Hence we have

$$\psi(x + dx, y) - \psi(x, y) = v\,dx.$$

But on account of the definition of a partial differential coefficient

$$\psi(x + dx, y) - \psi(x, y) = \frac{\partial\psi}{\partial x}\,dx.$$

Hence we obtain

$$\frac{\partial\psi}{\partial x} = v. \tag{2.4,1}$$

Next let us find the increment of ψ in passing from *P* to *P'* where *PP'* is parallel to *OY* and equal to the infinitesimal dy. By the same argument as before we have

$$d\psi = \text{flux across } PP' = -u\,dy$$

since the normal component of velocity across *PP'* in the sense from right to left is $-u$. But now

$$d\psi = \psi(x, y + dy) - \psi(x, y) = \frac{\partial\psi}{\partial y}\,dy$$

and we accordingly derive

$$\frac{\partial \psi}{\partial y} = -u. \tag{2.4,2}$$

Thus, whenever we know ψ we may obtain the two components of velocity from the formulae

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x} \tag{2.4,3}$$

The reader's attention is drawn to the fact that we derive the component of velocity in the direction OX by differentiating the stream function with respect to y and introduce the negative sign whereas we derive the component of velocity in the direction OY by differentiating the stream function with respect to x.†

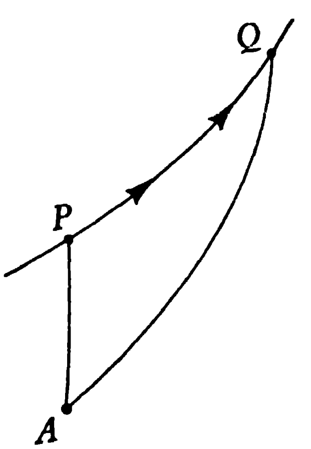

Fig. 2.4,2. Diagram illustrating the constancy of the stream function on a streamline.

We shall now show that the stream function is constant along any streamline at the instant considered. In Fig. 2.4,2 P is a fixed point on the streamline and Q is any other point on the same streamline. We join P and Q to the fixed point A where ψ vanishes. We then have a closed region enclosed by AP, the segment PQ of the streamline and QA. The total flux out of this closed region is zero and the flux across PQ is zero for, by definition, the normal component of velocity is everywhere zero on the streamline. Hence

$$\text{flux inward across } AP = \text{flux outward across } AQ$$

or

$$\psi_P = \psi_Q$$

by the definition of the stream function. Since P is a fixed point and Q is any point on the streamline, it follows that ψ is constant on the streamline. Hence when we are given the stream function the equations to the streamlines are

$$\psi(x, y) = C \tag{2.4,4}$$

where C is a constant characteristic of the streamline. In order to display the field of flow it is convenient to plot streamlines for the values 0, $\pm a$, $\pm 2a$, $\pm 3a$ etc. of the stream function, where a is some convenient constant. Then the flux in the channel or streamtube bounded by any streamline and its neighbour on either side has the constant value a. Hence the mean velocity in a channel anywhere at the instant considered is given by a/n, where n is the normal width of the channel.

---

† Some authors define the stream function as the flux across AP in the sense from left to right. This reverses the sign of the stream function, so the signs in equations (2.4,3) are reversed.

The equation of continuity (2.3,9) for an incompressible fluid becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.4,5}$$

for two-dimensional flow. Now from (2.4,3)

$$\frac{\partial u}{\partial x}=-\frac{\partial^2\psi}{\partial x\,\partial y}$$

and

$$\frac{\partial v}{\partial y}=\frac{\partial^2\psi}{\partial y\,\partial x}=\frac{\partial^2\psi}{\partial x\,\partial y}=-\frac{\partial u}{\partial x}$$

so the equation of continuity is satisfied. This is as it should be since we have deduced the existence of the stream function from the concept of continuity.

ILLUSTRATIVE EXAMPLES OF THE STREAM FUNCTION

Example 1. *Uniform stream*

First suppose that the velocity is U parallel to OX in the positive sense and let ψ vanish at O. Drop a perpendicular PN from the point P (whose coordinates are x,y) on OX and find the flux across the broken line ONP. The flux across ON is zero since the flow is along ON and the flux across NP from right to left is $-Uy$. Hence, in accordance with the definition, we have

$$\psi=-Uy. \tag{2.4,6}$$

This can be verified by use of equations (2.4,3). We find

$$u=-\frac{\partial\psi}{\partial y}=-(-U)=U$$

and

$$v=\frac{\partial\psi}{\partial x}=0.$$

It should be noted that we may add any constant to ψ without altering the velocities; the only result of this is to alter the streamline on which ψ vanishes. Hence the most general expression for the stream function in this case is

$$\psi=A-Uy \tag{2.4,7}$$

where A is an arbitrary constant. The streamlines are the loci on which ψ is constant, i.e. the straight lines $y=$ constant. This is otherwise obvious.

Next, suppose that the uniform stream has the velocity V parallel to OY in the positive sense. We now find that the flux across the broken line ONP is Vx and the most general expression for the stream function is

$$\psi=B+Vx. \tag{2.4,8}$$

The reader may prove from the definition or verify by use of equations (2.4,3) that

$$\psi = C + Vx \sin \alpha - Vy \cos \alpha \tag{2.4,9}$$

is the stream function for a uniform stream of velocity V inclined to OX at the positive angle α.

Example 2. *Forced vortex*

Suppose that the fluid rotates about O as a solid with anti-clockwise velocity ω. Let ψ vanish at O and let P be at the distance r from O. The flux across an element of OP of length ds at a distance s from O is $\omega s\, ds$ for the velocity is ωs normal to OP. Hence

$$\psi = \int_0^r \omega s\, ds = \tfrac{1}{2}\omega r^2 = \tfrac{1}{2}\omega(x^2 + y^2). \tag{2.4,10}$$

We derive

$$u = -\frac{\partial \psi}{\partial y} = -\omega y$$

$$v = \frac{\partial \psi}{\partial x} = \omega x$$

which are correct. The streamlines are the circles $r =$ constant. If the rotation takes place about a point whose coordinates are a, b the expression for the stream function becomes

$$\psi = \tfrac{1}{2}\omega[(x-a)^2 + (y-b)^2]. \tag{2.4,11}$$

Example 3. *Flow in a right-angled corner*

Let the stream function be

$$\psi = -kxy. \tag{2.4,12}$$

Then we find from (2.4,3) that

$$u = kx, \qquad v = -ky.$$

These components of velocity agree with those given in equation (2.2,5). The streamlines are the curves $\psi =$ constant, i.e. the rectangular hyperbolas

$$xy = \text{constant}.$$

We obtained this result in § 2.2 by integrating the differential equation of the streamlines when the components of velocity were given.

2.5 The Combination of Motions

Let us first consider two-dimensional flow of an incompressible fluid. The stream functions ψ_1, ψ_2 correspond to flows whose velocities are u_1, v_1 and u_2, v_2 respectively. Now let

$$\psi_3 = \psi_1 + \psi_2 \tag{2.5,1}$$

Then the component in the direction OX of the velocity corresponding to ψ_3 is

$$u_3 = -\frac{\partial \psi_3}{\partial y} = -\frac{\partial \psi_1}{\partial y} - \frac{\partial \psi_2}{\partial y} = u_1 + u_2 \tag{2.5,2}$$

while the component in the direction OY is

$$v_3 = \frac{\partial \psi_3}{\partial x} = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial x} = v_1 + v_2 \tag{2.5,3}$$

by equations (2.4,3). Thus the velocity corresponding to ψ_3 is the vector resultant of the velocities corresponding to ψ_1 and ψ_2 and we obtain the rule:—

To compound vectorially two fields of two-dimensional flow we add their stream functions. Important examples of this principle are given below.

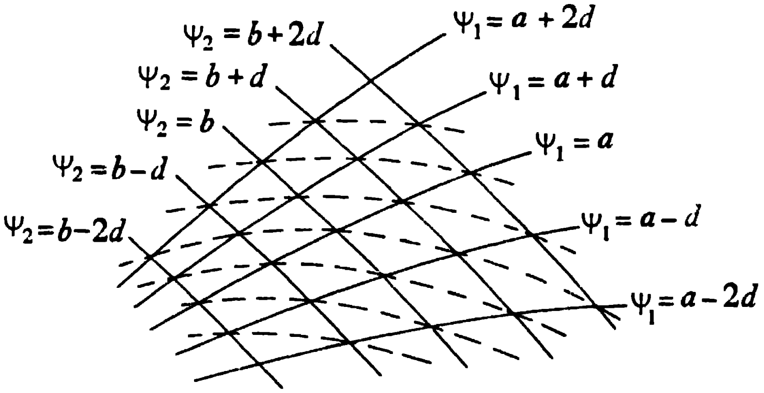

Fig. 2.5,1. Streamlines of a resultant flow.

Points on the streamlines of the resultant flow can easily be constructed by the following method which was first given by Rankine (see Fig. 2.5,1). Draw the streamlines of the first motion for the values a, $a \pm d$, $a \pm 2d$, $a \pm 3d$, etc. of ψ_1 and the streamlines of the second motion for the values b, $b \pm d$, $b \pm 2d$, $b \pm 3d$, etc. of ψ_2 where a, b and d are conveniently chosen constants; it should be noted that the increment of ψ between adjacent streamlines is d for both motions. The diagonals of the mesh drawn in such a manner that $\psi_1 + \psi_2$ is constant give the streamlines of the resultant motion, as shown in the figure.

For general three-dimensional flows we cannot employ a stream function. However, when the fluid is incompressible, the motion obtained by compounding two flows vectorially is a kinematically possible flow. This follows at once from the equation of continuity (2.3,9).

2.6 Conditions at the Boundary of the Fluid

Suppose that the fluid is in contact with a rigid body, which is itself moving or at rest. In the course of the motion the fluid does not penetrate into the rigid body nor does it separate from the surface of the body and leave an empty space (since we have supposed that cavitation does not occur). This implies the following kinematic condition: *the component of the velocity of the fluid at a point P on the surface of the body along the normal PN to the surface is equal to the component of the velocity of the body in the same direction.* The condition must be satisfied at all points where the body is in contact with the fluid and at all instants.

For an inviscid fluid there may be relative velocity of body and fluid at the common surface in a tangential direction, i.e. there may be relative tangential slip at the surface. However, *for a viscous fluid there is no slip at the surface and consequently the relative velocity of fluid and body is zero over the surface of contact.* This requires some qualifications where the fluid is a gas at a very low density (see § 9.14).

Let us now consider an inviscid fluid and at first suppose that the rigid body is at rest. Since the normal component of velocity of the body is zero it follows that the normal component of the velocity of the fluid is zero at the surface. Hence the surface is covered with streamlines. For two-dimensional flow this result takes the simple form that the boundary of the body is a streamline. When the fluid is incompressible, so that the stream function ψ exists, we accordingly find that the equation to the boundary is (see § 2.4)

$$\psi = \text{const.} \tag{2.6,1}$$

Next suppose that the rigid body is moving with velocity U in the direction OX and does not rotate. We may bring the body to rest *without altering the relative velocities* by superposing a velocity $-U$ on the whole system. Let ψ be the stream function for the original motion while ψ_1 corresponds to the uniform velocity $-U$ and ψ' is the stream function for the combined motion. By § 2.5 we then have

$$\psi' = \psi + \psi_1 \tag{2.6,2}$$

and the condition at the surface is

$$\psi' = \text{const.} \tag{2.6,3}$$

since the body is now at rest. Thus the boundary condition satisfied by ψ is

$$\psi = -\psi_1 + \text{const.} \tag{2.6,4}$$

By Example 1 of § 2.4 we have†

$$\psi_1 = Uy \tag{2.6,5}$$

and the condition to be satisfied by the original stream function becomes

$$\psi = -Uy + \text{const.} \tag{2.6,6}$$

at the surface of the moving body.

† Note that ψ_1 corresponds to a velocity $-U$.

For other types of motion of the rigid body we may again employ the foregoing method and have only to take ψ_1 as the stream function for the motion which, when superposed on the whole system, brings the body to rest. When the body moves without rotation in the direction OY with speed V we thus have ψ_1 corresponding to a uniform flow with speed $-V$ in the direction OY. By Example 1 of § 2.4

$$\psi_1 = -Vx + \text{const.}$$

and the boundary condition (2.6,4) becomes

$$\psi = Vx + \text{const.} \tag{2.6,7}$$

Lastly, suppose that the rigid body rotates about O with angular velocity ω. The superposed motion is now a rotation about O with angular velocity $-\omega$ and by Example 2 of § 2.4

$$\psi_1 = -\tfrac{1}{2}\omega(x^2 + y^2) + \text{const.}$$

Hence the boundary condition (2.6,4) becomes

$$\psi = \tfrac{1}{2}\omega(x^2 + y^2) + \text{const.} \tag{2.6,8}$$

There is no purely kinematic condition to be satisfied at any free surface of a fluid. However, as we shall see later, there is often the dynamical condition that the pressure is constant and this has implications regarding the velocity at the surface.

ILLUSTRATIVE EXAMPLES OF THE BOUNDARY CONDITIONS

Example 1. *Fixed circular cylinder in a uniform stream*

This is an example of two-dimensional flow of an incompressible fluid, so the stream function exists. The expression for the stream function is

$$\psi = Uy\left(1 - \frac{a^2}{x^2 + y^2}\right). \tag{2.6,9}$$

On the circle

$$x^2 + y^2 = a^2$$

we therefore have ψ zero. Hence (2.6,9) represents a kinematically possible flow about a fixed circular cylinder of radius a with OZ as axis.

Example 2. *Circular cylinder in rectilinear motion through fluid*

The stream function is here

$$\psi = -\frac{Ua^2y}{x^2 + y^2} \tag{2.6,10}$$

which we shall later see corresponds to the flow caused by a certain doublet (see equation (2.7,20)). On the circle

$$x^2 + y^2 = a^2$$

we have

$$\psi = -Uy$$

so equation (2.6,6) is satisfied.

Example 3. *Plane rotating about OZ*

Let us assume that

$$\psi = kxy \tag{2.6,11}$$

and enquire whether equation (2.6,8) can be satisfied on the straight line

$$y = x \tan \theta \tag{2.6,12}$$

by giving ω a suitable value. Now on this line $\psi = kr^2 \cos \theta \sin \theta$ and (2.6,8) will be satisfied when $\omega = 2k \cos \theta \sin \theta$

$$= k \sin 2\theta. \tag{2.6,13}$$

If ω be given, the corresponding expression for the stream function is

$$\psi = \omega xy \operatorname{cosec} 2\theta \tag{2.6,14}$$

This solution fails when θ is 90° since cosec 180° is infinite.

2.7 Sources, Sinks and Doublets

A *source* is a point within a fluid from which fluid issues symmetrically in all directions and the *strength* of the source is defined to be the volume of fluid which issues from it in unit time, i.e. the strength is equal to the total flux from the source. A source is an abstraction which can never be perfectly realised but, as we shall see, it is a very useful conception, as a real flow may often be regarded as due to a combination of sources and of *sinks*, which are sources of negative strength, i.e. points towards which the fluid converges symmetrically and from which it is supposed to be continuously removed. We could approximate very crudely to an ideal source by feeding fluid through a narrow tube into a spherical rose provided with numerous small holes, all of the same diameter and evenly distributed. This same arrangement approximates to an ideal sink when the fluid is sucked away through the tube.†

First let us consider an isolated three-dimensional point source of strength σ situated at the point S in an incompressible fluid. Let P be any other point and let r be the distance SP. If we describe a sphere of radius r with centre S the flow due to the source will be radial and have the same speed q at all points of the sphere. Hence the flux across the surface of the sphere will be $4\pi r^2 q$ and, by continuity, this must be equal to the strength σ of the source. Accordingly

$$q = \frac{\sigma}{4\pi r^2}. \tag{2.7,1}$$

---

† For reasons which cannot be discussed here the sink is represented in this manner much better than the source when we have to deal with a real viscous fluid.

Let S now be the origin of coordinates O. Then the cosine of the angle between OP and OX is x/r and the component of velocity in the direction OX is

$$u = \frac{qx}{r} = \frac{\sigma x}{4\pi r^3} \tag{2.7,2}$$

Similarly

$$v = \frac{\sigma y}{4\pi r^3} \tag{2.7,3}$$

and

$$w = \frac{\sigma z}{4\pi r^3}\,. \tag{2.7,4}$$

When it is necessary to show explicitly that u refers to the point whose coordinates are x, y, z we shall use the symbol $u(x, y, z)$; $v(x, y, z)$ and $w(x, y, z)$ will be similarly employed. Since the flow is everywhere radial the streamlines are straight lines through the source.

Now let us consider a *uniform line source*. We may regard this as built up from very numerous point sources distributed along a straight line so that the total strength of the sources in the length ds is equal to $m\,ds$, where m is constant. When the line source is infinitely long the fluid will issue radially at right angles to the line at all points. Hence the flow due to the line source will in fact be two-dimensional in planes perpendicular to the line. For this reason a uniform and unbounded line source is often called a *two-dimensional source* and its strength m is the flux from the source in a layer of unit depth. Take a circular cylinder of radius r having the line source for axis. The flux across unit length of the cylinder is $2\pi rq$ and this must equal the strength m. Consequently

$$q = \frac{m}{2\pi r}\,. \tag{2.7,5}$$

It follows that the two components of velocity in a plane perpendicular to the line source are

$$u = \frac{qx}{r} = \frac{mx}{2\pi r^2} \tag{2.7,6}$$

$$v = \frac{qy}{r} = \frac{my}{2\pi r^2} \tag{2.7,7}$$

where the origin O lies on the source. Let us now find the value of the stream function at the point P (see Fig. 2.7,1) whose polar coordinates are r, θ. Let ψ be taken to be zero on the streamline OX. Then the value of ψ at P is the flux from right to left across

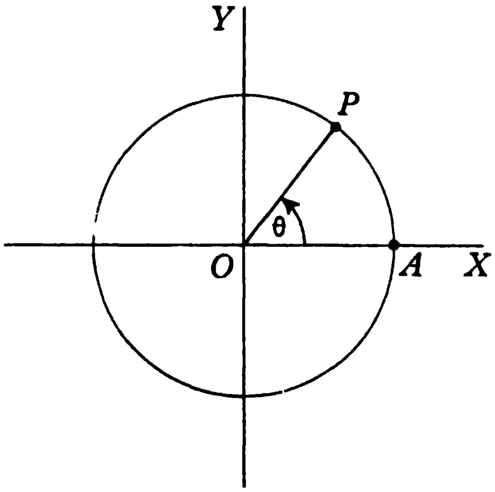

Fig. 2.7,1. Stream function of two-dimensional source.

the arc AP and this is equal to the fraction $\theta/2\pi$ of the whole flux from the source but with reversed sign. Thus

$$\psi = -\frac{m\theta}{2\pi} = -\frac{m}{2\pi}\tan^{-1}\left(\frac{y}{x}\right). \tag{2.7,8}$$

The streamlines are again straight lines through the source.

Next, let us consider a source-sink pair consisting of a three-dimensional source of strength σ at S (see Fig. 2.7,2) and a sink of numerically equal strength at S', where $S'O = OS = h$. The velocity at P due to the source at S is the same as the velocity at P' due to an equal source situated at O, where $P'P$ is equal and parallel to OS. Similarly the velocity at P due to the sink at S' is equal to that at P'' due to an equal sink at O, i.e. the velocity at P due to the sink at S' is equal and opposite to the velocity at P'' due to a source of strength σ at O, with PP'' equal and parallel to $S'O$. Let $u_1(x, y, z)$ be the velocity at x, y, z due to a *unit* source at O. Then we see from the above that the total component of velocity at P in the direction OX is

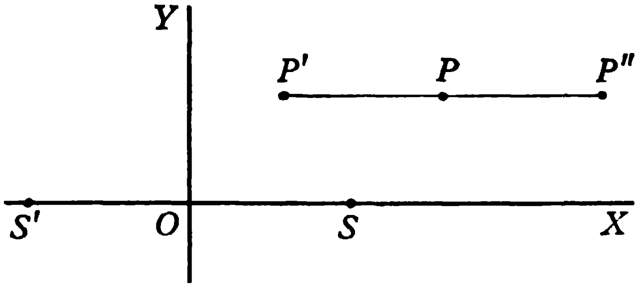

Fig. 2.7,2. Source-sink pair.

$$u = \sigma\{u_1(x - h, y, z) - u_1(x + h, y, z)\}. \tag{2.7,9}$$

Suppose now that h tends to zero while σ increases in such a manner that

$$2h\sigma = \mu, \tag{2.7,10}$$

a constant. The arrangement is then called a three-dimensional *doublet* or *dipole* of strength μ. Since now

$$u_1(x - h, y, z) - u_1(x + h, y, z) = -2h\frac{\partial}{\partial x}u_1(x, y, z) \tag{2.7,11}$$

it follows from (2.7,9) that

$$u = -\mu\frac{\partial}{\partial x}u_1(x, y, z). \tag{2.7,12}$$

Similarly

$$v = -\mu\frac{\partial}{\partial x}v_1(x, y, z) \tag{2.7,13}$$

and

$$w = -\mu\frac{\partial}{\partial x}w_1(x, y, z) \tag{2.7,14}$$

where $v_1(x, y, z)$ and $w_1(x, y, z)$ are respectively the components in the directions OY and OZ of the velocity due to a *unit* source at O.

If in Fig. 2.7,2 we regard S and S' as a two-dimensional source and sink respectively, each of strength m, and now define the strength of the doublet as

$$2hm = \mu', \tag{2.7,15}$$

we find by the same argument as before that the velocity due to the doublet has the components.

$$u = -\mu' \frac{\partial}{\partial x} u_1(x, y) \tag{2.7,16}$$

$$v = -\mu' \frac{\partial}{\partial x} v_1(x, y) \tag{2.7,17}$$

where $u_1(x, y)$ and $v_1(x, y)$ are the components of the velocity due to a *unit* source at O, in the directions OX, OY respectively. Likewise if ψ_1 is the stream function of a unit two-dimensional source at O, the stream function for the doublet of strength μ' is

$$\psi = -\mu' \frac{\partial \psi_1}{\partial x}. \tag{2.7,18}$$

By (2.7,8)

$$\psi_1 = -\frac{1}{2\pi} \tan^{-1}\left(\frac{y}{x}\right) \tag{2.7,19}$$

and (2.7,18) accordingly becomes

$$\psi = -\frac{\mu' y}{2\pi(x^2 + y^2)} = -\frac{\mu' \sin\theta}{2\pi r}. \tag{2.7,20}$$

Therefore the components of velocity are

$$u = -\frac{\partial \psi}{\partial y} = \frac{\mu'(x^2 - y^2)}{2\pi(x^2 + y^2)^2} = \frac{\mu' \cos 2\theta}{2\pi r^2} \tag{2.7,21}$$

$$v = \frac{\partial \psi}{\partial x} = \frac{2\mu' xy}{2\pi(x^2 + y^2)^2} = \frac{\mu' \sin 2\theta}{2\pi r^2}. \tag{2.7,22}$$

The resultant speed is

$$q = \sqrt{(u^2 + v^2)} = \frac{\mu'}{2\pi r^2} \tag{2.7,23}$$

and, since the streamlines are the curves on which the stream function is constant, they are the circles

$$r = -\left(\frac{\mu'}{2\pi\psi}\right) \sin\theta. \tag{2.7,24}$$

Whether the doublet is two- or three-dimensional, the line joining the source and sink is called the *axis* of the doublet and the positive sense of the axis is from the sink towards the source. In all the cases discussed above the axis of the doublet lies along OX and the positive sense of the axis agrees with that of OX when the strength of the doublet is positive.

When a source, sink or doublet is steady its position is fixed in the coordinate system adopted and its strength is constant. For a steady doublet the axis and its sense are also fixed.

Sources, sinks and doublets were introduced into hydrodynamic theory by Rankine. When we are dealing with an incompressible fluid, a source (or sink) is an exact analogue of a gravitating particle, electrically charged particle or magnetic pole and the velocity of the fluid corresponds to the field vector of the particle or pole. A doublet corresponds to a magnet whose length is very small and the strength of the doublet corresponds to the *magnetic moment* of the magnet.

ILLUSTRATIVE EXAMPLES ON SOURCES, SINKS AND DOUBLETS

Example 1. *The stream function and streamlines of a two-dimensional source-sink pair*

In Fig. 2.7,3 the source is at S and the sink at S' where these are points

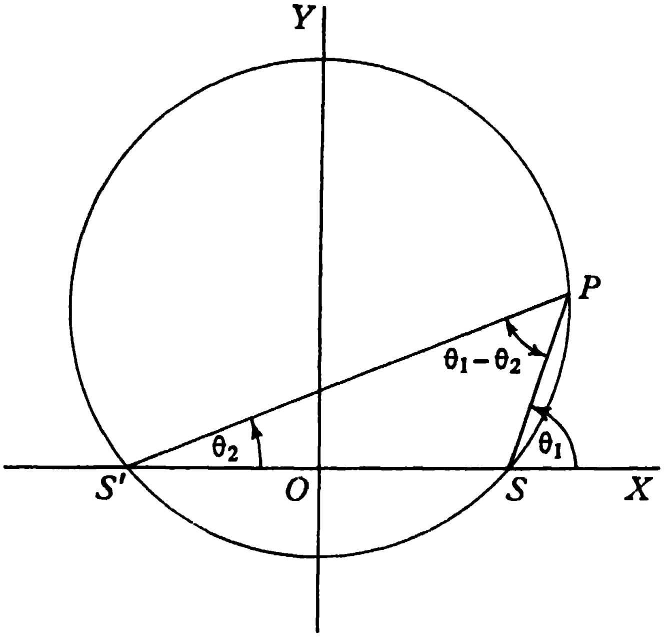

Fig. 2.7,3. Circular streamline of two-dimensional source-sink pair.

on OX with abscissae $\pm h$. By equation (2.7,8) the value of the stream function for the combined motion at any point P is

$$\psi = \frac{m}{2\pi}(\theta_2 - \theta_1). \tag{2.7,25}$$

Hence a streamline is a curve such that $(\theta_2 - \theta_1)$ is constant, i.e. such that the angle SPS' is constant. Therefore every streamline is a circle through S and S' (see Fig. 2.7,4).

Now $$\tan\theta_2 = \frac{y}{x+h} \quad \text{and} \quad \tan\theta_1 = \frac{y}{x-h}$$

$$\therefore \quad \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2\tan\theta_1} = \frac{-2hy}{x^2 + y^2 - h^2}$$

$$= \tan\left(\frac{2\pi\psi}{m}\right) \text{ by (2.7,25).}$$

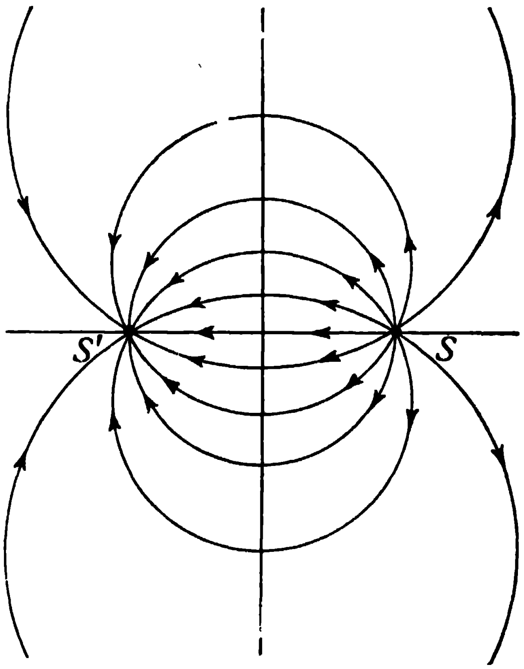

Fig. 2.7,4. Streamlines of two-dimensional source-sink pair.

Hence the Cartesian equation of the streamline is

$$x^2 + y^2 - h^2 = -2hy\cot\left(\frac{2\pi\psi}{m}\right),$$

or

$$x^2 + \left\{y + h\cot\left(\frac{2\pi\psi}{m}\right)\right\}^2 = h^2\operatorname{cosec}^2\left(\frac{2\pi\psi}{m}\right). \qquad (2.7,26)$$

Thus the centre of the circle is the point $[0, -h\cot\{(2\pi\psi/m)\}]$ and its radius is $h \operatorname{cosec}(2\pi\psi/m)$.

When h tends to zero while $2hm = \mu'$, we have, for a given point P, $\theta_2 - \theta_1$ tending to zero. Hence we may substitute the tangent for the measure of the angle in equation (2.7,25) and so obtain for the doublet

$$\psi = \frac{-2hmy}{2\pi(x^2 + y^2)} = \frac{-\mu' y}{2\pi(x^2 + y^2)}.$$

This agrees with equation (2.7,20). The streamlines (2.7,26) become circles touching OX at O (see Fig. 2.7,5). The ordinate of the centre, which is numerically equal to the radius, is $-\mu'/(4\pi\psi)$.

Example 2. Velocity due to a three-dimensional doublet

In the course of the work we require the values of $\dfrac{\partial r}{\partial x}, \dfrac{\partial r}{\partial y}$ and $\dfrac{\partial r}{\partial z}$.

Now
$$r^2 = x^2 + y^2 + z^2$$

and
$$2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}.$$

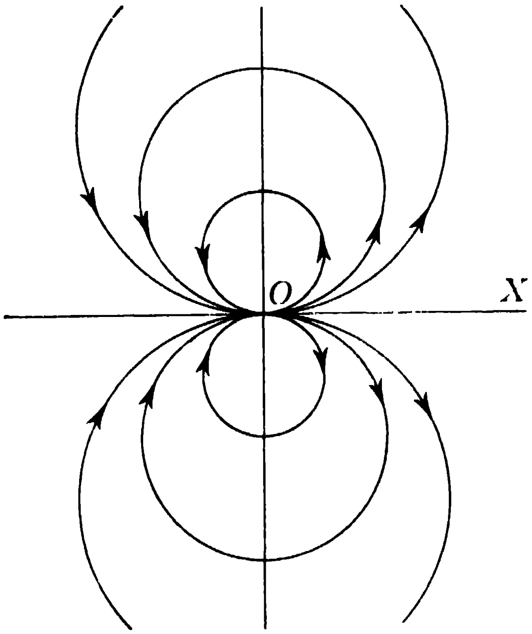

Fig. 2.7,5. Streamlines of two-dimensional doublet.

Similarly
$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \text{and} \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

From equations (2.7,2) and (2.7,12) we get

$$u = -\mu\frac{\partial}{\partial x}\left(\frac{x}{4\pi r^3}\right) \qquad \text{since } \sigma = 1$$

$$= \frac{\mu}{4\pi r^5}(3x^2 - r^2). \tag{2.7,27}$$

Similarly

$$v = -\mu\frac{\partial}{\partial x}\left(\frac{y}{4\pi r^3}\right) = \frac{3\mu xy}{4\pi r^5} \tag{2.7,28}$$

and

$$w = -\mu\frac{\partial}{\partial x}\left(\frac{z}{4\pi r^3}\right) = \frac{3\mu xz}{4\pi r^5}. \tag{2.7,29}$$

The radial component of the velocity is

$$u_r = \frac{ux + vy + wz}{r} = \frac{\mu x}{2\pi r^4}. \tag{2.7,30}$$

Now the radial component of the velocity of a uniform stream of speed U in the positive sense of OX is Ux/r. The total radial component will vanish on a sphere of radius $r = a$ if

$$x\left(\frac{\mu}{2\pi a^4} + \frac{U}{a}\right) = 0.$$

This requires that

$$\mu = -2\pi a^3 U. \tag{2.7,31}$$

Since the total radial velocity is zero over the surface of the sphere we may, when dealing with a perfect fluid, place a rigid and fixed sphere of radius a in the fluid without disturbing the flow external to it. Thus, at all external points, the modification of the uniform stream brought about by the presence of the sphere is exactly the same as that caused by a doublet at its centre and whose strength is given by equation (2.7,31). This doublet is the *image* of the uniform stream in the spherical surface (see also § 2.16). It should be noted that the positive sense of the axis of the doublet is opposed to that of the stream.

The foregoing important result can be stated as follows: the velocity of perturbation caused by placing a fixed rigid sphere in a uniform stream of perfect fluid is equal to that caused by a certain doublet situated at the centre of the sphere. The axis of this doublet lies along the stream but with opposite sense and its strength is equal to the velocity of the stream multiplied by 3/2 times the volume of the sphere (see § 2.17).

Now let us superpose on the whole system a uniform velocity equal and opposite to that of the stream. We then have a sphere of radius a advancing with velocity $-U$ into a perfect fluid which is at rest except where disturbed by the sphere. The velocity of the fluid is everywhere the same as that caused by the above specified doublet. (See also Appendix 2 to this chapter.)

Example 3. *Plane source*

Suppose that an infinite plane is covered with sources so that their total strength per unit area is s (constant). The flow is then normal to the plane and away from it on both sides. The flux per unit area on one side is $s/2$ and this is the constant speed of the flow. Hence a uniform stream of velocity U may be regarded as caused by a plane source of strength $2U$, the plane being perpendicular to the velocity vector. Alternatively, we may regard the field of uniform flow as existing *between* a plane source and a plane sink, each of strength $2U$.

2.8 Streamline Bodies

Following a method due to Rankine we shall show here how we can obtain the flow pattern in the neighbourhood of bodies of certain shapes placed in a uniform stream of perfect fluid, and we begin with the case of two-dimensional motion. The method consists in combining the motions due to sources and sinks with the uniform stream. There is then at least one *stagnation point* where the fluid is at rest. The particular streamline which passes through the stagnation point divides in two there, as shown in Fig. 2.8,1, and is called the *dividing streamline*. When the total strength of the sources and sinks is zero there will be, in typical cases, a second stagnation point on the same streamline which thus consists of curves extending to infinity and a closed loop. The fluid within the loop issues from the sources and is drained away at the sinks, so it does not mingle with the general stream. When the motion is steady we may place a thin fixed and rigid barrier on the loop of the dividing streamline. Alternatively, we may abolish the fluid within the loop and suppose the barrier to be the surface of a solid rigid body. It is now clear that we have only *imagined* the sources and sinks to exist; they were merely a device for obtaining the form of a body such that the flow round it could easily be calculated. The imaginary sources and sinks within the fixed boundary are the *image* of the uniform stream in the fixed boundary (see further § 2.16). Since the total strength of the sources and sinks is zero the system can always be supposed to be constructed from source-sink pairs or, possibly, doublets (see § 2.7).

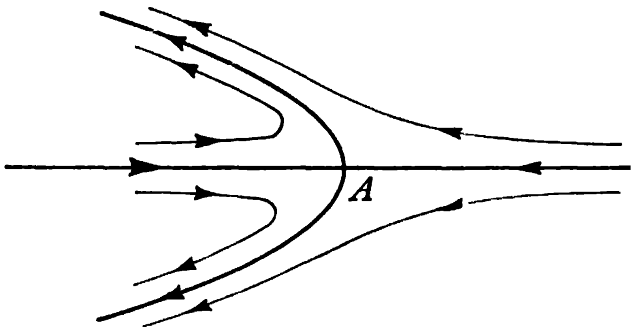

Fig. 2.8,1. Dividing streamline (stagnation point at A).

We shall approach the detailed study of this method by considering a single two-dimensional source of strength m at O in a uniform stream flowing with speed U in the direction OX (see Fig. 2.8,2) and we shall suppose that m and U are positive. First let us find the stagnation point or points. It is clear that any such points must lie on the axis OX for it is only on this line that the two contributions to the velocity have the same direction. There cannot be a stagnation point on OX to the right of O since both velocities then have the same sense. Suppose that the stagnation point lies at A to the left of O where $AO = a$. Then the velocity at A due to the source is towards the left and the condition for a zero resultant is

$$\frac{m}{2\pi a} = U$$

or

$$a = \frac{m}{2\pi U}. \tag{2.8,1}$$

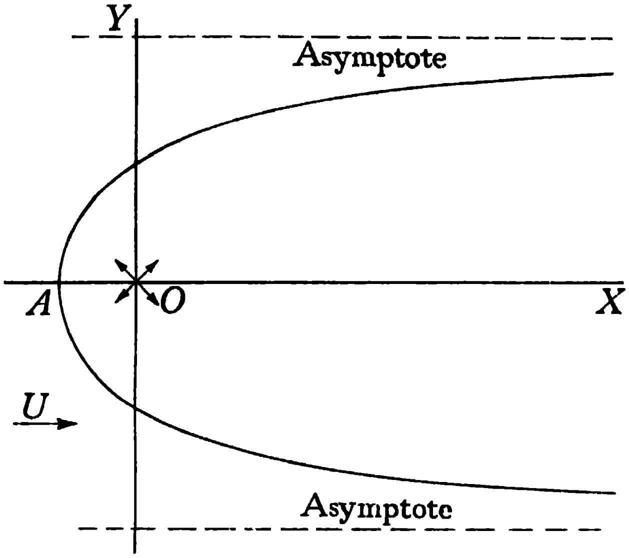

Fig. 2.8,2. Half-body (two-dimensional source in uniform stream).

There is thus only one stagnation point and the dividing streamline does not close again downstream. The solid whose profile is this dividing streamline has been called a *half body* and it takes the form of a slab with a rounded nose and extending to infinity downstream.

The stream function for the uniform stream is, by equation (2.4,6),

$$\psi_1 = -Uy$$

and the stream function for the source is, by equation (2.7,8),

$$\psi_2 = -\frac{m\theta}{2\pi}.$$

Hence the stream function for the combined motion is

$$\psi = -Uy - \frac{m\theta}{2\pi} = -Uy - \frac{m}{2\pi}\tan^{-1}\left(\frac{y}{x}\right). \tag{2.8,2}$$

The part of OX to the left of A is part of the dividing streamline and here $y = 0$ while $\theta = \pi$. Hence the constant value of ψ on the dividing streamline is

$$\psi_0 = -\frac{m}{2} \tag{2.8,3}$$

To the right of O at infinity θ is zero on the upper branch of the dividing streamline†. Hence (2.8,2) yields

$$y = \frac{m}{2U} \tag{2.8,4}$$

and the total thickness of the half body at infinity is m/U. We can easily prove this independently. At infinity the velocity due to the source is zero

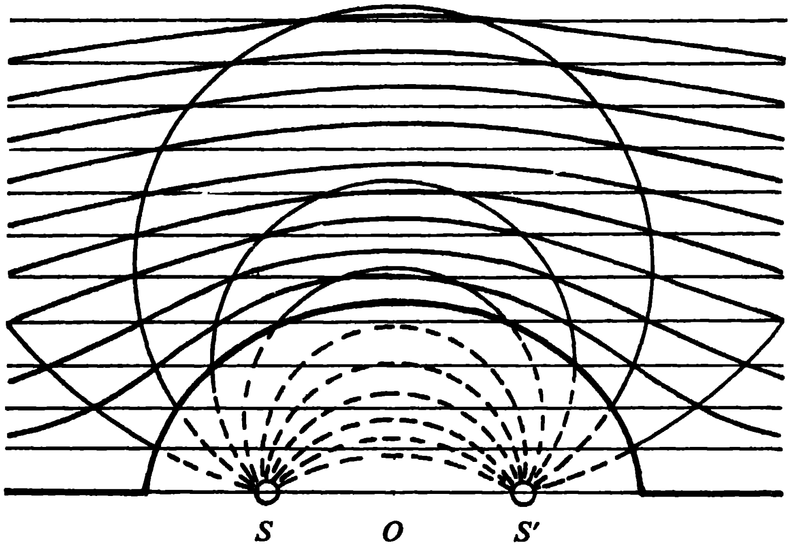

Fig. 2.8,3. Rankine oval. Constructed by combining flow of source-sink pair with uniform stream.

and the resultant velocity is uniform and equal to U. Now all the fluid issuing from the source flows away in a band of width $2y$, so

$$2yU = m$$

in accordance with (2.8,4). Since the resultant velocity tends to parallelism with OX at infinity the profile of the half body, being a streamline, also tends to parallelism with OX.

Instead of thinking of the complete half body in an unobstructed stream we may take the upper half only together with the part of OX to the left of A as the profile of a rigid barrier. We then have fluid flowing over a plane and meeting a plateau whose face is a cliff rounded in a particular curve.

Next let us take a two-dimensional source at S with a sink of the same strength at S' (see Fig. 2.8,3) and superpose a uniform stream of velocity U in the direction SS'. The loop of the dividing streamline is then called a *Rankine oval.* Take the midpoint of SS' as the origin O and the axis OX along OS. Let $SO = OS' = h$ and, as before, let m be the numerical strength of the source and sink. For the stagnation point A at the upstream end of the oval we have

$$\frac{m}{2\pi(a-h)} \quad \frac{m}{2\pi(a+h)} = U \tag{2.8,5}$$

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† For the lower branch of the curve $\theta = 2\pi$.

where $AO = a$. Hence we obtain

$$\frac{a}{h} = \sqrt{(1 + \beta)} \tag{2.8,6}$$

where

$$\beta = \frac{m}{\pi U h} \tag{2.8,7}$$

is the *shape parameter* of the oval and is non-dimensional (see § 4.7). Rankine ovals having the same value of the shape parameter are geometrically similar and their corresponding linear dimensions are in proportion to the distance separating the source and sink. The second stagnation point A' is also at the distance a from O, for equation (2.8,5) also expresses the balance of velocities at a point lying at a positive distance a to the right of O. The whole oval is, in fact, symmetrical about OY. The stream function for the flow is obtained by adding those for the uniform flow and for the source-sink pair, with the source on the negative side of the origin. Hence, by the results of Example 1 of § 2.7

$$\psi = -Uy + \frac{m}{2\pi} \tan^{-1} \left(\frac{2hy}{x^2 + y^2 - h^2}\right). \tag{2.8,8}$$

The dividing streamline is such that ψ vanishes on it and its equation can be written

$$\begin{aligned} x^2 + y^2 - h^2 &= 2hy \cot \left(\frac{2\pi U y}{m}\right) \\ &= 2hy \cot \left(\frac{2y}{\beta h}\right) \end{aligned} \tag{2.8,9}$$

or

$$\left(\frac{x^2 + y^2}{h^2}\right) = 1 + 2\left(\frac{y}{h}\right) \cot \left(\frac{2y}{\beta h}\right). \tag{2.8,10}$$

The oval can be plotted from this equation, for x/h can be obtained at once when y/h is given. For a given ordinate y of a point on the oval there is a pair of equal and opposite values of the abscissa x and these are unaltered when the sign of y is changed. Hence the oval is symmetrical about both axes.

The resultant velocity on the transverse axis OY is parallel to OX. Now the flux across OY within the oval is equal to the strength of the source. Hence

$$\int_{-b}^{b} u \, dy = m \tag{2.8,11}$$

where b is the semi-axis minor of the oval and u is measured on OY. Since both the source and sink give a positive contribution to u on OY it follows that

$$u > U$$

and equation (2.8,11) shows that

$$\int_{-b}^{b} U\,dy < m$$

or

$$b < \frac{m}{2U}. \tag{2.8,12}$$

When β is very small the oval is long and narrow; each half of the oval then approximates to the shape of a "half-body" as discussed above and the inequality (2.8,12) tends towards an equality as β tends to zero.†

Next, let the source-sink pair become a doublet of strength μ' with its axis along OX in the negative sense. The expression (2.8,8) for the stream function becomes (see § 2.7)

$$\begin{aligned}\psi &= -Uy + \frac{\mu' y}{2\pi(x^2 + y^2)}\\ &= -Uy\left\{1 - \frac{(\mu'/2\pi U)}{x^2 + y^2}\right\}.\end{aligned} \tag{2.8,13}$$

The equation to the loop of the dividing streamline $\psi = 0$ is therefore

$$x^2 + y^2 = \frac{\mu'}{2\pi U}. \tag{2.8,14}$$

The Rankine oval has therefore become a circle of radius

$$a = \sqrt{\frac{\mu'}{2\pi U}}. \tag{2.8,15}$$

We can interpret this result in the following way: when the fixed circle

$$x^2 + y^2 = a^2 \tag{2.8,16}$$

is placed in a uniform stream of velocity U in the positive sense of OX the stream function is

$$\psi = -Uy\left\{1 - \frac{a^2}{x^2 + y^2}\right\}. \tag{2.8,17}$$

The doublet of strength

$$\mu' = 2\pi U a^2 \tag{2.8,18}$$

situated at the origin with its axis along OX in the negative sense is the image of the uniform stream in the circle (see § 2.14).

The foregoing results can be generalized in several ways. First, we may have several sources and sinks or continuous distributions of these, always

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† It can easily be shown that, when β is small, b is approximately equal to

$$\frac{m}{2U}(1 - \beta + \beta^2).$$

subject to the condition that the algebraic total strength of the sources and sinks is zero. Second, the sources and sinks may not lie on a single straight line parallel to the direction of flow, and the oval is then unsymmetric, i.e. it is cambered. Clearly, a source offset from OX is equivalent to a source of the same strength situated at the foot of the perpendicular from the source on OX, together with a source-sink pair whose axis is perpendicular to OX. This pair is almost exactly equivalent to a doublet having the same axis and whose strength is equal to the "moment" of the pair.

We have hitherto considered two-dimensional sources, sinks and doublets placed in a uniform stream but there are analogous results for three-dimensional sources etc. When a single three-dimensional source is fixed in a steady uniform stream there is a fixed surface separating the fluid which has issued from the source from the rest of the fluid. This surface of separation is a surface of revolution whose axis is parallel to the stream and any plane through the axis cuts it in a streamline; it is known as a (three-dimensional) half-body and it takes the form of a cylinder with a rounded end upstream. At a great distance from the source the velocity is equal to that of the undisturbed stream and, if the radius of the body be r, we must have

$$\sigma = \pi r^2 U$$

or

$$r = \sqrt{\frac{\sigma}{\pi U}}. \qquad (2.8,19)$$

The surface of revolution corresponding to the Rankine oval is called a *Rankine ovoid.* When the source-sink pair becomes a doublet, the ovoid becomes a sphere, as proved in Example 2 of § 2.7, where it is shown that for a sphere of radius a the strength of the doublet is

$$\mu = -2\pi a^3 U. \qquad (2.8,20)$$

ILLUSTRATIVE EXAMPLES

Example 1. Velocity at the surface of a fixed circular cylinder in a uniform stream of perfect fluid

By equation (2.8,17) the stream function is

$$\psi = -Uy\left\{1 - \frac{a^2}{x^2 + y^2}\right\}.$$

Hence the components of velocity are

$$u = -\frac{\partial \psi}{\partial y} = U\left\{1 - \frac{a^2(x^2 - y^2)}{(x^2 + y^2)^2}\right\}$$

$$v = \frac{\partial \psi}{\partial x} = -\frac{2Ua^2xy}{(x^2 + y^2)^2}.$$

On the surface of the cylinder we have

$$x = a\cos\theta \qquad y = a\sin\theta.$$

Hence $\qquad u = 2U\sin^2\theta \qquad v = -2U\cos\theta\sin\theta$

and the resultant velocity, which is tangent to the cylinder, is

$$q = 2U\sin\theta.$$

The greatest velocity occurs on OY and is $2U$.

Example 2. *Approximate determination of the source distribution for a slender symmetrical body*

This problem is inverse to that considered in the text. Consider first two-dimensional flow and let y be the ordinate of the upper surface of the body. Since the body is very slender the sources and sinks are weak. Hence there is little error in taking the resultant velocity to be that of the undisturbed stream, namely, U. Let $m'(x)$ be the source strength per unit length. Then the total flux within the body at x is

$$\int_{x_0}^{x} m'(x)\,dx = 2Uy$$

where x_0 is the abscissa of the "nose" or leading edge. Hence we obtain by differentiation

$$m'(x) = 2U\frac{dy}{dx}.$$

This is a good approximation provided that dy/dx is small, but the formula fails at a rounded nose or tail. (Compare a slender Rankine oval where $m'(x)$ is zero at the ends.) An alternative proof is given in § 11.5,6.

For a slender body of revolution we obtain by a similar argument

$$\int_{x_0}^{x} \sigma'(x)\,dx = \pi U r^2$$

where $\sigma'(x)$ is the source strength per unit length and r is the radius at the abscissa x. Hence

$$\sigma'(x) = 2\pi U r\frac{dr}{dx}.$$

This is a good approximation when dr/dx is small.

Not all symmetric bodies can be accurately represented as Rankine ovals or ovoids.

2.9 Acceleration of a Fluid Particle

A knowledge of the acceleration of the fluid particles is required when the pressure in the fluid is under investigation. We shall begin by considering *steady motion* (see § 2.2). Then the streamlines are fixed and any particle on a given streamline moves along it. The resultant velocity is q, which is constant at any given point on the streamline but, in general, varies with the position of the point. Let s be the arc of the streamline measured from some fixed datum point on it. Then q is some function of s only, so long as we consider this particular streamline. It is known from elementary mechanics that, when a particle moves along a fixed curve, the component of its acceleration in the direction of the tangent to the path is

$$a_t = \frac{dq}{dt} = \frac{dq}{ds}\frac{ds}{dt} = q\frac{dq}{ds} = \frac{1}{2}\frac{dq^2}{ds}$$

since $q = ds/dt$. Thus the component of acceleration of a fluid particle in the direction of its motion is

$$a_t = \frac{1}{2}\frac{\partial q^2}{\partial s} \tag{2.9,1}$$

where we now use the notation for a partial differential coefficient, for q^2 is, in general, a function of all the spatial coordinates. The normal component of the acceleration is

$$a_n = \frac{q^2}{R_c} \tag{2.9,2}$$

where R_c is the radius of curvature of the streamline.† We shall see later (§ 3.4) that the expression (2.9,1) for the tangential component of the acceleration suffices for the establishment of Bernoulli's theorem.

In order to find the three components of the acceleration in the general case of unsteady motion we shall first find the time rate of change of the measure α of some arbitrary physical property of a particle of fluid as the particle moves and we shall suppose that α is given as a function of the four variables x, y, z and t. Then

$$\delta\alpha = \frac{\partial\alpha}{\partial x}\delta x + \frac{\partial\alpha}{\partial y}\delta y + \frac{\partial\alpha}{\partial z}\delta z + \frac{\partial\alpha}{\partial t}\delta t.$$

But we have also

$\delta x = u\delta t$, $\delta y = v\delta t$ and $\delta z = w\delta t$ since we are following a given particle of fluid. Therefore

$$\delta\alpha = \left(u\frac{\partial\alpha}{\partial x} + v\frac{\partial\alpha}{\partial y} + w\frac{\partial\alpha}{\partial z} + \frac{\partial\alpha}{\partial t}\right)\delta t$$

and

$$\frac{\delta\alpha}{\delta t} = u\frac{\partial\alpha}{\partial x} + v\frac{\partial\alpha}{\partial y} + w\frac{\partial\alpha}{\partial z} + \frac{\partial\alpha}{\partial t}. \tag{2.9,3}$$

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† For general steady three-dimensional motion this is the component of the acceleration along the *principal normal* to the streamline. The resultant acceleration lies in the osculating plane of the streamline, so the component of acceleration along the *binormal* is zero. For motion in the plane OXY the binormal is parallel to OZ.

Now the component of the acceleration of the particle in the direction OX is the rate at which its component of velocity u in the same direction changes with time. Hence we shall obtain this component of the acceleration by identifying α with u. Accordingly the component of the acceleration of the particle in the direction OX is

$$a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}. \tag{2.9,4}$$

Similarly we obtain

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \tag{2.9,5}$$

and

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}. \tag{2.9,6}$$

It is usual to write the differential operator appearing in these equations as

$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t} \tag{2.9,7}$$

and then we have the set of equations

$$a_x = \frac{Du}{Dt}, \qquad a_y = \frac{Dv}{Dt}, \qquad a_z = \frac{Dw}{Dt}. \tag{2.9,8}$$

When the motion is steady the partial derivatives with respect to t vanish while for two-dimensional motion in the plane OXY we have w and a_z zero everywhere.[†]

Examples on the Acceleration of Fluid Particles

Example 1. *Steady forced vortex*

The forced vortex has been defined in § 1.7 and, when it is steady, the fluid revolves about OZ as a rigid body with constant angular velocity ω. The components of velocity are therefore

$$u = -\omega y, \quad v = \omega x, \quad w = 0.$$

Hence

$$\begin{array}{llll}
\dfrac{\partial u}{\partial x} = 0 & \dfrac{\partial u}{\partial y} = -\omega & \dfrac{\partial u}{\partial z} = 0 & \dfrac{\partial u}{\partial t} = 0 \\[2ex]
\dfrac{\partial v}{\partial x} = \omega & \dfrac{\partial v}{\partial y} = 0 & \dfrac{\partial v}{\partial z} = 0 & \dfrac{\partial v}{\partial t} = 0 \\[2ex]
\dfrac{\partial w}{\partial x} = 0 & \dfrac{\partial w}{\partial y} = 0 & \dfrac{\partial w}{\partial z} = 0 & \dfrac{\partial w}{\partial t} = 0
\end{array}$$

[†] It is possible for the motion to be instantaneously two-dimensional (w zero everywhere) while a_z is not zero, in unsteady motion.

Therefore we obtain from equations (2.9,4) . . . (2.9,6)

$$a_x = -\omega^2 x$$
$$a_y = -\omega^2 y$$
$$a_z = 0.$$

These results agree with those obtained by elementary methods.

Example 2. *Derivation of the radius of curvature of the path of a particle*

We may, for steady motion, find the resultant and tangential accelerations and so deduce the value of the normal acceleration, from which the radius of curvature can be found.

As a simple example, take the steady two-dimensional motion for which $u = cx$, $v = -cy$, with c constant. We find at once that $a_x = c^2x$, $a_y = c^2y$.

Also $q^2 = c^2(x^2 + y^2)$ while by resolution of the acceleration in the direction of motion we get

$$a_t = \frac{ua_x + va_y}{q} = \frac{c^3(x^2 - y^2)}{q}.$$

Hence $$a_n^{\,2} = a_x^{\,2} + a_y^{\,2} - a_t^{\,2} = \frac{4c^4x^2y^2}{x^2 + y^2}$$

and, by equation (2.9,2),

$$R_c^{\,2} = \frac{q^4}{a_n^{\,2}} = \frac{(x^2 + y^2)^3}{4x^2y^2}.$$

In polar coordinates this becomes

$$R_c = r \operatorname{cosec} 2\theta.$$

The streamlines are here (see § 2.2) the rectangular hyperbolas $xy = k$. Hence the expression for the radius of curvature can be written alternatively

$$R_c = \frac{r^3}{2k}.$$

These expressions can be verified by the ordinary formulae for the radius of curvature of a plane curve.

2.10 Flow along a Curve. Circulation

In Fig. 2.10,1 APB is some curve joining the points A and B and lying wholly within the fluid. Let PP' be an element of the curve of length ds and let θ be the angle between the tangent to the curve at P and the velocity of the fluid there at the instant considered. Then the flow from A to B along the given curve is defined by the integral

$$\int_A^B q \cos\theta \, ds.$$

It should be noted that we pass along the curve in the sense from A to B and this also defines the positive sense of the tangent at any point. The velocity q is to be reckoned for the *same instant* at all points. We shall find it convenient to represent the flow from A to B by the symbol $\overrightarrow{AB}$. Hence we have by the definition

$$\overrightarrow{AB} = \int_A^B q \cos\theta \, ds. \qquad (2.10,1)$$

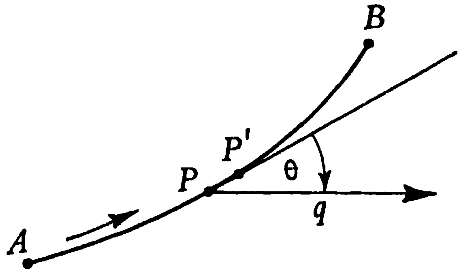

Fig. 2.10,1. Definition of flow along a curve.

It will be noted that there is an analogy between flow along a curve and work done on a particle as it moves along a curve. If we substitute the force on the particle for q in (2.10,1) the integral becomes equal to the work done on the particle. It is often convenient to represent the flow along a line $ABCDE$, say, by the symbol $ABCDE$. It is evident from the definition that

$$ABCDE = AB + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}. \qquad (2.10,2)$$

It is also evident that

$$\overrightarrow{BA} = -\overrightarrow{AB} \qquad (2.10,3)$$

for the sign of $\cos\theta$ in (2.10,1) is reversed when the positive sense of the tangent is reversed. In general, AB has a definite value at a given instant only when the curve joining A and B is specified, but there are important cases where the flow is independent of the choice of the curve. In steady motion the flow is the same at all instants. The curve along which the flow is reckoned need not, in general, be plane.

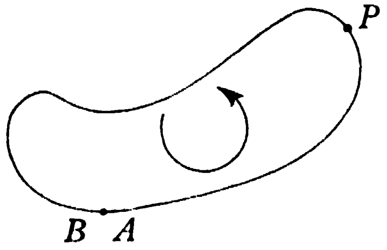

Fig. 2.10,2. Definition of circulation (flow along a closed circuit). The arrow shows the positive sense of the circulation.

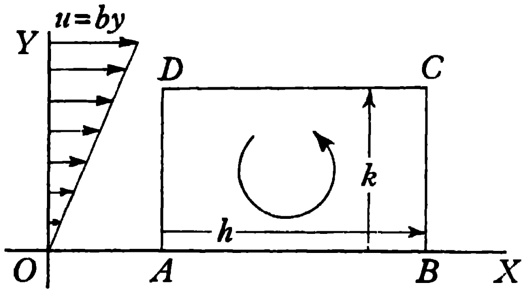

Fig. 2.10,3. Circulation in a rectangular circuit for shearing flow.

When A and B coincide, and the curve connecting them is accordingly a closed circuit, the flow is called the *circulation* in the circuit. It is very important for the reader to appreciate that the term "circulation" is here used in a special technical sense which has only a remote connection with the ordinary meaning of the word. In particular, the circulation in a given circuit may differ from zero while no particle of the fluid "circulates", i.e.

describes a closed curve. To illustrate this, let us consider the counterclockwise circulation in the rectangular circuit $ABCD$ (see Fig. 2.10,3) when the velocity at the instant considered is given by

$$u = by, \qquad v = 0, \qquad w = 0. \tag{2.10,4}$$

Then we have $\overrightarrow{AB}$ zero since the fluid is at rest on AB and $\overrightarrow{BC}$ is also zero since the component of velocity along BC is zero. On CD we have

$$q = u = bk \qquad \text{and} \qquad \cos\theta = -1$$

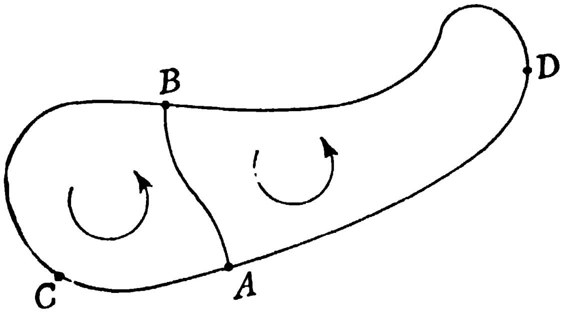

Fig. 2.10,4. Circulation in a divided circuit.

since the positive sense of the tangent to the circuit is opposite to that of the motion. Hence

$$\overrightarrow{CD} = -bkh$$

and $\overrightarrow{DA}$ is zero since the velocity is perpendicular to DA. Finally the circulation is†

$$\begin{aligned}\Gamma &= \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}\\ &= -bkh.\end{aligned} \tag{2.10,5}$$

It is notable that the circulation is here proportional to the area of the circuit. This is always true for sufficiently small circuits lying in a given plane and enclosing a fixed point.

We shall now show that, when a given circuit is divided into a pair of circuits, the circulation in the given circuit is equal to the sum of the circulations in the pair of circuits. In Fig. 2.10,4 the points A and B on the given circuit are joined by an arbitrary curve, which need not be plane but must lie within the fluid. Let Γ be the circulation in the given circuit $ADBCA$ while Γ_1 and Γ_2 are the circulations in the circuits $ABCA$ and $ADBA$ respectively, where the positive sense of circulation is the same for all the circuits. Then we have by the definition of circulation

$$\begin{aligned}\Gamma_1 + \Gamma_2 &= (\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}) + (\overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{BA})\\ &= \overrightarrow{AD} + \overrightarrow{DB} + \overrightarrow{BC} + \overrightarrow{CA}\end{aligned}$$

---

† The measure of the circulation is usually denoted by the symbol Γ (capital Greek gamma).

by equation (2.10,3). But the expression on the right of the last equation is the circulation in the given circuit. Hence

$$\Gamma = \Gamma_1 + \Gamma_2. \tag{2.10,6}$$

Evidently we can now carry on the sub-division of the circuits as far as we please and we shall always have Γ equal to the sum of the circulations in the sub-circuits.

When the process of sub-division of a circuit is carried to the extreme we eventually arrive at sub-circuits whose dimensions are all infinitesimal. As a first case, let the original circuit lie in the plane OXY and let the sub-circuits be infinitesimal rectangles lying in this plane, of which $ABCD$ in Fig. 2.10,5 is typical. The mid-point P of this rectangle has coordinates x, y and the components of velocity in the plane at this point are u, v. The side AB is parallel to OX and of length dx; its normal distance from OX is $y - \frac{1}{2}\,dy$. Hence the mean value of u on AB is

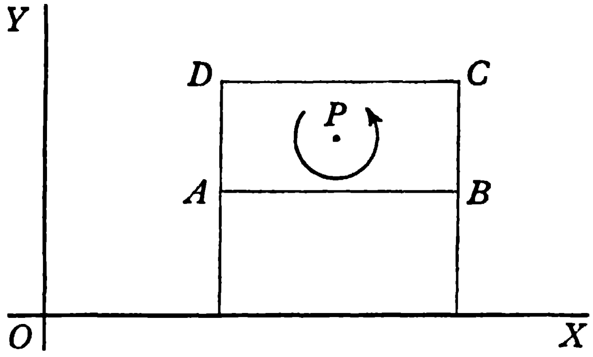

Fig. 2.10,5. Circulation in an elementary rectangle.

$$u - \frac{1}{2}\frac{\partial u}{\partial y}\,dy \qquad \text{and} \qquad \overrightarrow{AB} = \left(u - \frac{1}{2}\frac{\partial u}{\partial y}\,dy\right) dx.$$

Similarly the mean value of u on DC is $u + \dfrac{1}{2}\dfrac{\partial u}{\partial y}\,dy$

and
$$\overrightarrow{CD} = -\left(u + \frac{1}{2}\frac{\partial u}{\partial y}\,dy\right) dx.$$

Hence
$$\overrightarrow{AB} + \overrightarrow{CD} = -\frac{\partial u}{\partial y}\,dx\,dy.$$

By a similar argument we may show that

$$\overrightarrow{BC} + \overrightarrow{DA} = \frac{\partial v}{\partial x}\,dx\,dy$$

and we find that

$$\begin{aligned}\Gamma &= AB + BC + CD + DA \\ &= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy.\end{aligned} \tag{2.10,7}$$

The expression in the bracket is called the component of the *vorticity* of the fluid along the normal to the plane of the circuit $ABCDA$, i.e. OZ, and it is represented by the symbol ζ (zeta). When the motion is two-dimensional in the plane OXY this is *the* vorticity since the other components are zero (see § 2.11). We see that the circulation is proportional to the area of the circuit when it is infinitesimal and provided that ζ is not zero. This result is general since we can always take OZ to lie along the normal to any infinitesimal plane circuit. Let us take any circuit lying in the plane OXY and let Γ now be the circulation in this circuit. This is the sum of the circulations in the infinitesimal rectangular circuits into which we may suppose it to be divided and accordingly

$$\Gamma = \int\int \zeta \, dx \, dy \tag{2.10,8}$$

where

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{2.10,9}$$

This is a particular case of *Stokes' theorem* (see § 2.11).

EXAMPLES ON FLOW AND CIRCULATION

Example 1. *Circulation in a field of uniform shearing flow*

Suppose that $u = by$, $v = w = 0$.

Then by (2.10) $\zeta = -b$, and by (2.10,8) $\Gamma = -b\int\int dx\, dy$

$$= -b \times \text{(area enclosed by circuit)}.$$

The result for the circuit shown in Fig. 2.10,3 is a special case of this.

Example 2. *Flow along a radial line through a two-dimensional source*

The resultant velocity is radial and

$$q = \frac{m}{2\pi r}.$$

Hence

$$\overrightarrow{AB} = \int_a^b \frac{m\, dr}{2\pi r} = \frac{m}{2\pi} \ln \left(\frac{b}{a}\right)$$

where a and b are the distances of the points A and B respectively from the source O.

Now take a circuit $ABCDA$ where AD and CD are radial lines through O while BC and DA are circular arcs with O as centre. From the result just obtained

$$\overrightarrow{CD} = -\overrightarrow{AB} \quad \text{while} \quad \overrightarrow{BC} = \overrightarrow{DA} = 0.$$

Hence

$$\Gamma = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA} = 0.$$

If we select any point P in the field of flow we can describe a small circuit of the type here considered so as to enclose P and the circulation in this circuit is zero. Hence the vorticity is zero and the flow is of the type called irrotational (see § 2.12).

Example 3. *Vorticity in the "forced vortex"*

In the "forced vortex" of Rankine the fluid rotates about OZ as a rigid body with angular velocity ω. Hence the components of velocity are $u = -\omega y$, $v = \omega x$ and by (2.10,9)

$$\zeta = 2\omega.$$

This exemplifies the general theorem that the vorticity is twice the resultant angular velocity (spin) of the fluid.

Example 4. *General formulae for flow and circulation*

The direction cosines l_1, m_1, n_1 of the velocity are u/q, v/q, w/q respectively while the direction cosines l_2, m_2, n_2 of the tangent to a curve are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ respectively. Now the cosine of the angle between these directions is

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2\,.$$

Hence
$$q\cos\theta\, ds = u\,dx + v\,dy + w\,dz$$

and equation (2.10,1) becomes

$$\overrightarrow{AB} = \int_A^B (u\,dx + v\,dy + w\,dz). \qquad (2.10,10)$$

Therefore also the circulation in a circuit is

$$\Gamma = \oint (u\,dx + v\,dy + w\,dz), \qquad (2.10,11)$$

where $\oint$ indicates that a circuit integral is to be taken.

2.11 Vorticity and Vortices

We have already given the definition of vorticity for the special case of two-dimensional flow in § 2.10. In order to define this quantity in the general case of three-dimensional flow, let us consider the field of flow very near to a point P of fixed coordinates at a particular instant. Take a very small plane circuit, which for definiteness we may suppose to be a circle of very small fixed radius, with P as centre and let A be its area and PN the normal to its plane (see Fig. 2.11,1). The circulation Γ in this circuit depends on the direction of PN and there is a direction for which Γ reaches a numerical maximum Γ_m. Then the magnitude of the vorticity vector at P and at the instant considered is

$$\Omega = \frac{\Gamma_m}{} \tag{2.11,1}$$

and its direction is that of PN. The sense of the vector is related to that of the circulation by the corkscrew rule, i.e. the positive sense of the vorticity vector is in the direction of advance of a right-handed corkscrew which rotates in the direction of the circulation.†

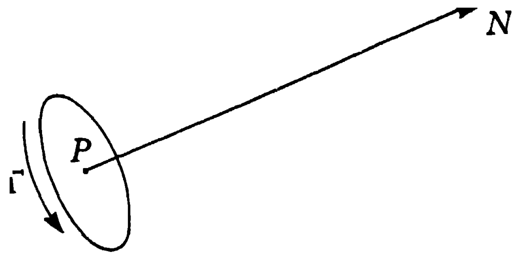

Fig. 2.11,1. Diagram to illustrate definition of the vorticity vector.

In order to find the three rectangular components ξ, η, ζ of the vorticity vector at P we take small circuits whose normals are parallel to OX, OY, OZ respectively. We have already seen in § 2.10 that the vorticity in the plane OXY, whose normal is OZ, is given by

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

By cyclic interchange of the symbols‡ we derive without further investigation that

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \quad \text{and} \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}.$$

Thus finally the vorticity has the components

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{2.11,2}$$

in the directions of the rectangular axes OX, OY, OZ respectively.¶

Having defined the vorticity we may then derive from it a *vortex line*; this is related to the vorticity vector in the same way that a streamline is related to the velocity vector. The formal definition is:—*a vortex line is a curve lying in the fluid such that its tangent at any point P is parallel to the vorticity at P at the instant considered.* For steady flow the vorticity at a point with fixed coordinates is constant, as follows from (2.11,2); hence a vortex line is then a fixed curve. As an example, for two-dimensional flow in the plane OXY the vortex lines are straight lines parallel to OZ. Next, take a simple closed curve lying in the fluid. Then the vortex lines through the points on this curve, all supposed drawn at the same instant, generate a tubular surface called a *vortex tube* or simply a *vortex*. A vortex tube whose cross-section is everywhere very small is often called a *vortex filament*. The *strength of a vortex* is defined to be the circulation in a circuit described on its surface and passing round the tube just once. It is proved below that the strength of a given vortex tube is the same everywhere at a given instant, so we are justified in speaking of *the* strength of the vortex. The strength of a vortex filament is equal to the product of its normal cross-sectional area and the measure of the vorticity.

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† This rule is established merely by convention.

‡ We suppose the symbols xyz to be arranged round a circle and always pass from one symbol to its neighbour in the same sense, thus $x \to y \to z \to x$. The velocities uvw are similarly taken in the cyclic order $u \to v \to w \to u$. Accordingly $\partial v/\partial x$, for example, becomes $\partial w/\partial y$ at the first step of the cyclic interchange and $\partial u/\partial z$ at the second. When we have set up a set of rectangular axes at a point, the *labelling* of the particular axes is a matter of choice. Thus, having established that the vorticity in the plane OXY is given by $\{(\partial v/\partial x) - (\partial u/\partial y)\}$, we may relabel the axes and derive without further investigation that the vorticity in the plane OYZ is given by $\{(\partial w/\partial y) - (\partial v/\partial z)\}$ while that in the plane OZX is $\{(\partial u/\partial z) - (\partial w/\partial x)\}$. This device of cyclic interchange is extremely useful and the reader should become familiar with it.

¶ It is proved below that vorticity is indeed a vector quantity.

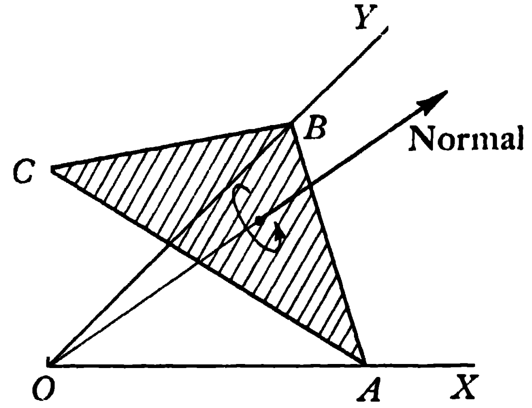

Fig. 2.11,2. Circulation in triangular circuit ABC.

We shall now prove that vorticity is a vector quantity. In the first place we note that the circulation round any very small plane area surrounding a point P and with a fixed normal PN is proportional to the measure of the area; this follows from the investigation given in § 2.10 since we may choose the axis OZ to be parallel to PN. In Fig. 2.11,2 let ABC be of area A and let the normal to its plane have direction cosines l, m, n. Then the area of the projection OAB on the plane OXY is nA, while the areas of OBC and of OCA are lA and mA respectively.† Let ξ, η, ζ be the measures of the vorticity, or circulation per unit area, in the planes OYZ, OZX, OXY respectively, whose normals are OX, OY, OZ respectively. Then the circulations round OAB, OBC, OCA are ζnA, ξlA, ηmA respectively. But

$$\overrightarrow{ABCA} = \overrightarrow{ABOA} + \overrightarrow{BCOB} + \overrightarrow{CAOC}$$

or

$$\Omega_{lmn} A = \zeta nA + \xi lA + \eta mA$$

where Ω_{lmn} is the vorticity, or circulation per unit area, in the plane ABC. Hence

$$\Omega_{lmn} = l\xi + m\eta + n\zeta. \tag{2.11,3}$$

---

† The cosine of the angle between the normal to the plane ABC and the normal OZ to the plane OXY is n, by definition.

Now this is the expression for the component in the direction defined by l, m, n of a vector whose rectangular components are ξ, η, ζ. Hence the vorticity is a vector quantity. The magnitude of the vorticity vector is

$$\Omega = \sqrt{(\xi^2 + \eta^2 + \zeta^2)} \tag{2.11,4}$$

and its direction cosines are

$$\lambda = \frac{\xi}{\Omega}, \qquad \mu = \frac{\eta}{\Omega}, \qquad \nu = \frac{\zeta}{\Omega}. \tag{2.11,5}$$

Now let us consider a surface bounded by a curve C; the surface has no other edges and is of finite area, so it has the form of a cap with C as edge. We may suppose the surface divided into small elements of area dS and the circulation round the element is

$$(l\xi + m\eta + n\zeta)\, dS$$

where l, m, n are the direction cosines of the normal to the element. The circulation Γ in C is the sum of the circulations in the elements and we therefore have

$$\Gamma = \int\!\!\int (l\xi + m\eta + n\zeta)\, dS. \tag{2.11,6}$$

When we substitute for Γ its expression as a circuit integral (see (2.10,11)) and use (2.11,2) this becomes

$$\oint (l_s u + m_s v + n_s w)\, dS = \int\!\!\int \left\{ l\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + m\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + n\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right\} ds, \tag{2.11,7}$$

where l_s, m_s, n_s are the direction cosines of the element of arc ds. This is Stokes' theorem in its general form. We can state this theorem in the following words: the circulation in a circuit C is equal to the surface integral of the normal component of vorticity taken over a finite surface lying wholly within the fluid and having C for its only edge. It should be understood that the circulation and vorticity refer to the same instant of time.

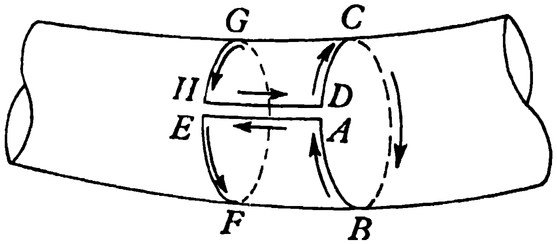

Fig. 2.11,3. Circuit described on surface of vortex tube.

With the help of Stokes' theorem we can now prove that the strength of a given vortex is the same everywhere. In Fig. 2.11,3 we have the circuit $AEFGHDCBA$ described on the surface of a vortex tube, where D and H are very near to A and E respectively. Since the surface of the tube is generated from vortex lines, it follows that the normal component of vorticity is everywhere zero. Then, by Stokes' theorem, the circulation in the circuit is zero. Now the circulation is

$$\overrightarrow{AE}+\overrightarrow{EFGH}+\overrightarrow{HD}+\overrightarrow{DCBA}=0.$$

When D tends towards A and H tends towards E we have in the limit

$$\overrightarrow{HD}=-\overrightarrow{AE}$$

and we obtain

$$\overrightarrow{EFGH}=-\overrightarrow{DCBA}=\overrightarrow{ABCD}$$

or, in the limit,

$$EFGE=ABCA.$$

Thus the circulation is the same, at the instant considered, in every simple circuit embracing the tube and described on its surface. The reader should note that we have *not* proved here that the circulation is independent of time; the time-dependence of the circulation can only be determined with the help of the dynamical equations.

Stokes' theorem can now be restated in the form: the circulation in any circuit which can be regarded as the edge of a finite surface lying wholly in the fluid and which has no other edge is equal to the algebraic sum of the strengths of the vortices which thread the circuit. For the vortices will intersect the surface in closed curves in each of which the circulation is equal to the strength of the vortex while outside the vortices the vorticity is zero and the contribution to the circulation is zero. This is exactly analogous to a theorem of electro-magnetism: the work done in carrying a unit magnetic pole round a closed circuit is equal to the algebraic sum of the currents threading the circuit, multiplied by a constant. There is, in fact an exact analogy between vortices and electric currents which will now be explained. An isolated vortex filament in an infinite and unbounded fluid has a definite field of velocity associated with it. Let the vortex filament be replaced by a thin wire occupying its site and carrying an electric current. Then the magnetic field vector has the same direction as the corresponding velocity vector and is a constant multiple of it. We shall return later to the consideration of the velocity field of a vortex.

We shall now show that a vortex cannot have an end *within* the fluid. Suppose that the vortex comes to an end as shown in Fig. 2.11,4. Let C be a circuit embracing the vortex and lying wholly in the fluid. With C as edge describe a surface S_1 intersecting the vortex in the circuit C_1. Then the circulation in C is equal to the circulation in C_1 and this is equal to Γ, the strength of the vortex. Let S_2 be a second surface having C as edge and lying wholly in the fluid but not intersecting the vortex. It then follows that the circulation in C is zero and we have arrived at a contradiction. Therefore the vortex cannot end in the fluid; thus a vortex is either a closed loop lying wholly in the fluid or has ends on the boundary of the fluid. In the case of two-dimensional motion, where the vortex lines are straight lines perpendicular to the plane of motion, we may suppose the fluid to be bounded by two infinitely distant planes upon which the vortices have their ends.

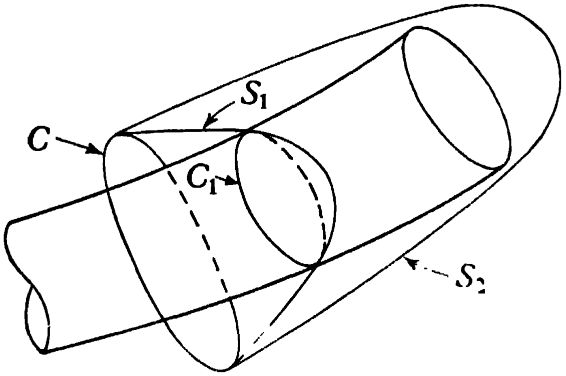

Fig. 2.11,4. Proof that a vortex cannot end in the fluid.

A given region in the fluid may be entirely free from vortices, in which case the flow is said to be *irrotational* (see § 2.12), or it may contain one or more isolated vortex filaments or, finally, it may be filled with vortices; an example of the last possibility is provided by the shearing motion in which $u = by$, $v = 0$ and $w = 0$. In many cases of technical importance the vorticity exists only in shallow layers of fluid, usually adjacent to the surface of an immersed body or in its wake. When the layer is exceedingly thin we have what is called a *vortex sheet*, which is, in effect, a surface within the fluid where there is a discontinuity of the tangential component of the velocity. In order to define and measure the strength of a vortex sheet (see Fig. 2.11,5) we take a small rectangular circuit of length *ds* and extremely small width in a plane normal to the sheet and enclosing an element of the sheet. If the circulation in this circuit is $d\Gamma$ the strength of the vortex sheet in the plane of the circuit is defined to be

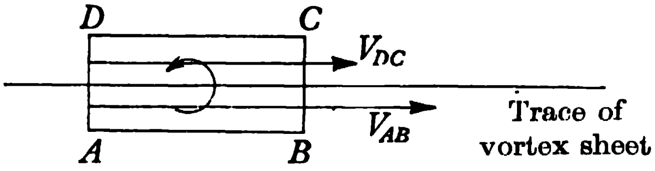

Fig. 2.11,5. Definition of strength of vortex sheet.

$$\sigma = \frac{d\Gamma}{ds} \tag{2.11,8}$$

$$= V_{AB} - V_{DC} \tag{2.11,9}$$

where V_{AB} and V_{DC} are the components of velocity of the fluid along *AB* and *DC* respectively. At a given point on the sheet there will be a certain direction of the normal to the circuit $ABCD$ for which σ reaches a maximum σ_m and this maximum is *the* strength of the sheet at the point. The direction of the normal, which is necessarily tangential to the sheet, is that of the axes of the vortices from which the sheet may be supposed to be built up. When the normal to the circuit makes an angle θ with the local axis of vorticity we have

$$\sigma = \sigma_m \cos \theta. \qquad (2.11,10)$$

Example 1. *Vorticity in a combined motion*

It follows at once from equations (2.11,2) that when two fields of flow are superposed the vorticity vector for the combined motion is the vector resultant of the individual vorticity vectors. When the motion added is irrotational the vorticity is unaltered.

Example 2. *The vorticity is a solenoidal vector*

We find at once from equations (2.11,2) that

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0, \quad \text{or div } \boldsymbol{\omega} = 0$$

where $\boldsymbol{\omega} = (\xi, \eta, \zeta)$ is the vorticity vector. By definition, a vector having components satisfying this equation is *solenoidal*; it has zero divergence. The velocity of a uniform and incompressible fluid is also solenoidal (see equation (2.3,9)).

The vorticity is the "curl" of the velocity. Let the vector $\mathbf{V}$ have the rectangular components v_x, v_y and v_z. Then curl $\mathbf{V}$ is defined as the vector whose rectangular components are

$$\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \quad \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}, \quad \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}.$$

also we have

$$\text{div curl } \mathbf{V} = \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) = 0.$$

Hence the curl of any vector has zero divergence, i.e. is solenoidal.

Example 3. *Hill's spherical vortex*

The components of velocity in the region within the fixed sphere

$$x^2 + y^2 + z^2 = a^2$$

are given by

$$u = A(x^2 + 2y^2 + 2z^2 - a^2), \qquad v = -Axy, \qquad w = -Axz.$$

The reader may verify that these expressions satisfy the equation of continuity of an incompressible fluid and that the radial component of the velocity at the spherical surface is zero, so the motion is kinematically possible inside the fixed sphere.

By equations (2.11,2) we find that the vorticity has the components

$$\xi = 0, \qquad \eta = 5Az, \qquad \zeta = -5Ay.$$

The differential equations of a vortex line are

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}$$

or

$$\frac{dx}{0} = \frac{dy}{5Az} = -\frac{dz}{5Ay}.$$

Hence $dx = 0$ or $x =$ constant, while the last equation of the set yields

$$y\,dy + z\,dz = 0.$$

Therefore $y^2 + z^2 =$ constant.

Thus the vortex lines are all circles whose centres lie on OX and whose planes are perpendicular to OX. (See Example 3 of § 3.5 for the distribution of pressure in this vortex.)

Example 4. Velocity induced by a uniform vortex sheet in the form of a circular cylinder

Suppose that the fluid inside a cylinder of radius a is at rest, while outside the velocity has constant circulation Γ about the cylinder. Just outside the cylinder the velocity is $\Gamma/2\pi a$ and this is the magnitude of the discontinuity of velocity and the strength of the vortex sheet. The postulated state of velocity can be no other than that induced by the vortex sheet. This can be verified by integrating the effects of the individual vortices in the sheet, but the work is relatively difficult.

2.12 Irrotational Flow and the Velocity Potential

The flow in a region is said to be *irrotational* when the three components of the vorticity are all zero throughout the region. By equations (2.11,2) we therefore have

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \qquad \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \qquad \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}. \tag{2.12,1}$$

These equations show that the three components of velocity are the partial differential coefficients of a function of x, y and z. We shall write†

$$u = -\frac{\partial \phi}{\partial x}, \qquad v = -\frac{\partial \phi}{\partial y}, \qquad w = -\frac{\partial \phi}{\partial z} \tag{2.12,2}$$

---

† The negative sign is introduced by a mere convention and by analogy with the gravitational potential. Some writers omit the negative sign.

where the function ϕ is called the *velocity potential.* The first of equations (2.12,1) is satisfied since

$$\frac{\partial^2\phi}{\partial y\,\partial x} = \frac{\partial^2\phi}{\partial x\,\partial y}$$

and the others are similarly satisfied. Whenever the flow is irrotational the complete state of velocity is given in terms of a *single* function of position, instead of three independent functions and there is a corresponding simplification of all the mathematical relationships. For steady motion ϕ is independent of time, but otherwise it depends on the four variables x, y, z and t.

Let u_s be the component of the fluid velocity in the direction of an element of length ds with direction cosines l, m, n. Then

$$\begin{aligned} u_s = lu + mv + nw &= -\left(l\frac{\partial\phi}{\partial x} + m\frac{\partial\phi}{\partial y} + n\frac{\partial\phi}{\partial z}\right) \\ &= -\frac{\partial\phi}{\partial s}. \end{aligned} \tag{2.12,3}$$

Now let ds lie in a surface on which ϕ is constant, i.e. an *equipotential surface.* Then $\partial\phi/\partial s$ is zero and we see that the component of velocity along any tangent to the equipotential surface is zero. This implies that the resultant velocity at any point P is normal to the equipotential surface through P at P. In other words, *the streamlines cut the equipotentials orthogonally.*

By equation (2.10,1) the flow from A to B along a given curve is

$$\overrightarrow{AB} = \int_A^B q\cos\theta\,ds = \int_A^B (u\,dx + v\,dy + w\,dz)$$

by Example 4 of § 2.10. When we use equation (2.12,2) we obtain

$$\begin{aligned} \overrightarrow{AB} &= -\int_A^B \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) \\ &= -\int_A^B d\phi = -(\phi_B - \phi_A). \end{aligned} \tag{2.12,4}$$

Provided that, at the instant considered, ϕ is a *single valued* function of position the flow from A to B will be the same for all paths joining these points and lying wholly within the fluid. The potential will certainly be single valued except in the following circumstances:

(a) When the fluid contains vortices embedded in the region of irrotational flow.

(b) When the region occupied by the fluid is not simply connected.

These cases are separately discussed below.

Suppose that

$$\overrightarrow{ACB} = M \qquad \text{and} \qquad \overrightarrow{ADB} = N.$$

Then
$$\overrightarrow{ACBDA} = \overrightarrow{ACB} - \overrightarrow{ADB} = M - N,$$

and the circulation in the circuit $ACBDA$ is not zero when M and N are unequal. Now suppose that $ACBDA$ can be regarded as the edge of a cap-shaped surface lying wholly in the fluid. Then we know by Stokes' theorem (see § 2.11) that the circulation in this circuit must be zero when no vortices thread the circuit; hence $M = N$ and the increment of the velocity potential in passing from A to B is the same for the paths ACB and ADB. When *all* circuits which can be described within the fluid can be reduced to a mere point by continuous variation of their points and without leaving the fluid, the space occupied by the fluid is called *simply connected.* For a simply connected space and in the absence of all vortices, the velocity potential must be single valued. However, when the space occupied by the fluid is *multiply connected,* the velocity potential may be many valued even in the absence of all vortices. The space is multiply connected, for example, when there is a rigid ring lying within the fluid. For, any circuit not threading the ring is *reducible* in the sense already explained, whereas one which threads the ring is *irreducible.*

Suppose now that the fluid is incompressible. On substitution from equations (2.12,2) in the equation of continuity (2.3,9) the latter becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \tag{2.12,5}$$

This is known as *Laplace's equation* and it is often written in the abbreviated form

$$\nabla^2\phi = 0 \tag{2.12,6}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{2.12,7}$$

is a second order differential operator. The gravitational potential in empty space also satisfies equation (2.12,6). Any function which satisfies Laplace's equation is called a *harmonic function.* Hence we see that the velocity potential of an incompressible fluid is necessarily a harmonic function.

For two-dimensional flow in the plane OXY the velocity potential is independent of z and the equation of continuity for an incompressible fluid becomes

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{2.12,8}$$

which is also commonly written in the abbreviated form (2.12,6), it being understood that the operator is now two-dimensional. Since the flow is irrotational we have

$$\zeta = 0$$

and on substitution from (2.4,3) and (2.11,2) this becomes

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{2.12,9}$$

Thus, both the velocity potential and the stream function are here harmonic functions; although they satisfy the same differential equation they are necessarily distinct.

EXAMPLES ON THE VELOCITY POTENTIAL

Example 1. *Addition of velocity potentials*

Let $$\phi_3 = \phi_1 + \phi_2.$$

Then it follows from equations (2.12,2) that

$$u_3 = u_1 + u_2, \qquad v_3 = v_1 + v_2, \qquad w_3 = w_1 + w_2.$$

Hence, when velocity potentials are added, the corresponding motions are added vectorially. As we have seen, the same is true when two stream functions are added.

Example 2. *Velocity potentials of sources and sinks*

For a two-dimensional source of strength m situated at the origin the components of velocity are (see § 2.7)

$$u = \frac{mx}{2\pi r^2}, \quad v = \frac{my}{2\pi r^2}$$

where $r^2 = x^2 + y^2$. Hence

$$\phi = -\frac{m}{2\pi} \ln r$$

since $$\frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r}.$$

For a three-dimensional source of strength σ we have (see § 2.7)

$$u = \frac{\sigma x}{4\pi r^3}, \quad v = \frac{\sigma y}{4\pi r^3}, \quad w = \frac{\sigma z}{4\pi r^3}.$$

Therefore $\phi = \dfrac{\sigma}{4\pi r}$ where now $r^2 = x^2 + y^2 + z^2$.

The fact that the velocity is derivable from a potential shows that the motion is irrotational.

Example 3. *Velocity potentials of doublets*

We may derive the expressions for the velocity potentials of doublets from those for the velocities given in § 2.7 or we may obtain them by differentiation of the potentials of sources (see § 2.7 where the stream function for a two-dimensional doublet is obtained by a similar method).

We find for a two-dimensional doublet of strength μ' at the origin and with its axis in the positive direction of OX that

$$\phi = -\mu' \frac{\partial}{\partial x}\left(-\frac{1}{2\pi}\ln r\right) = \frac{\mu'}{2\pi}\frac{x}{r^2} = \frac{\mu' \cos\theta}{2\pi r}.$$

For the corresponding three-dimensional doublet

$$\phi = -\mu \frac{\partial}{\partial x}\left(\frac{1}{4\pi r}\right) = \frac{\mu x}{4\pi r^3}.$$

The reader should verify that the expressions for the components of velocity derived from these potentials agree with those given in equations (2.7,21), (2.7,22) and (2.7,27) . . . (2.7,29).

Let θ be the angle between OP and the axis OX of the three-dimensional doublet. Then the value of the velocity potential at P is

$$\phi = \frac{\mu \cos\theta}{4\pi r^2}.$$

2.13 The Velocity Field of an Isolated Vortex

A given isolated vortex filament situated in an unbounded fluid has a definite velocity field associated with it. The velocity at a point P is usually known as the *induced velocity* at P. It should be understood that the fluid is at rest except in so far as it is disturbed by the vortex; in particular, the velocity tends to zero at great distances from the vortex.

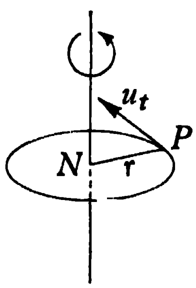

Fig. 2.13,1. Velocity induced by infinite straight vortex.

We begin by considering an infinitely long straight vortex filament of strength Γ (see Fig. 2.13,1). As proved in § 2.11, the strength is the same at all points on the vortex and there is, consequently, nothing to distinguish one point on the vortex from another. Hence the flow must be the same in all planes perpendicular to the vortex. The flow will, in fact, be two-dimensional in such planes, on account of the complete symmetry of the system about any such plane. Again by symmetry, the radial and tangential components of the velocity at a point P can depend only on the normal distance $r = PN$ of the point from the vortex. But the radial component must be zero for, if not, we should have a constant radial flow on a concentric cylinder of radius r, implying the existence of a line source at the vortex. Hence the resultant velocity at P is tangential, i.e. perpendicular to the plane containing P and the vortex. Let u_t be the magnitude of this velocity. Then the circulation in the circle through P with its centre at N is $2\pi r u_t$, since u_t is constant on the circle. But (see § 2.11) the circulation in any circuit embracing the vortex is equal to its strength Γ. Hence we find that

$$u_t = \frac{\Gamma}{2\pi r}. \tag{2.13,1}$$

This result is so important that we shall give it in words, thus: the velocity induced by an infinite straight vortex at a point situated at a perpendicular distance r from the vortex is normal to the plane containing the vortex and the point while its magnitude is equal to the strength of the vortex divided by the circumference of a circle of radius r. The sense of the velocity agrees with that of the spin of the vortex. A real vortex cannot be of infinitesimal sectional area, so the expression (2.13,1) is only applicable at points such that r is greater than the radius of the vortex. We may, if we please, suppose that the vortex is of radius a, and that the fluid within rotates with constant angular velocity ω. The circulation at the radius a is then

$$\omega a \times 2\pi a = \Gamma$$

or

$$\omega = \frac{\Gamma}{2\pi a^2}. \tag{2.13,2}$$

This result can also be obtained from the consideration that the vorticity $\zeta = 2\omega$. Hence the circulation is

$$\Gamma = \text{area} \times \zeta = 2\pi\omega a^2.$$

The smaller the core radius a, the higher must be the rate of spin in the core for a given Γ. Rankine called the system just discussed the *combined vortex* and the irrotational motion outside, where equation (2.13,1) is valid, the *free vortex*. The core he called the *forced vortex* (see also § 1.7).

We shall now obtain the expressions for the stream function and velocity potential of the motion induced by the vortex and corresponding to the expression (2.13,1) for the velocity. For convenience we shall take the stream function to be zero at unit distance from the vortex, which lies along OZ. Take A on OX at unit distance from O and let B be the point where a circle through a point P and with centre O cuts OX to the right of O. Then
flux across AP = flux across AB + flux across arc BP
= flux across AB since the arc BP is a streamline. Hence

$$\psi_v = \text{flux across } AB \quad \frac{\Gamma}{2\pi}\int_1^r \frac{dx}{x}$$

$$= \frac{\Gamma}{2\pi}\ln r \tag{2.13,3}$$

where $OP = r$. This expression yields

$$u = -\frac{\partial\psi}{\partial y} = -\frac{\Gamma y}{2\pi r^2} \tag{2.13,4}$$

$$v = \frac{\partial\psi}{\partial x} = \frac{\Gamma x}{2\pi r^2}. \tag{2.13,5}$$

The resultant of these is perpendicular to OP and equal to u_t as given by (2.13,1). We easily find from the last equations that the velocity potential is

$$\begin{aligned}\phi &= -\frac{\Gamma}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)\\ &= -\frac{\Gamma\theta}{2\pi}.\end{aligned} \tag{2.13,6}$$

On comparison with equation (2.7,8) we see that this is the same as the stream function for a line source of strength Γ (see further in § 2.14).

Next, let us consider a *vortex pair* consisting of a vortex of strength Γ at the point $(h, 0)$ and a vortex of strength $-\Gamma$ at $(-h, 0)$.† Let r_1, θ_1 and r_2, θ_2 be polar coordinates with the centres of these vortices as origins. Then the potential is

$$\phi = \frac{\Gamma}{2\pi}(\theta_2 - \theta_1) \tag{2.13,7}$$

and the equipotentials are therefore circles through the centres of the vortices (compare Example 1 of § 2.7). The stream function is by (2.13,3)

$$\psi = \frac{\Gamma}{2\pi}\ln\left(\frac{r_1}{r_2}\right). \tag{2.13,8}$$

Now the locus of a point whose distances from fixed poles are in a constant ratio is a circle. The streamlines are in fact a family of coaxial circles. The equipotentials and streamlines are shown in Fig. 2.13,2.

Next, let the separation $2h$ of the vortices tend to zero while $2h\Gamma$ has the constant value μ'. By the argument used in § 2.7 in dealing with source-sink doublets we find that the stream function for the two-dimensonal vortex doublet of strength μ' is

$$\begin{aligned}\psi &= -\frac{\mu'}{2\pi}\frac{\partial}{\partial x}\ln r = -\frac{\mu'}{4\pi}\frac{\partial}{\partial x}\ln(x^2 + y^2)\\ &- \frac{\mu' x}{2\pi(x^2 + y^2)}\end{aligned} \tag{2.13,9}$$

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† The reversal of sign indicates reversal of the direction of spin.

while the velocity potential is

$$\phi = \frac{\mu'}{2\pi}\frac{\partial}{\partial x}\tan^{-1}\frac{y}{x}$$

$$= \frac{-\mu' y}{2\pi(x^2+y^2)}. \tag{2.13,10}$$

This vortex doublet has its axis (i.e. the line joining the vortices) along *OX* and by comparison of equations (2.7,20) and (2.13,9) we see that the stream function of the vortex doublet is identical with that of a two-dimensional source-sink doublet of strength μ' whose axis lies along *OY* in the negative sense. The vortex doublet is, in fact, equivalent to a source-sink doublet of equal strength whose axis is inclined at $-90°$ to that of the vortex doublet.

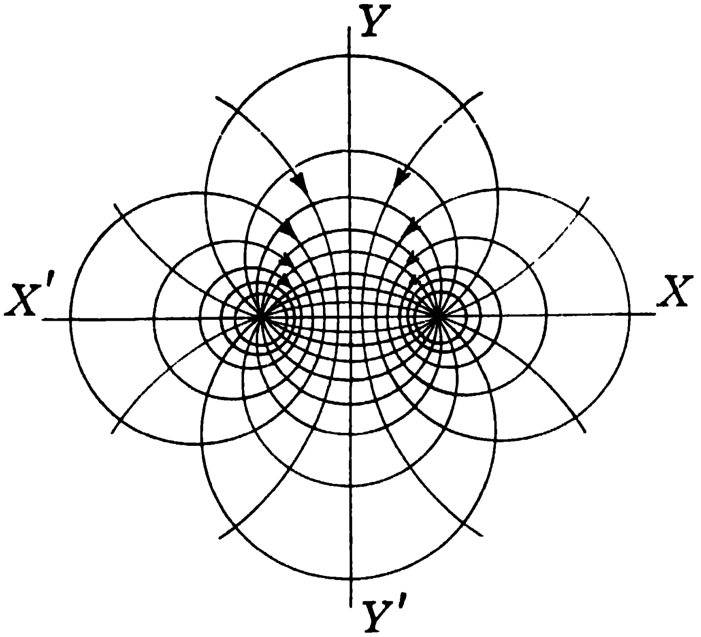

Fig. 2.13,2. Streamlines and equipotentials for vortex pair.

We can suppose the plane connecting the vortices of a vortex pair filled with vortex pairs in such a manner that the negative vortex of any pair coincides with the positive vortex of its neighbour; all the vortices then cancel each other with the exception of the original ones at the ends. The pair of vortices of strength $\pm\Gamma$ at a distance apart dx are equivalent to a vortex doublet of strength $\Gamma\,dx$, i.e. the strength of the vortex doublets per unit length is Γ. As we have just seen, this is equivalent to a sheet of source-sink doublets with their axes parallel to *OY'* and of strength Γ per unit length. Thus we have reached the important conclusion that the vortex pair is equivalent to a sheet of two-dimensional source-sink doublets connecting the vortices of the pair and of strength Γ per unit length. The argument also shows that this sheet need not be plane, but it must have the constant strength Γ per unit of arc and have its edges at the vortices of the pair while the axes of the doublets are always normal to the arc. We can go further, for a two-dimensional source-sink doublet is in fact a line doublet which may be regarded as constructed from point doublets. Hence the two-dimensional doublet of strength Γ per unit of arc is equivalent to an infinitely long strip covered with point doublets of strength Γ per unit area. Thus, finally, the vortex pair is equivalent to a sheet of source-sink point doublets of strength Γ per unit area covering a cylindrical surface whose edges are the vortices of the pair and the axes of the doublets are at −90° to the surface. We shall use the term doublet-sheet to describe a surface covered with source-sink doublets whose axes are normal to the surface. In the present case the doublet sheet is cylindrical and uniform.

Next let us consider a vortex of strength Γ in the form of a closed ring of any shape (the ring need not be a plane curve). Take a surface which has the ring as edge and cover this with a net of curves so that the surface is cut up into small areas. Let the boundary of each mesh of the net be a vortex ring of strength Γ. Where these vortices meet their neighbours they cancel one another since their directions of spin are opposed where they meet and it is only at the edge of the surface that cancellation does not occur. Hence the whole system of small vortex rings is exactly equivalent to the original vortex ring. Now a very small vortex ring of area dA and strength Γ is equivalent to a source-sink doublet† whose axis is normal to the ring such that the vortex ring is clockwise about it and whose strength is $\Gamma\, dA$. Therefore the vortex ring is equivalent to a uniform doublet sheet whose edge is the vortex and whose strength per unit area is Γ. This is the generalization of the result obtained for the vortex pair which may be regarded as a rectangular ring with its closing ends at infinity.

We shall now obtain the velocity potential ϕ for the ring vortex. First take the elementary ring which is equivalent to a doublet of strength $\Gamma\, dA$. Let P be the point where ϕ is to be found. Then we know from Example 3 of § 2.12 that the contribution due to the small ring is

$$d\phi = \frac{\Gamma\, dA \cos\theta}{4\pi r^2} \qquad (2.13,11)$$

where θ is the angle between the axis of the doublet and the line joining it to P. Now describe a sphere of unit radius with P as centre and let the cone whose vertex is P and which passes through the ring cut an area $d\omega$ from the sphere; this is the element of solid angle subtended at P by the elementary ring. The projection of dA on a plane perpendicular to the line joining it to P is $dA \cos\theta$ and we therefore have

$$d\omega = \frac{dA \cos\theta}{r^2}.\ddagger \qquad (2.13,12)$$

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† It is clear that the vortex ring and the doublet both give rise locally to a jet of fluid.

‡ Note $d\omega < 0$ if vortex is anticlockwise about radius vector from ring to $P(\theta > \pi/2)$.

Hence (2.13,11) becomes

$$d\phi = \frac{\Gamma\, d\omega}{4\pi}$$

and by integration

$$\phi = \frac{\Gamma\omega}{4\pi} \tag{2.13,13}$$

where ω is the solid angle subtended at P by the ring. Let P now describe a circuit which threads the ring once. Then the total increment of the solid angle is numerically equal to the area of the unit sphere, namely 4π, and the increment of the velocity potential is accordingly $\pm\Gamma$. But if the circuit described by P does not thread the ring the increment of solid angle is zero. These results show that the circulation in any simple circuit which threads the ring is $\pm\Gamma$, where the sign depends on the sense of the displacement of P, while the circulation in any circuit not threading the ring is zero. These results confirm the formula (2.13,13). In any given case we could use equation (2.13,13) to find the velocity potential and then derive the components of velocity by differentiation. It can be shown† that the same result is obtained in the following manner: Take an element QQ' of the vortex of length ds and let r be the distance QP while θ_r is the angle between QQ' and QP. Then the velocity induced at P by the element is perpendicular to the plane containing QQ' and QP and its magnitude is

$$dV = \frac{\Gamma\, ds \sin\theta_r}{4\pi r^2}. \tag{2.13,14}$$

The induced velocity is then obtained by summing vectorially the contributions of the elements. However, it should be remembered that an *isolated* element of a vortex cannot exist and it is only the *resultant* induced velocity which is significant. There is a formula corresponding to (2.13,14) for the magnetic field (measured in gauss) due to an element ds of a thin wire carrying a current I, in amperes, namely

$$dF = \frac{I\, ds \sin\theta_r}{10r^2} \tag{2.13,15}$$

where the medium has unit permeability and the number 10 in the denominator arises from the fact that the ampere is one tenth of the electromagnetic unit of current. Equations (2.13,14) and (2.13,15) express completely the analogy between vortices and electric currents already alluded to in § 2.11.

Historical Note. The theory of vortices, which is of fundamental importance in the mechanics of fluids, was first given by Helmholtz in 1858. His work included the kinematic theory and the very remarkable dynamical theory which is briefly discussed in Chapter 3 of this book. The concept of circulation was introduced by Kelvin in 1869 and he made this the basis for recasting the whole theory of vortices. Kelvin's treatment is now generally followed as it helps the student to understand the physical basis of the theory.

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† See Lamb's *Hydrodynamics*, Chapter VII. The reader should consult this book for a complete and rigorous treatment of vortex motion.

EXAMPLES ON THE VELOCITY INDUCED BY A VORTEX

Example 1. *Velocity potential of a vortex pair*

The expression for the potential has been given in equation (2.13,7) but we shall now obtain the same result from the general formula (2.13,13) for ϕ in terms of the solid angle. We can regard the vortex pair as a rectangular vortex ring whose closing sides are at infinity. Hence the vortex is the edge of a plane strip of width $2h$ and infinite length. The solid angle ω subtended at any point P is the area of a complete lune of a unit sphere whose centre is P, i.e. it is twice the plane angle at P between planes through P and the two vortices. Call this angle β. Then

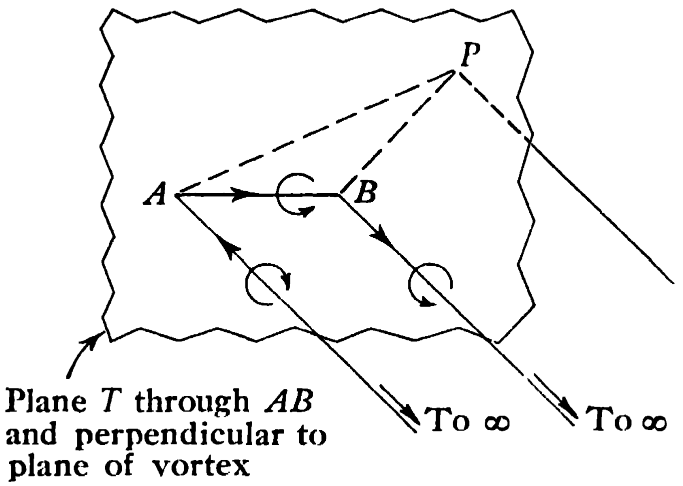

Fig. 2.13,3. 'Horse-shoe' vortex.

$$\omega = 2\beta \qquad \text{and} \qquad \phi = \frac{\Gamma\beta}{2\pi}$$

in agreement with equation (2.13,7).

Example 2. *Velocity induced by a "horse-shoe vortex"*

The name "horse-shoe vortex" is applied (rather absurdly) to a rectangular vortex, *one* of whose closing sides is at infinity (see Fig. 2.13,3). We can at once establish the following important fact: the induced velocity in a plane T through AB and perpendicular to the plane of the vortex is half that induced at the same point by a vortex pair through A and B. For the solid angle subtended at any point P in the plane T is clearly just half that subtended by the vortex pair. Hence the velocity potential at P is half that for the vortex pair and therefore the velocity *in the plane T* is half that due to the vortex pair. There is also, however, a component of the induced velocity perpendicular to the plane T. See further Example 3 below.

Example 3. *Velocity induced by a vortex composed of straight segments*

We shall obtain the contribution of one straight segment as given by equation (2.13,14). The resultant induced velocity is then obtained by vector summation; due attention must be paid to the directions and senses of the contributions. In Fig. 2.13,4 AB is a segment of the vortex and P the point at which the velocity is to be found. We take OX to lie along AB while the plane OXY passes through P, which has the coordinates x, y. The point Q lies within AB and $OQ = s$. From the figure we have

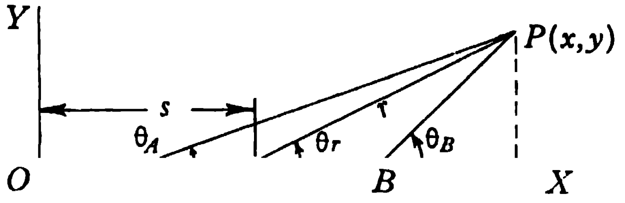

Fig. 2.13,4. Velocity induced by a straight vortex segment.

$$s = x - y \cot \theta_r$$

so $\qquad ds = y \operatorname{cosec}^2 \theta_r \, d\theta_r \qquad$ while $\qquad r = QP = y \operatorname{cosec} \theta_r.$

Hence the contribution to the induced velocity at P is normal to the plane ABP and is given by

$$V_{AB} = \frac{\Gamma}{4\pi} \int_{\theta_A}^{\theta_B} \frac{\sin \theta_r \dfrac{ds}{d\theta_r} \, d\theta_r}{r^2} = \frac{\Gamma}{4\pi y} \int_{\theta_A}^{\theta_B} \sin \theta_r \, d\theta_r$$

$$= \frac{\Gamma}{4\pi y} (\cos \theta_A - \cos \theta_B) = \frac{\Gamma}{4\pi y} (\cos PAB + \cos PBA).$$

The following special cases are important:

(1) P lies on the perpendicular to AB at A and B is at infinity. Then

$$V_{AB} = \frac{\Gamma}{4\pi y}.$$

Compare Example 2 above.

(2) The vortex extends from infinity to infinity; this is the infinite straight vortex considered at the beginning of § 2.13. The general formula gives

$$V_{AB} = \frac{\Gamma}{2\pi y} \qquad .$$

in agreement with equation (2.13,1).

Example 4. Velocity induced by a circular vortex at a point on its axis

We shall obtain the velocity by use of the velocity potential and from equation (2.13,14). The ring (see Fig. 2.13,5) is of radius a, its centre is the origin O and the normal to the plane of the ring is OX about which the vortex is clockwise. The induced velocity is to be found at P on OX.

The solid angle ω at P is cut from the unit sphere by a right circular cone of semi-vertical angle θ. Hence

$$\omega = 2\pi(1 - \cos\theta) = 4\pi \sin^2 \frac{\theta}{2}.$$

By equation (2.13,13) the velocity potential at any point on OX is

$$\phi = \frac{\Gamma}{2}(1 - \cos\theta) = \frac{\Gamma}{2}\left(1 - \frac{x}{\sqrt{(a^2 + x^2)}}\right).$$

The component of velocity u in the direction OX at P is therefore

$$u = -\frac{\partial\phi}{\partial x} = \frac{\Gamma a^2}{2(a^2 + x^2)^{3/2}}.$$

This is, in fact, the resultant velocity at P for, by symmetry, the component perpendicular to OX is zero. At the centre of the ring the velocity is

$$u_0 = \frac{\Gamma}{2a}.$$

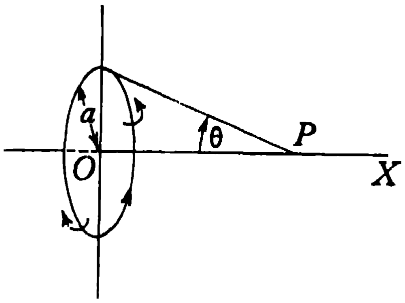

Fig. 2.13,5. Velocity induced by a circular vortex ring at a point on its axis.

Next, apply equation (2.13,14) and take an element of the ring of length ds at Q. The velocity induced at P is perpendicular to QP and $\theta_r = \frac{\pi}{2}$, for ds is also perpendicular to QP. Hence

$$dV = \frac{\Gamma\, ds}{4\pi(a^2 + x^2)}$$

and the component of this along OX in the positive sense is

$$dV \sin\theta = \frac{\Gamma a\, ds}{4\pi(a^2 + x^2)^{3/2}}.$$

Therefore the total induced velocity along OX at P is

$$u = \frac{\Gamma a^2}{2(a^2 + x^2)^{3/2}}$$

since the circumference of the ring is $2\pi a$. We see that the two methods are concordant.

If P were off the axis of the ring, the cone with the ring as base would be oblique and would cut a unit sphere at P in a curve called a sphero-conic. The measure of the solid angle is the area of this sphero-conic and this requires elliptic integrals for its expression.

2.14 The Complex Potential

In this place we are solely concerned with the *irrotational* motion of an *incompressible* fluid in two dimensions. Since the motion is irrotational a velocity potential ϕ exists and, since the motion is two-dimensional and the fluid incompressible, a stream function ψ also exists. Let the motion be in the plane OXY. Then by equations (2.4,3) and (2.12,3) the components of velocity are given by

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x} \tag{2.14,1}$$

$$v = \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} \tag{2.14,2}$$

Hence the functions ϕ and ψ are connected by the equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{2.14,3}$$

Functions related as in these last equations are said to be *conjugate*. Conjugate functions are necessarily harmonic for we have

$$\left.\begin{aligned} &\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \\ \text{and}\quad &\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x} = 0. \end{aligned}\right\} \tag{2.14,4}$$

The last equations are in conformity with (2.12,8) and (2.12,9). Another immediate deduction from (2.14,3) is that the families of curves ϕ = constant and ψ = constant cut at right angles everywhere, i.e. they are *orthogonal trajectories*. In accordance with this, we have already seen in § 2.12 that the streamlines, on which ψ is constant, cut the equipotentials, on which ϕ is constant, at right angles. We shall proceed to show that the conjugate functions ϕ and ψ are the real and imaginary parts of a function of the complex variable

$$z = x + iy \tag{2.14,5}$$

This function is called the *complex potential.*

It is clear from equations (2.14,4) that the solution of a problem in two-dimensional irrotational flow depends, in the first place, on obtaining a suitable solution of Laplace's equation. We shall obtain the general solution of the (two-dimensional) form of this equation and shall approach this by considering first the allied differential equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 f}{\partial t^2} \tag{2.14,6}$$

where a is a real constant. This is called *the equation of wave propagation in one dimension*; in the physical interpretation of the equation x is a spatial coordinate and t is the time. Let

$$f = F_1(x - at) \tag{2.14,7}$$

where the function F_1 is arbitrary, except that it can be differentiated twice with respect to its variable. Then we have

$$\frac{\partial^2 f}{\partial x^2} = F_1''(x - at) \qquad \text{and} \qquad \frac{\partial^2 f}{\partial t^2} = a^2 F_1''(x - at),$$

so f is a solution of (2.14,6). In the same way we can show that

$$f = F_2(x + at) \tag{2.14,8}$$

is a solution of (2.14,6), where F_2 is another arbitrary function. It can be shown that the most general solution of (2.14,6) is

$$f = F_1(x - at) + F_2(x + at) \tag{2.14,9}$$

This is d'Alembert's solution of the equation of wave propagation in one dimension.

Now substitute $i = \sqrt{-1}$ for a in equation (2.14,6) and write y in place of t. Equation (2.14,6) becomes Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial^2 y} = 0 \tag{2.14,10}$$

while the general solution (2.14,9) becomes

$$f = F_1(x - iy) + F_2(x + iy) \tag{2.14,11}$$

For our purpose it will suffice to take only a function of $(x + iy)$ and we shall put

$$w = F(x + iy) = F(z) \tag{2.14,12}$$

Now $F(x + iy)$ will have a real part and a pure imaginary part so we shall write

$$w = f_1(x, y) + i f_2(x, y) \tag{2.14,13}$$

and we shall show that f_1 and f_2 are necessarily conjugate functions.
For

$$\frac{\partial w}{\partial x} = F'(x + iy) = \frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x} \tag{2.14,14}$$

while

$$\frac{\partial w}{\partial y} = iF'(x + iy) = \frac{\partial f_1}{\partial y} + i\frac{\partial f_2}{\partial y} \tag{2.14,15}$$

Hence

$$\frac{\partial f_1}{\partial y} + i\frac{\partial f_2}{\partial y} = i\left(\frac{\partial f_1}{\partial x} + i\frac{\partial f_2}{\partial x}\right)$$

and when we equate real and imaginary parts we obtain

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}, \qquad \text{and} \qquad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x} \tag{2.14,16}$$

Thus, in accordance with the definition already given, f_1 and f_2 are conjugate functions and, as already proved, they are therefore both harmonic functions. On comparison of equations (2.14,3) and (2.14,16) it is evident that we can put

$$\phi = f_1, \qquad \psi = f_2 \tag{2.14,17}$$

and then (2.14,13) becomes

$$w = \phi + i\psi \tag{2.14,18}$$

The function w is the *complex potential* and, by (2.14,12), it is a function of the single complex variable z (see (2.14,5)). We find from the last equation by differentiation that

$$\frac{\partial w}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = -u + iv \tag{2.14,19}$$

But

$$\frac{\partial w}{\partial x} = \frac{dw}{dz} \tag{2.14,20}$$

for the differential coefficient $\frac{dw}{dz}$ has the same value for *all* infinitesimal values of dz and (2.14,20) follows when we identify dz with dx. Hence finally

$$\frac{dw}{dz} = -u + iv \tag{2.14,21}$$

We see that we can obtain both components of the velocity directly from the complex potential by differentiation but the reader should observe that the complex number on the right of (2.14,21) is *not* equal to the velocity vector on account of the presence of the negative sign attached to u.

The convenience of the complex potential will be shown in the examples which follow. Before dealing with these we point out the following facts:

(1) The velocity associated with the potential $(w_1 + w_2)$ is the vector resultant of the velocities associated with w_1 and w_2 separately.

(2) The velocity vector associated with aw, where a is a *real* constant, is a times that associated with w.

(3) When $w' = iw$, we have

$$\phi' + i\psi' = i(\phi + i\psi).$$

Hence $$\phi' = -\psi \quad \text{and} \quad \psi' = \phi.$$

The equipotentials and streamlines of the original motion become, respectively, the streamlines and equipotentials of the new motion. When dealing with complex quantities it is very important to state clearly whether constants are real or not. We shall always take all constants (coefficients, indices, etc) to be *real* unless the contrary is distinctly stated.

EXAMPLES OF THE COMPLEX POTENTIAL

Example 1. *Uniform flow*

Let $$w = -Uz, \qquad \frac{dw}{dz} = -U = -u + iv.$$

Hence $u = U$ and $v = 0$, so we have uniform flow in the direction OX.

$$w = \phi + i\psi = -U(x + iy).$$

$$\therefore \quad \phi = -Ux \quad \text{and} \quad \psi = -Uy$$

Next let $$w = iVz.$$

$$\therefore \quad -u + iv = iV, \quad u = 0, \quad v = V.$$

This potential represents uniform flow in the direction OY.

$$\phi + i\psi = iV(x + iy)$$

$$\therefore \quad \phi = -Vy, \quad \psi = Vx.$$

Finally let

$$w = -Wze^{-i\alpha} = z(-W\cos\alpha + iW\sin\alpha).$$

By the results already obtained, the velocity has the components $u = W\cos\alpha$, $v = W\sin\alpha$. Thus the resultant velocity is W inclined at the angle α to OX.

Example 2. *Flow in a corner*

Let $$w = Wz^2 = W(x + iy)^2 = W(x^2 - y^2) + 2iWxy.$$

Hence $$\phi = W(x^2 - y^2) \quad \text{and} \quad \psi = 2Wxy.$$

also $$\frac{dw}{dz} = -u + iv = 2Wz = 2W(x + iy)$$

so $$u = -2Wx, \quad v = 2Wy.$$

This potential represents flow in a right angled corner, for the streamline $\psi = 0$ consists of the axes OX and OY.

Next, let $w = Wz^n$. It is here convenient to transform to polar coordinates by use of the important relation

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$$

Hence $$w = Wr^n e^{in\theta} = Wr^n(\cos n\theta + i\sin n\theta).$$

Therefore $$\phi = Wr^n\cos n\theta \qquad \text{and} \qquad \psi = Wr^n\sin n\theta.$$

The streamline $\psi = 0$ consists of the pencil of radial straight lines

$$n\theta = m\pi$$

where m is any integer. The angle between adjacent lines is π/n, as exemplified above. The velocity is given by

$$-u + iv = \frac{dw}{dz} = nWz^{n-1}.$$

Hence

$$u = -nWr^{n-1}\cos(n-1)\theta,$$

$$v = nWr^{n-1}\sin(n-1)\theta.$$

Also the radial component of velocity is

$$u_r = u\cos\theta + v\sin\theta = -nWr^{n-1}\cos n\theta$$

while the transverse or circumferential component is

$$u_t = -u\sin\theta + v\cos\theta = nWr^{n-1}\sin n\theta.$$

Example 3. *Doublet*

This is the case of the last where $n = -1$ but it is worth while to work out the results afresh by another method.

Put $w = \dfrac{W}{z}$ and let $\bar{z}$ be the *conjugate* of z, i.e., let $\bar{z} = x - iy$.

Then $$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = r^2.$$

Therefore we have

$$w = \frac{W\bar{z}}{z\bar{z}} = \frac{W(x - iy)}{x^2 + y^2}$$

Hence $$\phi = \frac{Wx}{x^2 + y^2} = \frac{W\cos\theta}{r}$$

and $$\psi = -\frac{Wy}{x^2 + y^2} = -\frac{W\sin\theta}{r}$$

Now a two-dimensional doublet of strength μ' at 0 with its axis along OX has the stream function (see equation (2.7,20))

$$\psi = -\frac{\mu' \sin \theta}{2\pi r}$$

Therefore the complex potential of this doublet is $w = \dfrac{\mu'}{2\pi z}$.

The method of reduction employed here can be applied to any integral negative power of z.

Example 4. *Two-dimensional source*

$$\text{Let} \qquad w = W \ln z = W \ln (re^{i\theta}) = W \ln r + iW\theta$$

$$\text{Then} \qquad \phi = W \ln r \qquad \text{and} \qquad \psi = W\theta.$$

But we already know (see equation (2.7,8)) that the stream function of a source of strength m is

$$\psi = -\frac{m\theta}{2\pi}.$$

Hence the complex potential of this source is $w = -\dfrac{m}{2\pi} \ln z$.

Example 5. *Infinite straight vortex*

$$\text{Let} \qquad w = iW \ln z \quad = -W\theta + iW \ln r. \quad \text{(See Example 4.)}$$

$$\text{Then} \qquad \phi = -W\theta \qquad \text{and} \qquad \psi = W \ln r.$$

But we already know (see equation (2.13,3)) that the stream function of a vortex of strength Γ at 0 is

$$\psi = \frac{\Gamma}{2\pi} \ln r.$$

Hence the complex potential of this vortex is $w = \dfrac{i\Gamma}{2\pi} \ln z$.

Example 6. *Circular cylinder in uniform stream*

$$\text{Let} \qquad w = U\left(z + \frac{a^2}{z}\right)$$

By Examples 1 and 3 we see that this represents the resultant of a uniform flow of speed U in the negative direction of the axis OX and the field of a doublet at O of strength $2\pi a^2$ with its axis along OX. By use of Example 3 we find that

$$\phi = Ux + \frac{Ua^2x}{x^2 + y^2}$$

and

$$\psi = Uy - \frac{Ua^2y}{x^2 + y^2}.$$

The streamline $\psi = 0$ consists of the straight line $OX(y = 0)$ and the circle

$$x^2 + y^2 = a^2.$$

whose centre is O and radius a.

2.15 Conformal Transformation and Orthogonal Coordinates

The method of conformal transformation, which we are about to describe, is of great utility as it enables us to obtain the flow in relatively complex cases from the known results in simple cases. In particular, it enables us to find the flow about various aerofoils from the known flow in the neighbourhood of a circle. Conformal transformation is a particular kind of mapping and, when it is applied to fluid motion, the map of the equipotentials and streamlines for the flow about a body A is remapped in such a way that the equipotentials and streamlines for the flow about another body B are obtained.

Let us take two planes which we distinguish as I and II. A point P_1 in I has the rectangular coordinates x, y and there is a *corresponding point* P_2 in II which has the rectangular coordinates ξ, η. The relation of correspondence between P_1 and P_2 is established by making

$$\zeta = f(z) \tag{2.15,1}$$

where

$$\zeta = \xi + i\eta \tag{2.15,2}$$

and

$$z = x + iy \tag{2.15,3}$$

The plane I is the plane of the complex variable z and is usually called the z plane. Similarly the plane II is the plane of the complex variable ζ and is usually called the ζ *plane*.† Once the function f has been chosen, the relation between the points P_1 and P_2 becomes definite and we have a method for converting any figure or family of curves in the z plane into a corresponding figure or family of curves in the ζ plane. This particular method of relating the points in the two planes has the *conformal property*; this means that the figure in the ζ plane which corresponds to any infinitesimal figure in the z plane is geometrically similar to it.

Let ζ correspond to z, so that equation (2.15,1) is satisfied and let $\zeta + \Delta\zeta$ correspond to $z + \Delta z$. Then we have by Taylor's theorem

$$\begin{aligned}\zeta + \Delta\zeta &= f(z + \Delta z)\\ &= f(z) + \Delta z f'(z) + \tfrac{1}{2}(\Delta z)^2 f''(z) + \text{etc.}\end{aligned}$$

and by (2.15,1) we obtain

$$\frac{\Delta\zeta}{\Delta z} = f'(z) + \tfrac{1}{2}\Delta z f''(z) + \text{etc.} \tag{2.15,4}$$

---

† The variables ξ, η, ζ here have nothing to do with the vorticity.

Provided that $f'(z)$ is not zero we obtain when Δz is exceedingly small,

$$\frac{\Delta\zeta}{\Delta z} = \frac{d\zeta}{dz} = f'(z) \tag{2.15,5}$$

So long as z is constant, $\Delta\zeta$ is a constant multiple of Δz but the multiplier is in general, a complex number. Hence the length of the line represented by $\Delta\zeta$ is a constant multiple of the length of the line represented by Δz and the angle between the lines is constant. Therefore any infinitesimal figure is transformed into a geometrically similar figure and the conformal property is proved. By the properties of complex numbers we have

$$\left|\frac{\Delta\zeta}{\Delta z}\right| = |f'(z)| \tag{2.15,6}$$

where the bars placed round a number indicate that its *modulus* is to be taken. Also the angle between $\Delta\zeta$ and Δz is $\arg f'(z)$.

When $f'(z)$ vanishes the ratio of $\Delta\zeta$ to Δz depends on Δz (see equation (2.15,4)) and the conformal property fails. Failure also occurs when $f'(z)$ is infinite, for then $dz/d\zeta$ is zero. The points where $f'(z)$ is zero or infinite are called the *singular points* of the transformation.

Now let $w(z)$ be the complex potential of a flow in the z plane and let

$$z = F(\zeta) \tag{2.15,7}$$

Then

$$w(z) = w(F(\zeta)) \tag{2.15,8}$$

is the complex potential of the *corresponding flow* in the ζ plane. It should be noted that, on account of these equations, *the complex potential has the same value at corresponding points in the z and ζ planes.* An equipotential in the z plane is a curve such that the real part of w has a certain constant value on it and we see that the real part of w has the same constant value on the corresponding curve in the ζ plane. Hence *the equipotentials in the z and ζ planes are corresponding curves.* Similarly, *the streamlines in the z and ζ planes are corresponding curves.* Let the velocity components in the ζ plane in the directions of the axes of ξ and η be u' and v' respectively. Then we have

$$\frac{dw}{dz} = -u + iv \quad \text{and} \quad \frac{dw}{d\zeta} = -u' + iv'.$$

But

$$\frac{dw}{d\zeta} = \frac{dw}{dz}\frac{dz}{d\zeta} = F'(\zeta)\frac{dw}{dz}.$$

Hence

$$-u' + iv' = F'(\zeta)(-u + iv), \tag{2.15,9}$$

and an equivalent expression is

$$-u' + iv' = \frac{dw}{dz}\bigg/\frac{d\zeta}{dz} = (-u + iv)/f'(z) \tag{2.15,10}$$

Let
$$q'^2 = u'^2 + v'^2 \tag{2.15,11}$$
or
$$q' = |-u + iv| \tag{2.15,12}$$
so that q' is the resultant speed of flow in the ζ plane. By equations (2.15,9) and (2.15,10) we then have

$$q' = q|F'(\zeta)| = q/|f'(z)|. \tag{2.15,13}$$

These relations are of great utility in the applications of conformal transformation.

We shall illustrate the foregoing general theory by the special case of the *Joukowsky transformation* which has many useful applications to the theory of aerofoils and other matters. Let

$$\zeta = z + \frac{c^2}{z} \tag{2.15,14}$$

so that
$$\frac{d\zeta}{dz} = 1 - \frac{c^2}{z^2}. \tag{2.15,15}$$

The singular points of the transformation are $z = \pm c$ and $z = 0$. When z is very large ζ differs little from z so the very distant parts of the z plane are unaffected by the transformation; it follows that the flow at infinity is the same in the z and ζ planes (see equation (2.15,10)). By (2.15,14)

$$\xi + i\eta = z + \frac{c^2\bar{z}}{z\bar{z}} = x + iy + \frac{c^2(x - iy)}{x^2 + y^2}.$$

Hence
$$\xi = \frac{x(x^2 + y^2 + c^2)}{x^2 + y^2} \tag{2.15,16}$$

and
$$\eta = \frac{y(x^2 + y^2 - c^2)}{x^2 + y^2}. \tag{2.15,17}$$

For example, for points on the circle

$$x^2 + y^2 = a^2 \tag{2.15,18}$$

we have
$$\frac{\xi}{x} = \frac{a^2 + c^2}{a^2}, \quad \text{and} \quad \frac{\eta}{y} = \frac{a^2 - c^2}{a^2}. \tag{2.15,19}$$

Hence the circle (2.15,18) is transformed into the ellipse

$$\frac{\xi^2}{\left(a + \dfrac{c^2}{a}\right)^2} + \frac{\eta^2}{\left(a - \dfrac{c^2}{a}\right)^2} = 1. \tag{2.15,20}$$

When $a = c$ this becomes a "line ellipse" of major axis $4c$, i.e., the segment of the ξ axis lying between the points $(-2c, 0)$ and $(2c, 0)$. It now follows that when we know the solution to a problem of two-dimensional flow about a circular cylinder we can at once derive the corresponding flows about elliptic cylinders and flat plates of finite chord.

The Joukowsky transformation can be performed graphically as shown in Fig. 2.15,1. Here P is the point $z = x + iy$ and $OP = r$. $PM = PO$ and $PP' = c^2/r$. Hence P' is the point into which P is transformed. When the locus of P is the circle $r = c$, we have $PP' = PO = PM$, so P' lies at M and

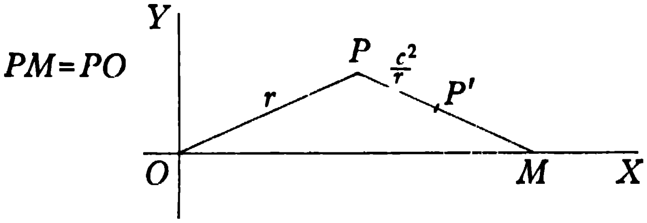

Fig. 2.15,1. Graphical representation of Joukowsky transformation.

the circle is transformed into a segment of OX, as we have already seen. The equation (2.15,14) can be written in an alternative form which immediately leads to interesting deductions. We obtain from this equation

$$\zeta + 2c = \frac{(z+c)^2}{z} \quad \text{and} \quad \zeta - 2c = \frac{(z-c)^2}{z}.$$

Hence
$$\frac{\zeta + 2c}{\zeta - 2c} = \left(\frac{z+c}{z-c}\right)^2 \tag{2.15,21}$$

Now let A_1 and A_2 be the singular points $z = \pm c$ in the z plane and let B_1 and B_2 be the corresponding singular points $\zeta = \pm 2c$ in the ζ plane. Further, let a point P have the polar coordinates r_1, θ_1 and r_2, θ_2 with A_1 and A_2 respectively as poles while the corresponding point P' has the polar coordinates ρ_1, ϕ_1 and ρ_2, ϕ_2 with B_1 and B_2 respectively as poles. Then we have

$$z - c = r_1 e^{i\theta_1}, \quad z + c = r_2 e^{i\theta_2}$$
$$\zeta - 2c = \rho_1 e^{i\phi_1}, \quad \zeta + 2c = \rho_2 e^{i\phi_2}$$

and equation (2.15,21) becomes

$$\frac{\rho_2}{\rho_1} e^{i(\phi_2 - \phi_1)} = \left(\frac{r_2}{r_1}\right)^2 e^{2i(\theta_2 - \theta_1)}$$

Therefore

$$\frac{\rho_2}{\rho_1} = \left(\frac{r_2}{r_1}\right)^2 \tag{2.15,22}$$

and
$$\phi_2 - \phi_1 = 2(\theta_2 - \theta_1). \tag{2.15,23}$$

Suppose now that the locus of P is such that r_2/r_1 is constant; it follows from (2.15,22) that the locus of P is such that ρ_2/ρ_1 is constant. Thus a family of coaxial circles with A_1 and A_2 as limiting points is transformed into a family of circles with B_1 and B_2 as limiting points. Again, let the locus of P be a circle through A_1 and A_2. Then $(\theta_2 - \theta_1)$ is constant and by (2.15,23) we have $(\phi_2 - \phi_1)$ also constant. Therefore P' lies on a circle through B_1 and B_2. A detailed investigation based on equations (2.15,16) and (2.15,17) shows that the complete circle through A_1 and A_2 is transformed into a single arc through B_1 and B_2; when the centre of the circle through A_1 and A_2 lies at O the arc becomes a segment of a straight line as we already know.

Consider now any curve passing through the singular point A_1 and not having a node or cusp at A_1. We can move along the curve from A_1 in either direction and the angle between these directions is π. On account of equation (2.15,23) the angle between the corresponding elements of the transformed curve at B_1 is 2π, for, when the point considered is very near to A_1, we have θ_2 and ϕ_2 zero and the equation gives $\phi_1 = 2\theta_1$. Hence the transformed curve has a cusp at B_1. Similarly, a curve passing through A_2 is transformed into one having a cusp at B_2. If we take a circle which passes through A_1 but encloses A_2 it is transformed into a closed oval with a sharp (cusped) tail at B_2. This is called a Joukowsky profile and could be used for the section of an aerofoil. In applying the Joukowsky transformation the singular points must not lie within the region occupied by fluid but they may lie on the boundary, as already exemplified.

There is another useful way of regarding the relation

$$z = F(\zeta) = F(\xi + i\eta) \tag{2.15,24}$$

which we shall now explain. We dispense altogether with the ζ plane and regard ξ and η as variable parameters. Let

$$F(\xi + i\eta) = F_1(\xi, \eta) + iF_2(\xi, \eta). \tag{2.15,25}$$

Then by (2.15,24) we have

$$\left.\begin{aligned} x &= F_1(\xi, \eta) \\ y &= F_2(\xi, \eta) \end{aligned}\right\} \tag{2.15,26}$$

so the coordinates of a point are determined by ξ and η. But we may regard ξ and η as themselves the curvilinear coordinates of a point, for, given their values, the position of the point is fixed. We can describe on the plane OXY a family of curves $\xi =$ constant and a second family of curves $\eta =$ constant. We then have a network in which we can immediately locate a point when we are given its curvilinear coordinates ξ and η. In the present case the curves in the network cut everywhere at right angles. This follows at once from the fact that the lines $\xi =$ constant and $\eta =$ constant are perpendicular in the ζ plane, together with the conformal property of the transformation. Thus ξ and η are *orthogonal curvilinear coordinates.* They are not, however, the most general coordinates of this nature. In fact if we put

$$\alpha = f_1(\xi) \qquad \text{and} \qquad \beta = f_2(\eta) \tag{2.15,27}$$

then α and β are orthogonal curvilinear coordinates. The special coordinates ξ and η are, nevertheless, the most convenient because Laplace's equation retains its usual form when these coordinates are used, whereas this is not so in general for α and β.

Let

$$\begin{aligned}\phi + i\psi = w &= f(z)\\ &= f\{F(\xi + i\eta)\}.\end{aligned} \tag{2.15,28}$$

Thus ϕ and ψ are conjugate functions of ξ and η and it immediately follows that

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} = 0 \tag{2.15,29}$$

and

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = 0. \tag{2.15,30}$$

The fact that Laplace's equation retains its form in this transformation is known as the theorem of Stokes and Lamé. It follows that if $\phi(x, y)$ is a velocity potential expressed in terms of the rectangular coordinates x, y then the function $\phi(\xi, \eta)$ is also a possible velocity potential when ξ and η are our special orthogonal coordinates; similar remarks apply to the stream function.

As a first example let

$$\begin{aligned}z &= e^{\xi + i\eta}\\ &= e^{\xi}(\cos\eta + i\sin\eta).\end{aligned} \tag{2.15,31}$$

Then

$$x = e^{\xi}\cos\eta\,, \qquad y = e^{\xi}\sin\eta \tag{2.15,32}$$

and it immediately follows that the polar coordinates are

$$r = \sqrt{(x^2 + y^2)} = e^{\xi}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) = \eta. \tag{2.15,33}$$

The orthogonal network consists of the concentric circles $\xi =$ constant and the radial straight lines $\eta =$ constant. The velocity potential satisfies (2.15,29) and on account of the equations (2.15,33) we can easily change the variables to r and θ. We have

$$\frac{\partial \phi}{\partial \xi} = \frac{\partial \phi}{\partial \ln r} = \frac{\partial \phi}{\partial r}\frac{dr}{d\ln r} = r\frac{\partial \phi}{\partial r}$$

Hence
$$\frac{\partial}{\partial \xi} = r\frac{\partial}{\partial r}$$

and
$$\frac{\partial^2 \phi}{\partial \xi^2} = r\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = r\frac{\partial \phi}{\partial r} + r^2\frac{\partial^2 \phi}{\partial r^2}.$$

Accordingly the equation satisfied by the velocity potential becomes

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \tag{2.15,34}$$

which is Laplace's equation in plane polar coordinates; the stream function satisfies the same equation. The original equation (2.15,29) is, however, far more convenient than (2.15,34). As a second example, let

$$z = c\cosh(\xi + i\eta). \tag{2.15,35}$$

Then

$$x = c\cosh\xi\cos\eta, \qquad y = c\sinh\xi\sin\eta. \tag{2.15,36}$$

When we eliminate η we find

$$\frac{x^2}{c^2\cosh^2\xi} + \frac{y^2}{c^2\sinh^2\xi} = 1 \tag{2.15,37}$$

and when we eliminate ξ we get

$$\frac{x^2}{c^2\cos^2\eta} - \frac{y^2}{c^2\sin^2\eta} = 1. \tag{2.15,38}$$

The family (2.15,37) consists of confocal ellipses and the family (2.15,38) of confocal hyperbolas; all the curves have the points ($\pm c$, 0) as foci. These coordinates are accordingly known as *plane confocal coordinates* or *elliptic coordinates*. When ξ is very large this system becomes substantially identical with that first described for we then have

$$\cosh\xi = \sinh\xi = \tfrac{1}{2}e^{\xi}$$

very nearly. Thus when ξ tends to infinity we have

$$r \to \tfrac{1}{2}ce^{\xi}, \qquad \theta \to \eta. \tag{2.15,39}$$

EXAMPLES OF THE APPLICATIONS OF CONFORMAL TRANSFORMATION AND ORTHOGONAL COORDINATES

Example 1. *Flow normal to a flat plate*

We take a fixed circular cylinder whose section is the circle

$$x^2 + y^2 = c^2$$

placed in a uniform stream of velocity V in the direction OY. The complex potential for this flow is

$$w = iV\left(z - \frac{c^2}{z}\right)$$

for this gives

$$\psi = Vx\left(1 - \frac{c^2}{x^2 + y^2}\right)$$

which satisfies all the kinematic conditions. We now apply the Joukowsky transformation (2.15,14) which transforms the circle into a segment of the ξ axis, of length $4c$, with its mid-point at the origin. There are now two possible procedures:

(a) We may use the transformation forwards only. Thus we take the velocity at a given point (x, y) and derive the coordinates (ξ, η) of the transformed point and the velocity at it.

(b) We may express the complex potential in terms of ζ by using the transformation in reverse and then derive expressions for the components of velocity in terms of ξ and η.

Both these procedures will now be illustrated.

The general formulae for method (a) are equations (2.15,16), (2.15,17) and the relation

$$-u' + iv' = \frac{dw}{dz}\Big/\frac{d\zeta}{dz} = \frac{-u + iv}{d\zeta/dz} = \frac{-u + iv}{1 - (c^2/z^2)}.$$

These will be applied to points on the circle which are transformed into points on the plate. It is convenient to put $x = c \cos \theta$ and $y = c \sin \theta$ and then $\xi = 2c \cos \theta$, $\eta = 0$. Also

$$1 - \frac{c^2}{z^2} = 1 - e^{-2i\theta} = 1 - \cos 2\theta + i \sin 2\theta = 2 \sin \theta(\sin \theta + i \cos \theta).$$

The general expressions for the components of the velocity are

$$u = -\frac{\partial \psi}{\partial y} = -\frac{2Vc^2xy}{(x^2 + y^2)^2}$$

$$v = \frac{\partial \psi}{\partial x} = \frac{V}{(x^2 + y^2)^2}[(x^2 + y^2)^2 + c^2(x^2 - y^2)]$$

and on the circle these become

$$u = -2V \cos \theta \sin \theta, \qquad v = 2V \cos^2 \theta.$$

Hence $$-u' + iv' = \frac{2V \cos \theta(\sin \theta + i \cos \theta)}{2 \sin \theta(\sin \theta + i \cos \theta)} = V \cot \theta.$$

Finally $$u' = -V \cot \theta \quad \text{and} \quad v = 0.$$

For the upper surface of the plate θ ranges from 0 to π; for the lower surface we may take the range to be 0 to $-\pi$. The velocity beneath is equal in magnitude to that above, but reversed in direction.

The velocity u can be expressed in terms of ξ. For the upper surface of the plate we have

$$u' = -V \cot \theta = -\frac{V\xi}{\sqrt{(4c^2 - \xi^2)}}$$

and on the lower surface

$$u' = \frac{V\xi}{\sqrt{(4c^2 - \xi^2)}}\,.$$

In method (b) we express z in terms of ζ. We find from equation (2.15,14) that

$$z = \tfrac{1}{2}[\zeta \pm \sqrt{(\zeta^2 - 4c^2)}]$$

and therefore

$$w = iV\left(z - \frac{c^2}{z}\right) = iV(2z - \zeta) = \pm iV\sqrt{(\zeta^2 - 4c^2)}.$$

Hence

$$-u' + iv' = \frac{dw}{d\zeta} = \pm \frac{iV\zeta}{\sqrt{(\zeta^2 - 4c^2)}}\,.$$

Let us take the positive sign and apply this to a point on the plate. Then we have $\zeta = \xi$ (real) with $|\xi| < 2c$, so

$$\sqrt{(\zeta^2 - 4c^2)} = i\sqrt{(4c^2 - \xi^2)}.$$

Therefore

$$u' = -\frac{V\xi}{\sqrt{(4c^2 - \xi^2)}} \quad \text{and} \quad v' = 0.$$

This agrees with the result previously obtained for the velocity above the plate. Next take a point on the plane of the plate but outside it. We now have $\zeta = \xi$ with $\xi > 2c$ and obtain

$$u' = 0, \qquad v' = \frac{V\xi}{\sqrt{(\xi^2 - 4c^2)}}\,.$$

It will be seen that the velocity tends to infinity as the edge of the plate is approached from either side. This could not occur in a real fluid, but the infinity arises from the infinitesimal thickness of the plate in the theory. There is an unrealizable abstraction here.

In the course of the treatment of this problem by the method (b) we have come across a double-valued function of the complex variable, namely the square root of a quadratic function. The reader should consult a book on the theory of the complex variable for a discussion of multiply-valued functions and the manner in which they are represented on a Riemann surface so that they become effectively single valued. It must be emphasised that in the present and other similar problems of fluid motion the velocities are certainly single valued. Some remarks on the uniqueness of solutions are given in § 2.17.

Example 2. Circulatory flow about a fixed elliptic cylinder

Consider the stream function

$$\psi = \frac{\Gamma\xi}{2\pi}.$$

This satisfies equation (2.15,30) and therefore represents an irrotational flow. Also the curves $\xi = \text{constant}$, which are the streamlines, are confocal ellipses as we have seen. Any one of these ellipses can be taken as a fixed barrier in contact with the fluid. We also have from equation (2.15,39) that when ξ tends to infinity it tends to equality with

$$\ln\left(\frac{2r}{c}\right) = \ln r + \text{constant}.$$

Hence at great distances from the origin

$$\psi \sim \frac{\Gamma \ln r}{2\pi} + \text{constant}.$$

This expression is the stream function for a vortex of strength Γ at 0 and the circulation in any circuit embracing the origin once is Γ. Consequently the stream function with which we started represents an irrotational circulatory flow of circulation Γ about a fixed elliptic cylinder.

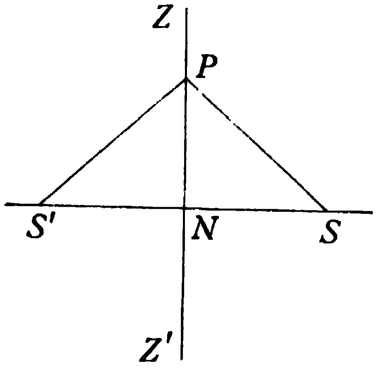

Fig. 2.16,1. Image of a point source in a fixed plane wall.

The ellipse $\xi = 0$ is the line ellipse $y = 0$, $|x| \leq c$ by equations (2.15,36). Hence our solution also gives the circulatory flow about a flat plate of chord $2c$ lying along OX and with its mid-point at O.

2.16 The Method of Images

An optical image is a very familiar concept which is indeed forced upon our attention by direct observation. However, it was shown by Lord Kelvin that a method based on the idea of images could be used to solve many and varied problems of mathematical physics. Kelvin's first applications of the method were to problems of static electricity but he later gave many applications to fluid motion. We shall now give some examples of the method, which is of very great utility and of beautiful simplicity.

As a first example let us consider the problem of finding the flow in a perfect fluid due to a point source of strength σ when the fluid is bounded by a fixed plane wall. In Fig. 2.16,1 let the source be at S and let ZZ' be the trace of the surface of the wall. Drop SN perpendicular to ZZ' and make $S'N = NS$, so S' is the optical image of S in the surface. Now let us imagine that a second source of strength σ is placed at S', that the wall is removed and that the space which was behind the wall is filled with fluid. We have seen in § 2.7 that the velocity due to a point source of strength σ is radial and equal to $\sigma/4\pi r^2$. Now $S'P = SP$ so the velocities at P due to the sources at S and at S' are equal in magnitude. Further, the components perpendicular to NP are equal in magnitude but opposite in direction, for the angles NPS and NPS' are equal. Hence the resultant velocity at P perpendicular to the plane is zero and, the fluid being supposed frictionless, we could replace the fixed plane surface at ZZ' without influencing the motion. Consequently the motion to the right of the plane is the same as that due to the combined effects of the source and its image at S', in the absence of the plane boundary.

It is easy to generalize the foregoing. Let us suppose that we now have a system of sources and sinks in the presence of a fixed plane wall. Then the motion is the same as that which would occur, in the absence of the wall, if the images of all the sources and sinks were added to the system; the image in every case is situated at the optical image of the source or sink in the plane and has the same strength. This follows from the fact, already demonstrated, that any source (or sink) together with its image gives zero resultant normal velocity everywhere on the plane. We may particularly note the following important special cases of this general result:

The image of a line source in a fixed plane wall to which it is parallel is an equal line source situated at the optical image of the source. In other words, the image of a two-dimensional source in a fixed straight line is an equal source situated at the optical image of the source in the line.

The image of a doublet in a fixed plane wall is a doublet of equal strength situated at the optical image of the doublet in the plane and the axis of the image doublet has the direction and sense of the optical image of the axis of the original doublet. For example, the image of a doublet which is normal to the fixed plane and with its source end adjacent to the plane is also a doublet normal to the plane and with its source end adjacent to the plane.

Next, let S in Fig. 2.16,1 be the trace of an infinite straight vortex which is parallel to the fixed boundary whose trace is ZZ'. Then it is easy to verify that the image is a vortex whose trace is S', whose strength is equal to that of the original vortex but whose direction of spin is reversed. For the vortex and its image give equal and opposite components of velocity at P perpendicular to ZZ'. We can then generalize this and obtain the image of any system of vortices which are all parallel to the fixed wall.

It is a familiar fact that an object has a set of optical images when there are two or more reflecting planes. Similarly a source, for example, will have a set of images (possibly infinite in number) when the fluid is bounded by two or more fixed plane walls. As a simple example, take a point source S in the region between a pair of perpendicular fixed plane walls, as shown in Fig. 2.16,2. The complete system of images consists of three equal sources S_1, S_2, S_3 as shown in the figure, where S_1 is the optical image of S in YY' while S_2 and S_3 are the images of S and S_1 respectively in XX'. It is clear from the earlier discussion that the pair of sources S and S_1 give zero resultant normal velocity on YY' and similarly the pair of sources S_2 and S_3 gives zero resultant normal velocity on this line. Hence the complete system of source and images gives zero normal velocity on YY'. In the same manner we can show that the normal velocity on XX' is also zero. If, next, S is the trace of a vortex parallel to the fixed planes, the image system consists of three vortices of equal strengths, that at S_3 having the same direction of spin as the original vortex while those at S_1 and S_2 have the reversed direction of spin. When we have a pair of parallel fixed plane walls there is an infinite set of images. Thus we have first the primary images in each wall; then each primary image has a secondary image in the other wall, and so on *ad infinitum*. Although there is an infinity of images the resultant induced velocity will be finite. The method of images will fail when any of the images falls in the region occupied by the fluid.

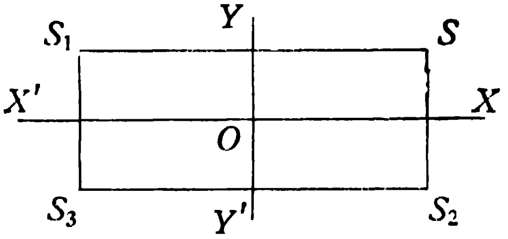

Fig. 2.16,2. Images of a point source in a pair of perpendicular fixed plane walls.

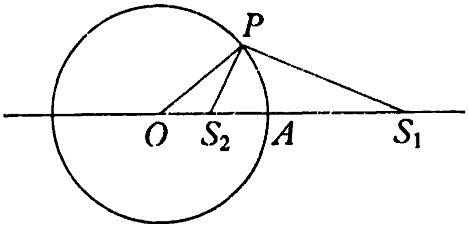

Fig. 2.16,3. Images of a vortex in a fixed circular cylinder. (S_1 and S_2 are inverse points).

The method of images can often be applied when the fixed boundary of the fluid is a circular cylinder or sphere. Many such applications depend on the properties of *inverse points* in a circle. In Fig. 2.16,3 the inverse points S_1 and S_2 lie on a line OA through the centre O of the circle and

$$OS_1 . OS_2 = OA^2. \tag{2.16,1}$$

Since $OP = OA$ we have from the last equation

$$OS_1 : OP = OP : OS_2.$$

Hence the triangles OS_1P and OPS_2 are similar, for they have a common angle at O. Consequently

$$S_2P : S_1P = OP : OS_1. \tag{2.16,2}$$

Now let the circle be the trace of a circular cylinder and let S_1 be the trace of a vortex parallel to the axis of the cylinder. Suppose first that there is an equal vortex at S_2 but with reversed direction of spin. Then the stream function for the pair of vortices at S_1 and S_2 is

$$\psi = \frac{\Gamma}{2\pi}(\ln r_1 - \ln r_2) = \frac{\Gamma}{2\pi}\ln\left(\frac{r_1}{r_2}\right), \tag{2.16,3}$$

where r_1 and r_2 are the distances of a point from S_1 and S_2 respectively.

By (2.16,2) ψ is constant on the circle. Hence the circle is a streamline and a fixed boundary can be placed there without altering the flow. The vortex at S_2 is not the complete image system, however, for the vortices at S_1 and S_2 give zero circulation at infinity, since the sum of their strengths is zero, whereas S_1 alone gives the circulation Γ at infinity. To restore this circulation we place a vortex equal to that at S_1 along the axis of the cylinder. The stream function for this vortex is constant on the circle, so all the conditions are satisfied. To sum up, the image system consists of an equal vortex at O and an equal and oppositely spinning vortex at the inverse point S_2. The stream function for the flow outside the fixed cylinder is accordingly

$$\psi = \frac{\Gamma}{2\pi}(\ln r + \ln r_1 - \ln r_2) = \frac{\Gamma}{2\pi}\ln\left(\frac{rr_1}{r_2}\right), \qquad (2.16,4)$$

where r is the radius vector measured from O.

Images in fixed circles can be found at once by applying *Milne-Thomson's circle theorem* which is stated by its author as follows:

"Let there be irrotational two-dimensional flow of incompressible inviscid fluid in the z plane. Let there be no rigid boundaries and let the complex potential be $f(z)$ where the singularities of $f(z)$ are all at a greater distance than a from the origin. If a circular cylinder typified by its cross-section the circle, C, $|z| = a$, be introduced into the field of flow, the complex potential becomes

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right)."$$

Here $\bar{f}(\ \ )$ represents the function obtained by substituting in $f(\ \ )$ the complex conjugate of every complex (or pure imaginary) *constant* which occurs in it. If $f(\ \ )$ only contains real constants, then $\bar{f}(\ \ ) = f(\ \ )$. It follows from this theorem that $\bar{f}(a^2/z)$ is the complex potential of the image system.

We can prove this theorem as follows. On the circle C we have

$$z\bar{z} = a^2 \qquad \text{or} \qquad \frac{a^2}{z} = \bar{z}$$

Hence on C we have

$$w = f(z) + \bar{f}(\bar{z}) \qquad = \text{a real quantity},$$

for $\bar{f}(\bar{z})$ is the complex conjugate of $f(z)$.

But

$$w = \phi + i\psi$$

and ψ is therefore zero on C. Hence C is a streamline.

For any point z outside C we have

$$\left|\frac{a^2}{z}\right| < a.$$

Now the singularities of $f(z)$ are such that $|z| > a$. Hence any singularities of $\bar{f}(a^2/z)$ lie inside C, i.e., outside the field of flow. Moreover $\bar{f}(a^2/z)$ tends to the constant value $\bar{f}(0)$ when $z \rightarrow \infty$, so the flow at infinity is the same for the complex potentials w and $f(z)$.

Example. Image of a two-dimensional source in a circle

Suppose that the source of strength m is at the point $z = b$ (real), where $|b| > a$. The complex potential for this source alone is

$$f(z) = -\frac{m}{2\pi} \ln (z - b).$$

All the constants in this are real. Hence $\bar{f}(\ ) = f(\ )$ and we get

$$w = -\frac{m}{2\pi}\left\{\ln (z - b) + \ln \left(\frac{a^2}{z} - b\right)\right\}.$$

But
$$\frac{a^2}{z} - b = -\frac{b}{z}\left(z - \frac{a^2}{b}\right)$$

and
$$\ln \left(\frac{a^2}{z} - b\right) = \ln \left(z - \frac{a^2}{b}\right) - \ln z + \ln (-b).$$

The last term can be neglected as it is a constant and we have as the complex potential of the *image system alone*

$$-\frac{m}{2\pi} \ln \left(z - \frac{a^2}{b}\right) + \frac{m}{2\pi} \ln z.$$

We immediately recognise the first of these terms as the potential of a source of strength m at $z = a^2/b$ while the second term is the potential of an equal sink at the origin. Hence the image system of the external source consists of an equal source at the inverse point and an equal sink at the centre.

2.17 The Status of Kinematic Theory

In kinematic theory we study the motions of fluids in a very general way but we do not attempt to answer the question as to how a fluid with given physical properties does move in given circumstances. The answer to this question must be based on dynamical arguments, as discussed in Chapter 3. However, dynamical theory shows the importance of the type of motion called irrotational, i.e., that in which the vorticity is zero. It appears that when a body moves through a fluid which is at rest, except as disturbed by the body itself, the flow is irrotational except in the *boundary layer* adjacent to the surface of the body and in the *wake*, which is really an extension of the boundary layer (see Chapter 6). Thus the special attention given to irrotational flow in the present chapter is justified; we can go further and say truly that the theory of irrotational flow is an essential foundation for the dynamics of fluids.

An important property of the irrotational motion of an incompressible fluid is its *uniqueness*, subject to the satisfaction of certain conditions at the boundary. The basic proposition is as follows: an incompressible fluid contained in a closed simply-connected region with fixed boundaries must be at rest unless it contains vortices. It follows from this that, if the motion is irrotational, the velocity field within a closed simply-connected region with a specified velocity of the boundary is unique; the specification of the velocity over the boundary must be compatible with the constancy of the enclosed volume.†

CHAPTER 2. APPENDIX 1

ANALYSIS OF THE MOTION OF A FLUID NEAR A POINT

The following investigation is due to Stokes and shows that the motion of a fluid in the neighbourhood of a point and at a given instant can be regarded as made up of the following parts:

(a) a velocity of translation
(b) a spin
(c) a motion of distortion.

Let the velocity at time t have the components u, v, w at the point whose coordinates are x, y, z. Take a neighbouring point whose coordinates are $x + \Delta x, y + \Delta y, z + \Delta z$. Then the velocity at this point at time t is given by

$$u + \Delta u = u + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z$$

$$v + \Delta v = v + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \frac{\partial v}{\partial z}\Delta z$$

$$w + \Delta w = w + \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z$$

by Taylor's theorem, when small quantities of the second order are neglected. It follows that the expression for the component Δu of the relative velocity can be written in the form

$$\begin{aligned}\Delta u &= \frac{\partial u}{\partial x}\Delta x + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\Delta y + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\Delta z \\ &\quad - \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Delta y + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\Delta z \\ &= \frac{\partial u}{\partial x}\Delta x + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\Delta y + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\Delta z - \tfrac{1}{2}\zeta\,\Delta y + \tfrac{1}{2}\eta\,\Delta z.\end{aligned}$$

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† For proofs and further details the reader may consult Lamb's *Hydrodynamics*.

Similarly

$$\Delta v = \frac{\partial v}{\partial y}\Delta y + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\Delta z + \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\Delta x - \tfrac{1}{2}\xi\,\Delta z + \tfrac{1}{2}\zeta\,\Delta x$$

and

$$\Delta w = \frac{\partial w}{\partial z}\Delta z + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\Delta x + \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\Delta y - \tfrac{1}{2}\eta\,\Delta x + \tfrac{1}{2}\xi\,\Delta y.$$

The terms $(\partial u/\partial x)\,\Delta x$, $(\partial v/\partial y)\,\Delta y$, $(\partial w/\partial z)\,\Delta z$ in Δu, Δv, Δw respectively represent velocities of stretching in the directions of the three coordinate axes. Next, the terms containing the quantities

$$\frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right),\quad \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),\quad \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

represent velocities of shearing while the remaining terms represent velocities of spin, the rectangular components of the angular velocity being $\tfrac{1}{2}\xi$, $\tfrac{1}{2}\eta$, $\tfrac{1}{2}\zeta$. Hence the angular velocity vector is half the vorticity. In irrotational motion the relative velocity is one of *pure strain.*

CHAPTER 2. APPENDIX 2

STOKES' STREAM FUNCTION FOR SYMMETRIC FLOW IN PLANES THROUGH A FIXED AXIS

The flow considered is the same in any plane through the axis OX; this is the kind of motion which occurs when a body of revolution is placed in a uniform stream parallel to the axis of the body. We shall take the fluid to be uniform and incompressible. The axis OY is normal to OX and the components of velocity at a point in the plane OXY are u, v parallel to OX, OY respectively, while the component perpendicular to OXY is zero.

In Fig. 2.4,1 we now regard the curves AMP, ANP as the traces of complete ring-shaped surfaces of revolution having OX as axis. The total flux across the surface of the closed annular region whose trace is $AMPNA$ is zero. Hence the flux across the annulus ANP = the flux across the annulus AMP. Thus the flux across any such annulus depends only on the positions of A and P. We now regard A as a fixed reference point and define the stream function to be

$$\psi = (\text{flux across annulus whose trace is } AP)/2\pi. \tag{1}$$

If ds is an element of arc of a curve, the area of the ring-shaped element generated by the rotation of ds is $2\pi y\,ds$ and the flux across this element is

$2\pi y u_n\,ds$, where u_n is the component of velocity normal to ds. In accordance with the definition we therefore have

$$d\psi = u_n y\,ds,$$

and we regard u_n as positive in the sense from right to left. Let $ds = dy$, so that $u_n = -u$. Then we get

$$u = -\frac{1}{y}\frac{\partial\psi}{\partial y}. \tag{2}$$

Next, let $ds = dx$, so that $u_n = v$. We now obtain

$$v = \frac{1}{y}\frac{\partial\psi}{\partial x}. \tag{3}$$

The formulae are the same as for two-dimensional motion except that the factor $1/y$ is introduced. It may be noted that the present stream function has the measure formula (see Chapter 4) L^3T^{-1} whereas the two-dimensional stream function has the measure formula L^2T^{-1}. By an argument similar to that used in § 2.4 we see that the streamlines are the curves $\psi =$ constant lying in any plane through OX.

The motion associated with a three-dimensional source at O is the same in all planes through OX. Let θ be the vectorial angle of a current point P in the plane OXY. Then the total flux from right to left across a circular area whose centre is N, the foot of the perpendicular from P on OX, and whose radius is NP is

$$-\frac{\sigma}{2}(1 - \cos\theta)$$

where σ is the strength of the source. Hence

$$\psi = -\frac{\sigma}{4\pi}(1 - \cos\theta) \tag{4}$$

$$= -\frac{\sigma}{4\pi}\left(1 - \frac{x}{r}\right). \tag{5}$$

Therefore $$\frac{\partial\psi}{\partial x} = \frac{\sigma y^2}{4\pi r^3} \quad\text{and}\quad \frac{\partial\psi}{\partial y} = -\frac{\sigma x y}{4\pi r^3}.$$

Hence $$u = -\frac{1}{y}\frac{\partial\psi}{\partial y} = \frac{\sigma x}{4\pi r^3} \tag{6}$$

and $$v = \frac{1}{y}\frac{\partial\psi}{\partial x} = \frac{\sigma y}{4\pi r^3}. \tag{7}$$

These results are in agreement with those obtained in § 2.7.

By the argument used in § 2.7 we find that the stream function for a point doublet of strength μ situated at O with its axis along OX in the positive sense is

$$\psi = -\mu \frac{\partial \psi_1}{\partial x}$$

where ψ_1 is the stream function of a unit point source at O. Hence

$$\psi = -\frac{\mu y^2}{4\pi r^3}. \tag{8}$$

Next, the stream function for a uniform stream of speed U in the positive sense of OX is

$$\psi = -\tfrac{1}{2}Uy^2 \tag{9}$$

as follows from equation (2) or directly from the defining equation (1). Now let us place the doublet whose stream function is (8) in a stream of speed U running from right to left. The stream function for the combined motion is obtained by addition and is accordingly

$$\psi = \tfrac{1}{2}Uy^2\left(1 - \frac{\mu}{2\pi U r^3}\right). \tag{10}$$

Part of the streamline $\psi = 0$ is the circle

$$r = \sqrt[3]{\frac{\mu}{2\pi U}}. \tag{11}$$

Suppose that the radius of this circle is given as a. Then we have as the strength of the doublet

$$\mu = 2\pi U a^3 \tag{12}$$

and equation (10) becomes

$$\psi = \tfrac{1}{2}Uy^2\left\{1 - \left(\frac{a}{r}\right)^3\right\}. \tag{13}$$

This is accordingly the stream function when a sphere of radius a is placed at rest in a uniform stream of speed U in the negative direction of OX. (See also Example 2 of § 2.7.)

When the motion is irrotational we have

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

or, by equations (2) and (3),

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{y}\frac{\partial \psi}{\partial y} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{14}$$

When the motion is not irrotational the vortex lines are necessarily circles whose centres lie on OX and whose planes are normal to OX.

EXERCISES. CHAPTER 2

1. Which of the following motions are kinematically possible for an incompressible fluid? (k is constant.)

(a) $u = kx \quad v = -ky \quad w = 0$
(b) $u = kx \quad v = -ky \quad w = kx$
(c) $u = kx \quad v = -ky \quad w = kz$
(d) $u = kx \quad v = ky \quad w = -2kz$
(e) $u = kx \quad v = ky \quad w = kz$ (Answer. (a), (b) and (d).)

Is the motion

$$u = \frac{kx}{(x^2 + y^2)}, \qquad v = \frac{ky}{(x^2 + y^2)}, \qquad w = 0$$

kinematically possible for an incompressible fluid? (k is constant.)
(Answer. Yes.)

3. Is the motion

$$u = \frac{kx}{(x^2 + y^2 + z^2)^{3/2}}, \quad v = \frac{ky}{(x^2 + y^2 + z^2)^{3/2}}, \quad w = \frac{kz}{(x^2 + y^2 + z^2)^{3/2}}$$

kinematically possible for an incompressible fluid? (k is constant.)
(Answer. Yes.)

4. A two-dimensional fluid motion is specified in the Lagrangean manner by the equations $x = x_0 e^{kt}$, $y = y_0 e^{-kt}$. Find (a) the path of a particle (b) the expressions for the velocities in the Eulerian form. State (c) whether the motion is steady and (d) whether it is kinematically possible for an incompressible fluid.
(Answers. (a) $xy = x_0 y_0$ (b) $u = kx$, $v = -ky$. (c) Yes. (d) Yes.)

5. A two-dimensional motion is specified in the Lagrangean manner by the equations $x = x_0 e^{kt}$, $y = y_0 e^{-kt} + x_0(1 - e^{-kt})$. Find (a) the path of a particle (b) the expressions for the velocities in the Eulerian form. State (c) whether the motion is steady and (d) whether it is kinematically possible for an incompressible fluid. (Answers. (a) $xy = x_0 y_0 + x x_0 - x_0^2$.
(b) $u = kx$, $v = -ky + kxe^{-kt}$.
(c) No. (d) Yes.)

6. *Equation of Continuity in the Lagrangean Form.* Show that the equation of continuity for an incompressible fluid is

$$\frac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)} \equiv \begin{vmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial y}{\partial x_0} & \dfrac{\partial z}{\partial x_0} \\ \dfrac{\partial x}{\partial y_0} & \dfrac{\partial y}{\partial y_0} & \dfrac{\partial z}{\partial y_0} \\ \dfrac{\partial x}{\partial z_0} & \dfrac{\partial y}{\partial z_0} & \dfrac{\partial z}{\partial z_0} \end{vmatrix} = 1$$

where x_0, y_0, z_0 are the initial coordinates of the point x, y, z. [*Note.* The Jacobian $\dfrac{\partial(x, y, z)}{\partial(x_0, y_0, z_0)}$ is the ratio of a small element of volume at time t to the corresponding element at time $t = 0$.]

7. Use the last result to prove that the displacements given in Exercise (5) are kinematically possible for an incompressible fluid.

8. The velocities for a two-dimensional flow are $u = k(y^2 - x^2)$, $v = 2kxy$. Verify that these are possible for an incompressible fluid and find the stream function (which is zero at 0) (a) by calculating the flux across a line connecting 0 to the point (x, y) and (b) by integrating the equations for u and v in terms of ψ. (Answer. $\psi = k(x^2y - \frac{1}{3}y^3)$.)
9. Find the streamlines when $u = ax$, $v = -ay$, $w = c$.
   (Answer. The intersections of the cylindrical surfaces $xy = k_1$ and $x = k_2e^{az/c}$.)
10. When the components of velocity are

$$u = \frac{-ay}{x^2 + y^2}, \qquad v = \frac{ax}{x^2 + y^2}, \qquad w = c,$$

show that the streamlines are helices described on the circular cylinders $x^2 + y^2 = \text{const.}$

11. The rectangular components of the velocity induced by a three-dimensional doublet at O with axis OX are

$$u = k\frac{y^2 + z^2 - 2x^2}{r^5}, \qquad v = \frac{-3kxy}{r^5}, \qquad w = \frac{-3kxz}{r^5}.$$

Find the equation of the streamlines. [*Note*. There is complete symmetry about the axis OX so it suffices to obtain the equation to the streamlines in the plane OXY.] (Answer. $(x^2 + y^2)^3/y^4 = \text{const.}$)

12. The velocity potential of a two-dimensional motion is $\phi = kxy$. Find the streamlines. (Answer. $x^2 - y^2 = \text{const.}$)
13. The velocity potential of a two-dimensional motion is $\phi = k(x^3 - 3xy^2)$. Find the streamlines (a) by integration of their differential equation, which is of the homogeneous type, and (b) by obtaining the expression for the stream function, after verification that the equation of continuity is satisfied. (Answer. $y^3 - 3x^2y = \text{const.}$)
14. An incompressible fluid moves irrotationally in two-dimensions. Given that $u = kxy$, find the most general expression for v.
   (Answer $v = \frac{1}{2}k(x^2 - y^2) + \text{const.}$)
15. An incompressible fluid is in three-dimensional irrotational motion. Show that each rectangular component of the velocity is a harmonic function.
16. An incompressible fluid moves in two dimensions so that OX is a streamline while $u = \partial f/\partial y$, where f is a function of x and y. Find the most general expressions for v and for the stream function.

$$\left(\begin{aligned}\text{Answer. } \psi &= -f(x, y) + f(x, 0) + \text{const.}\\ v &= -\frac{\partial f}{\partial x} + \left(\frac{\partial f}{\partial x}\right)_{y=0}.\end{aligned}\right)$$

17. In a two-dimensional motion of an incompressible fluid the velocity in the direction OX is

$$u = u_0 \sin\left(\frac{y}{h}\right)$$

where u_0 is constant and h is a given function of x, while v is always zero when $y = 0$. Find the most general expression for v.

$$\left(\text{Answer. } v = u_0\frac{dh}{dx}\left[\frac{y}{h}\sin\left(\frac{y}{h}\right) + \cos\left(\frac{y}{h}\right) - 1\right].\right)$$

18. Flow takes place in the plane OXY and is steady. The stream function is $\psi = kxy$ and the temperature $\theta = a - by$, where a and b are independent of time. Find the rate of change of the temperature of a particle of fluid.

$$\left(\text{Answer. } \frac{D\theta}{Dt} = -kby.\right)$$

19. The velocity potential of a steady two-dimensional motion is $\phi = c(x^3 - 3xy^2)$. Find the rectangular components of the acceleration, also the radial and transverse components and the component in the direction of motion.

$$(\text{Answer. } A_x = 18c^2xr^2 \quad A_y = 18c^2yr^2 \quad A_r = 18c^2r^3$$
$$A_t = 0 \quad A_s = 18c^2x(3y^2 - x^2).)$$

20. When polar coordinates are used and the flow is two-dimensional, show that the rate of change of a quantity α appertaining to a particle of fluid is

$$\frac{D\alpha}{Dt} = u_r\frac{\partial\alpha}{\partial r} + \frac{u_t}{r}\frac{\partial\alpha}{\partial\theta} + \frac{\partial\alpha}{\partial t}$$

where u_r, u_t are the radial and transverse (or circumferential) components of velocity, respectively.

21. Use the result of the last Exercise to find the radial and transverse components of the acceleration of a fluid particle when the motion is two-dimensional.

$$\left(\text{Answer. } A_r = \frac{Du_r}{Dt} - \frac{u_t^2}{r},\ A_t = \frac{Du_t}{Dt} + \frac{u_ru_t}{r}.\right)$$

22. Apply the general formulae of the last Exercise to find the radial and transverse components of the acceleration of the fluid when the stream function is

$$\psi = Wr^n \sin n\theta$$

where W is constant in space and time.

$$(\text{Answer. } A_r = n^2(n-1)W^2r^{2n-3},\ A_t = 0.)$$

23. Two-dimensional sources and sinks of equal strengths are placed alternately on the circumference of a circle C with arbitrary spacings between them. Show that C is a streamline of the motion.

24. Show that the shape of a Rankine ovoid depends only on the value of the non-dimensional parameter $2\pi s^2U/\sigma$.

25. Show that the length of the minor semi-axis of a Rankine ovoid is less than $(\sigma/\pi U)^{1/2}$ and approximates to this when the shape parameter is large (*see* last Exercise).

26. The circle $x^2 + y^2 - 2ay = 0$ is situated in a two-dimensional shear flow with $u = cy$, $v = 0$. Find the circulation in the circle (a) by direct calculation and (b) by integration of the vorticity. (Answer. $-\pi a^2c$.)

27. The components of velocity in a certain vortex motion inside the fixed elliptic cylinder $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ are $u = -\dfrac{2ky}{b^2}$, $v = \dfrac{2kx}{a^2}$. Verify that the velocity is tangential to the ellipse and find the circulation in the ellipse.

(Answer. $2\pi k(a^2 + b^2)/ab$.)

28. An infinite rectilinear vortex intersects a circle whose plane is normal to the vortex. Show that the component of the induced velocity at any point P on the circle in the direction of its tangent is independent of the position of P. Prove that the circulation in the circle is half the strength of the vortex.

29. An infinite rectilinear vortex intersects a closed plane circuit C at a point where C has a unique tangent and the plane of C is normal to the vortex. Show that the circulation in C is half the strength of the vortex.

30. The two semi-infinite legs of a horse-shoe vortex of strength Γ are at a distance $2s$ apart and the sense of the spin is such that there is a downwash between them. Find the induced velocity in magnitude and direction at a point P lying on the cross vortex (or this produced) at a distance y from the centre line. What does this become at a point Q infinitely far "downstream" from P?

(Answer. Upward velocity at P normal to plane of vortex $\dfrac{\Gamma s}{2\pi(y^2 - s^2)}$.
Velocity at Q in same direction and twice as great).

31. The complex potential of a two-dimensional motion is

$$w = k(z^n + a^{2n}z^{-n})$$

where k and a are real constants. Obtain the expressions for the velocity potential and stream function and show that the motion can take place in the region between the fixed radial walls $\theta = 0$, $\theta = \pi/n$ and the fixed circle $r = a$. (Answer. $\phi = k(r^n + a^{2n}r^{-n}) \cos n\theta$
$\psi = k(r^n - a^{2n}r^{-n}) \sin n\theta$.)

32. The velocity potential and stream function of one flow are ϕ_1, ψ_1 respectively and ϕ_2, ψ_2 refer similarly to a second flow. Show that

$$\phi = \phi_1\phi_2 - \psi_1\psi_2, \quad \psi = \psi_1\phi_2 + \psi_2\phi_1$$

are the velocity potential and stream function of a third flow.

33. Show that if $z = nae^{i\theta}$ the Joukowsky transformation yields

$$\zeta = a(n + n^{-1}) \cos \theta + ia(n - n^{-1}) \sin \theta.$$

Hence show that the circle n = constant is transformed into an ellipse.

$$\left(\text{Answer. } \frac{\xi^2}{a^2(n + n^{-1})^2} + \frac{\eta^2}{a^2(n - n^{-1})^2} = 1.\right)$$

34. Show that the Joukowsky transformation yields the relations

$$x\eta + y\xi = 2xy$$
$$x\xi - y\eta = x^2 - y^2 + a^2.$$

Hence prove that x is given in terms of ξ and η as a root of the quartic equation

$$4x^4 - 8\xi x^3 + (5\xi^2 + \eta^2 + 4a^2)x^2 - (\xi^2 + \eta^2 + 4a^2)\xi x + a^2\xi^2 = 0.$$

35. Two-dimensional sources are distributed along OX between A_1 $(a, 0)$ and A_2 $(-a, 0)$ with constant strength s per unit length. Show that the stream function at the point $P(x, y)$ is given by

$$\psi = -\frac{s}{2\pi}\left\{a(\theta_1 + \theta_2) - x(\theta_1 - \theta_2) + y \ln\left(\frac{r_2}{r_1}\right)\right\}$$

where r_1, θ_1 are the polar coordinates of P with A_1 as origin while r_2, θ_2 similarly refer to A_2.

36. A line source of constant strength per unit length extends from the origin O to $-\infty$ on the axis OX. Show that the streamlines lie in planes through OX and are confocal parabolas with O as common focus. (Use the Stokes stream function).

37. By general reasoning, show that the streamlines in the last Exercise must be homothetic curves with O as centre of similitude.

38. A straight line source of constant strength per unit length extends from A to B. Show that the streamlines are confocal hyperbolas with A and B as foci.

39. A line source of constant strength per unit length extends from O to $-\infty$ along OX and is situated in a uniform stream of velocity U in the negative direction of OX. Show that the streamlines lie in planes through OX and that one streamline consists of OX and a parabola with O as focus.

40. A Rankine ovoid is generated by placing a source of strength σ at $(-s, 0)$ and a sink of strength σ at $(s, 0)$ in a uniform stream of velocity U in the positive direction of OX. Show that the equation of the meridian curve of the ovoid can be put in the form

$$y^2 = \frac{\sigma}{2\pi U}\left(\frac{x+s}{r_1} - \frac{x-s}{r_2}\right)$$

where r_1, r_2 are the distances from the source and sink respectively. Show also that this equation can be written in the rationalised form

$$\lambda^2 {r_1}^4 {r_2}^4 (4s^4 - \lambda^2 y^4) - 4\lambda^2 s^4 {r_1}^2 {r_2}^2 y^2 (x^2 + y^2 + s^2) = 16 s^{10} x^2$$

where λ is the shape parameter $2\pi s^2 U/\sigma$ while

$${r_1}^2 = (x+s)^2 + y^2 \qquad \text{and} \qquad {r_2}^2 = (x-s)^2 + y^2.$$

41. Let the ends of the major axis (stagnation points) of a Rankine ovoid be on OX at $x = \pm a$. Show that $\lambda = 2as^3/(a^2 - s^2)^2$ (*see* last Exercise) and that the equation to the meridian curve can now be written

$$a^2 {r_1}^4 {r_2}^4 \{(a^2 - s^2)^4 - a^2 s^2 y^4\} - a^2 (a^2 - s^2)^4 {r_1}^2 {r_2}^2 y^2 (x^2 + y^2 + s^2) = (a^2 - s^2)^8 x^2.$$

Show also that when s tends to zero while a is constant the last equation yields $x^2 + y^2 = a^2$, so the ovoid becomes a sphere of radius a.

42. An infinite rectilinear vortex of strength Γ is perpendicular to the plane OXY and meets it at the point $(a, 0)$. The fluid is bounded by a plane fixed wall, parallel to the vortex, whose trace is OY. Find the velocity at the point (x, y).

$$\left(\text{Answer. } u = -\frac{2\Gamma axy}{\pi[(x-a)^2 + y^2][(x+a)^2 + y^2]}\right.$$

$$\left. v = \frac{\Gamma a(x^2 - y^2 - a^2)}{\pi[(x-a)^2 + y^2][(x+a)^2 + y^2]}.\right)$$

43. An infinite rectilinear vortex is situated inside a fixed circular cylinder and is parallel to its axis. Show that the image is a vortex of equal strength and reversed direction of spin through the point which is the inverse of the trace of the original vortex.

44. An infinite rectilinear vortex of strength Γ is parallel to the line of intersection of fixed perpendicular planes whose traces are OX, OY and the trace of the vortex is the point (x, y). Find the resultant induced velocity at the vortex and obtain the equation to its path on the assumption that it moves with the fluid.

$$\left(\text{Answer. } u = \frac{\Gamma x^2}{4\pi y(x^2 + y^2)} \qquad v = \frac{-\Gamma y^2}{4\pi x(x^2 + y^2)}.\right.$$

Path $a^2(x^2 + y^2) = x^2 y^2$ or $r \sin 2\theta = 2a$ where a is a constant of integration.)

(This is one species of Cotes' spiral).

45. In the last Exercise find the acceleration of the vortex in its path on the assumption that Γ is constant. (Answer. The acceleration is radial from the origin and of magnitude $3\Gamma^2/16\pi^2r^3$.)

46. Show that the image in a fixed circle of radius a of a radial two-dimensional doublet of strength μ at an external point S_1 distant b from the centre is a radial doublet, situated at the inverse point S_2, of strength $\mu a^2/b^2$ and with the sense of the axis reversed.

47. Show that the image in a fixed sphere of radius a of a radial doublet of strength μ at an external point S_1 distant b from the centre is a radial doublet, situated at the inverse point S_2, of strength $\mu a^3/b^3$ and with the sense of the axis reversed.

48. *Flow about a fixed ellipse in a uniform stream.* Use the elliptic coordinates defined by equations (2.15,36) with $c = \sqrt{(a^2 - b^2)}$. Then the ellipse $x^2/a^2 + y^2/b^2 = 1$ is the locus $\xi = \xi_0$ where $\cosh \xi_0 = a/c$ and $\sinh \xi_0 = b/c$. Verify that the stream function

$$\psi = U\sqrt{\left(\frac{a+b}{a-b}\right)}\,(a \sinh \xi - b \cosh \xi) \sin \eta$$

represents an irrotational flow about the fixed ellipse $\xi = \xi_0$. Show that at a great distance from the ellipse ($\xi \to \infty$) the flow is a uniform stream of velocity U in the negative direction of the axis OX.

49. *Components of velocity with general plane orthogonal coordinates in use.* The coordinates are α, β (*see* § 2.15). Show from the original definition of the stream function that, with due attention to the sense of the velocity, the component along the *normal* to the contour $\alpha = \alpha_0$ is

$$u_\alpha = -\frac{\partial\psi}{\partial\beta}\bigg/\frac{ds}{d\beta}$$

where $\partial\psi/\partial\beta$ is evaluated at the point considered and s is the arc of the contour $\alpha = \alpha_0$. Likewise

$$u_\beta = \frac{\partial\psi}{\partial\alpha}\bigg/\frac{ds}{d\alpha}$$

where s is now the arc of the contour $\beta = \beta_0$.

50. Use the results of Exercises 48 and 49 to find the velocity on the ellipse $x^2/a^2 + y^2/b^2 = 1$ when this is at rest in a uniform stream of velocity $-U$ parallel to OX. $\left(\text{Answer. } -\dfrac{U(a+b)\sin\eta}{\sqrt{(a^2\sin^2\eta + b^2\cos^2\eta)}}.\right)$

CHAPTER 3

ELEMENTS OF THE DYNAMICS OF INCOMPRESSIBLE FLUIDS

3.1 Introduction

In this chapter we consider in an elementary way the dynamics of uniform and incompressible fluids and we make full use of the kinematic techniques already explained in Chapter 2. One section (§ 3.10) of this chapter is devoted to the treatment of some simple problems of the motion of viscous incompressible fluids but elsewhere the fluid is supposed to be inviscid. Several of the discussions cover compressible fluids, but special attention is drawn to this in each instance.

3.2 The Basic Dynamical Principles

All the investigations of fluid motion given in this book are based on Newtonian dynamics and it is convenient to have an explicit statement of those principles which will be applied hereafter. These are:

(a) *The Law of Acceleration*

The resultant force applied to a particle is equal, as a vector, to the product of the mass of the particle and its acceleration.

(b) *The Principle of Linear Momentum*

The increase of the total linear momentum of a body or system in a given interval of time is equal, as a vector, to the total impulse of the *external* forces acting on the body or system during the interval. An equivalent statement is: The increment of the total linear momentum per unit time is equal to the resultant external force.

(c) *The Principle of Angular Momentum*

The increase of the total angular momentum, measured about any fixed axis, of a body or system in a given interval of time is equal to the total impulsive moment about the axis of the *external* forces acting on the body or system during the interval. An equivalent statement is: The increment of the total angular momentum about a fixed axis per unit time is equal to the resultant moment of the external forces about the axis.

(d) *The Principle of Energy*

The increase of the total energy of a body or system in a given interval of time is equal to the total work done on the body or system by the forces acting on it during the interval, plus the mechanical equivalent of the heat supplied during the interval.

(e) *The Principle of Relativity*

The phenomena occurring in any physical system are independent of any velocity, constant in time and space, which may be supposed to be applied to the whole system.

This implies that it is impossible to establish any absolute velocity by physical means. In the present context the principle is often applied where it is convenient to regard a particular body, which has a constant velocity, as being at rest by superposing a velocity equal and opposite to that of the body on the whole system.

It is beyond our scope to discuss these principles in detail, but we add here a few words of explanation. The law (a) is Newton's Second Law of Motion and it can be immediately applied to a finite body provided that all its particles have the same acceleration. This law is concerned with one instantaneous state of a particle, in contrast with the next three principles, which are all concerned with happenings in a finite interval of time and with bodies or systems of any kind. The reader may be reminded that the impulse of a constant force throughout an interval of time is defined to be the product of the force and the interval of time. Let X be the component of a force in some fixed direction. Then the component in this direction of the impulse of the force is

$$I_x = \int_{t_1}^{t_2} X\,dt, \tag{3.2,1}$$

where the time interval begins at the instant t_1 and ends at the instant t_2. For a single particle of mass m we have by the law of acceleration

$$m\frac{d^2x}{dt^2} = X, \tag{3.2,2}$$

where x is the displacement of the particle in the direction of X. When the last equation is integrated with respect to time throughout the interval $t_1 \ldots t_2$ we get

$$\left(m\frac{dx}{dt}\right)_{t=t_2} - \left(m\frac{dx}{dt}\right)_{t=t_1} = \int_{t_1}^{t_2} X\,dt = I_x \tag{3.2,3}$$

and this is the expression of the principle of linear momentum as it applies to a single particle. When we have to deal with a system we have merely to add the equations, similar to the last, for all the particles of the system. By Newton's Third Law of Motion, the force which a particle A of the system exerts on a particle B, also of the system, is equal and opposite to the force which B exerts on A. Hence these two forces give a total contribution to the impulse on the *system* which is zero. Thus the total impulse which appears on the right hand side of the equation of impulse for the system contains only the contributions of the *external* forces applied to the system. It is this fact which gives the Principle of Linear Momentum its great importance.

The angular momentum of a particle about a fixed axis is defined to be the product of its mass and the moment of its velocity about the axis. This is equal to the moment of its momentum, supposed localized at the particle, about the axis and *moment of momentum* is used as a synonym for *angular momentum*. For a single particle we obtain, on multiplication of both sides of equation (3.2,2) by the appropriate effective arm, that mass × moment of acceleration = moment of force.

It is a general theorem that the moment of the resultant of two vectors about any axis is equal to the sum of the moments of the vectors about the axis. This enables us to deduce from the last equation that for a particle,

increment of angular momentum
= mass × increment of moment of velocity
= impulsive moment of applied force.

When we apply this to a system we find, as before, that the internal forces give a zero total contribution and the Principle of Angular Momentum is established.[†]

In applying the Principle of Energy to the dynamics of fluids we have to take account of any interconversions of heat and mechanical energy. Such exchanges are of importance when the fluid changes its density (if compressible) and when heat is generated as the result of fluid friction. For a *perfect fluid*, which is incompressible and inviscid, there can be no interconversion of heat and mechanical energy and the Principle of Energy takes the simple form that the gain of kinetic energy is equal to the work done by the body forces on the portion of fluid considered plus the work done by the pressures acting over its surface. When, as usual, the body forces are derivable from a potential, the principle can be stated as follows:

(The gain in the sum of the kinetic and potential energies of a portion of fluid is equal to the work done by the pressures acting over its surface. The pressures acting over any stationary surface give a zero contribution to the work.)

Example 1. *Jet reaction*

Suppose that a jet of fluid issues from a reservoir (which is at rest) with velocity v. Let A be the sectional area of the jet and ρ the density of the fluid. Then the mass of fluid discharged in unit time is ρvA and its momentum is ρv^2A. This is the rate at which the system consisting of the reservoir and contained fluid is losing momentum in the direction of the jet. Consequently the thrust in the direction opposite to that of the jet is

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[†] Since the principle of angular momentum is valid for *any* fixed axis it implies the principle of linear momentum. This follows on taking the difference of the results for a pair of parallel axes.

$$T = \rho v^2A.$$

Example 2. Jet impact and deflection

Suppose that a jet of sectional area A, speed v and density ρ hits a fixed vane and is deflected. The mass of fluid affected in unit time is ρvA and the momentum gained by the fluid in unit time is ρvA multiplied by the vector difference of the velocities after and before impact. Therefore the thrust on the vane is (as a vector)

$$\mathbf{T} = \rho vA\,(\mathbf{V}_1 - \mathbf{V}_2)$$

where $\mathbf{V}_1$, $\mathbf{V}_2$ are the velocity vectors before and after impact.

3.3 Pressure at a Point in a Moving Fluid

There are no shearing stresses (see § 1.2) in an inviscid fluid, even when it is in motion. This enables us to prove that for such a fluid the pressure is the same on all planes at a point at a given instant. The proof of this theorem given in § 1.2 for the case of a fluid at rest can be immediately adapted to the more general case of an inviscid fluid in motion by regarding the body force g as the resultant of the actual body force and the reversed acceleration of the fluid at the point. Otherwise, the mass-acceleration of the wedge-shaped element is proportional to l^3 whereas the forces due to the pressures are proportional to l^2. Hence, in the limit when l tends to zero, the equilibrium of the element is uninfluenced by the acceleration, and the equality of the pressures on the base and inclined face of the element follows.

Although it is beyond our scope to give a general discussion of the stresses in a viscous fluid in motion, the reader's attention is drawn to the fact that the normal stress on a plane at a point in such a fluid is not, in general, independent of the direction of the normal to the plane. The quantity called the "pressure" in the theory of viscous fluids and given the symbol p is the mean of the normal compressive stresses on three mutually perpendicular planes through the point considered.† The theory shows that there is a unique normal stress at a point when

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0. \tag{3.3,1}$$

This is true for simple shearing flow (see § 3.10) and always nearly true for the flow very close to the surface of an immersed solid.

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† The reader may consult Lamb's *Hydrodynamics*, Chap. XI, for the theory. See also § 11.1.

3.4 Bernoulli's Theorem

We shall now investigate the distribution of pressure in a moving inviscid and incompressible uniform fluid and shall begin with the case where the flow is *steady* (see § 2.2). The result of this first investigation is summed up in the very important *Theorem of Bernoulli.*

Proof of Bernoulli's theorem

Consider a short and very slender cylindrical element of fluid whose axis is parallel to the flow at the element. Let the normal cross-sectional area of the cylinder be A and let its length be ds, where s is the arc of the streamline on which the element lies, measured from a fixed point on it. The pressures on the curved surface of the cylinder contribute nothing to the resultant force in the direction of motion. The thrust on the rear end of the element, and therefore in the direction of motion, is pA while the opposing thrust on the other end is $(p + (\partial p/\partial s)\, ds)A$. The resultant forward thrust is therefore $-(\partial p/\partial s)\, A ds$. Next, let the component of the body force in the direction of motion be f_s. Then the total body force on the element which is of mass $\rho A\, ds$, is $f_s \rho A\, ds$. When we equate the total propelling force to the mass-acceleration of the element we get

$$a_t \rho A\, ds = f_s \rho A\, ds - \frac{\partial p}{\partial s} A\, ds,$$

where a_t is the component of acceleration of the fluid in the direction of motion. The last equation can be rewritten

$$a_t + \frac{1}{\rho}\frac{\partial p}{\partial s} - f_s = 0. \tag{3.4,1}$$

But we have by equation (2.9,1)

$$a_t = \frac{1}{2}\frac{\partial q^2}{\partial s} \tag{3.4,2}$$

while

$$f_s = -\frac{\partial \chi}{\partial s} \tag{3.4,3}$$

on account of the relation between a body force and its potential. Hence equation (3.4,1) becomes

$$\frac{\partial}{\partial s}\left(\tfrac{1}{2}q^2 + \frac{p}{\rho} + \chi\right) = 0 \tag{3.4,4}$$

since ρ is constant. The rate of change of the quantity inside the bracket along the streamline is zero so for any streamline

$$\tfrac{1}{2}q^2 + \frac{p}{\rho} + \chi = \text{constant}. \tag{3.4,5}$$

In practice the body force is nearly always gravity and the expression for the potential is then

$$\chi = gz \tag{3.4,6}$$

where z is the height above some fixed horizontal datum plane. Bernoulli's theorem now takes the form

$$\tfrac{1}{2}q^2 + \frac{p}{\rho} + gz = \text{constant}. \tag{3.4,7}$$

The last equation can be written alternatively

$$\frac{q^2}{2g} + \frac{p}{w} + z = \text{constant} \tag{3.4,8}$$

where $w = g\rho$ is the specific weight (weight per unit volume) of the fluid. All the quantities appearing in the last equation have the physical dimension of length (see Chapter 4) and are known in hydraulics as *heads*. Thus $q^2/2g$ is called the *velocity head* while p/w is the *pressure head*. Accordingly, Bernoulli's theorem is sometimes expressed in such applications in the form:

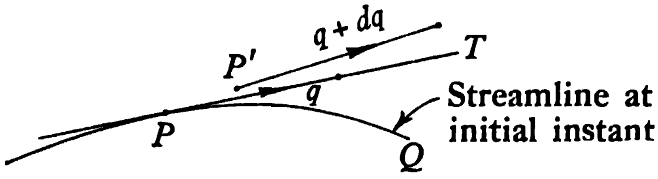

Fig. 3.4,1. Diagram to illustrate tangential acceleration in unsteady flow.

$$\text{velocity head} + \text{pressure head} + \text{position head} = \text{total head} \tag{3.4,9}$$

where *the total head is constant along any streamline*. The symbols appearing in equation (3.4,7) are sometimes regarded as quantities of energy (or work) *per unit mass* of the fluid. The British unit for such quantities is the *foot pound force per slug*, the SI unit is the metre Newton (or Joule) per kilogramme. However, the heads which appear in equation (3.4,8) can be regarded as quantities of energy (or work) *per unit weight* of the fluid; these would be measured in feet (British units) or metres (SI units).

Bernoulli's theorem can be extended to cover *unsteady motions* as follows. We continue to use equation (3.4,1) but we now have a modified expression for the component a_t of the acceleration in the direction of motion, namely

$$a_t = \frac{1}{2}\frac{\partial q^2}{\partial s} + \frac{\partial q}{\partial t} \tag{3.4,10}$$

where $\partial q/\partial t$ is the time rate of increase of the resultant velocity at a fixed point. To prove the last equation let us consider the diagram (Fig. 3.4,1), where PQ is a streamline drawn at the instant t. In the interval dt the fluid particle which was at P moves to a point P' where $PP' = ds = q\,dt$. Now the velocity of the particle at P' is $(q + dq)$ inclined at the angle $d\phi$ to the tangent PT, where $d\phi$ is a small quantity of the first order. Accordingly, when we neglect small quantities of the second order, the component of the velocity at P' in the direction of the tangent PT is $(q + dq)$. Thus the tangential component of acceleration is

$$a_t = \frac{dq}{dt}$$

while

$$dq = \frac{\partial q}{\partial s}\,ds + \frac{\partial q}{\partial t}\,dt$$

since, for points on a given curve,† q is a function of the two variables s and t. Hence

$$a_t = \frac{ds}{dt}\frac{\partial q}{\partial s} + \frac{\partial q}{\partial t} \quad = q\frac{\partial q}{\partial s} + \frac{\partial q}{\partial t} \quad = \frac{1}{2}\frac{\partial q^2}{\partial s} + \frac{\partial q}{\partial t}$$

in accordance with (3.4,10). Equation (3.4,1) can now be written, in view of (3.4,3) and (3.4,10),

$$\frac{\partial}{\partial s}\left(\tfrac{1}{2}q^2 + \frac{p}{\rho} + \chi\right) = -\frac{\partial q}{\partial t}. \tag{3.4,11}$$

When we integrate this along the streamline we get

$$\tfrac{1}{2}q^2 + \frac{p}{\rho} + \chi = C - \int_0^s \frac{\partial q}{\partial t}\,ds \tag{3.4,12}$$

where C is the value of $[\frac{1}{2}q^2 + (p/\rho) + \chi]$ at the reference point $s = 0$. Whenever q and $\partial q/\partial t$ are known the last equation can be used to find the pressure. Applications are easy to problems of the accelerated motion of incompressible fluids in rigid pipes or slender ducts where, by continuity, the velocities at all points must vary in proportion. Thus on account of continuity

$$qA = q_0A_0 \tag{3.4,13}$$

where the cross-sectional areas A and A_0 are functions of s but independent of t. Hence

$$\frac{\partial q}{\partial t} = \frac{A_0}{A}\frac{dq_0}{dt}, \tag{3.4,14}$$

so

$$\int_0^s \frac{\partial q}{\partial t}\,ds = A_0\frac{dq_0}{dt}\int_0^s \frac{ds}{A}, \tag{3.4,15}$$

and the integral on the right can be calculated from the geometrical particulars of the duct. Equation (3.4,12) accordingly becomes

$$\tfrac{1}{2}q^2 + \frac{p}{\rho} + \chi = \tfrac{1}{2}q_0^2 + \frac{p_0}{\rho} + \chi_0 - A_0\frac{dq_0}{dt}\int_0^s \frac{ds}{A}. \tag{3.4,16}$$

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† P' is not exactly on the streamline but its distance from the streamline is of second order and negligible in the limit.

When the pressures at the two ends of the duct are given as functions of the time this becomes a first-order differential equation for the determination of q_0, in view of (3.4,13).

The above proof of Bernoulli's Theorem has been based on momentum considerations. The form of equation (3.4,5) suggests however that it should be possible to derive it from energy considerations. This is certainly so but the derivation requires a full consideration of the thermodynamics of the flow if mistakes are to be avoided. This point is developed in some detail in § 9.3.

ILLUSTRATIVE EXAMPLES OF BERNOULLI'S THEOREM

Example 1. *Discharge from a reservoir under gravity*

We consider a reservoir containing liquid with its surface exposed to the atmosphere. A small opening is provided at a vertical height h below the free surface and the liquid escapes through this to atmosphere (atmospheric pressure p_0 assumed to be equal to that at the free surface). The reservoir and the free surface are so large that the velocity of the liquid at the free surface is very small. Since the orifice is small, the free surface sinks very slowly and the motion at any instant can be regarded as steady.

We consider a streamline starting at a point A in the free surface and passing through the orifice. The point B on the streamline is just outside the orifice and the pressure at B is atmospheric. Take the datum level at B. Then we have at A:

$$q = 0, \quad p = p_0, \quad z = h,$$

while at B:

$$p = p_0, \quad z = 0.$$

We apply Bernoulli's theorem and find that the velocity q at B is given by the equation

$$\tfrac{1}{2}q^2 + \frac{p_0}{\rho} = \frac{p_0}{\rho} + gh,$$

or

$$q = \sqrt{(2gh)}. \tag{3.4,17}$$

This velocity is that which would be acquired by a particle falling freely from rest through the height h under gravity (Torricelli's theorem). The velocity is quite independent of the *direction* in which the liquid escapes; thus the orifice may, for example, be in the bottom of the reservoir, in a vertical side or opening upwards from a horizontal arm of the reservoir. In practice the velocity of escape of a liquid of low viscosity, such as water, is only slightly less than that given by (3.4,17). The calculation of the volume discharged in unit time is a much more difficult problem, in general, on account of the usual formation of a *vena contracta* (see further § 8.6).

Example 2. The Venturi meter

The Venturi meter is inserted in a pipe line to measure the velocity of flow or rate of discharge. Constructional and operational details are discussed in § 5.10, together with the influence of viscosity and of irregularities in the flow of the fluid approaching the meter. Here we give the theory in outline and regard the flow as uniform in the approaching stream and at the throat of the instrument.

Essentially the Venturi meter consists of a converging cone whose larger (upstream) end registers with the supply pipe and whose downstream end merges into a parallel throat where the cross-sectional area is a minimum; this is followed by a diverging cone or diffuser whose downstream end registers with the pipe. We shall, for simplicity, take the axis of the instrument to be horizontal and shall consider the streamline which coincides with the axis. Let A_1 be the cross-sectional area of the supply pipe and A_2 that at the throat; also let the velocity and pressure at the upstream end of the converging cone be q_1 and p_1 respectively, while q_2 and p_2 are the corresponding quantities at the throat. By Bernoulli's theorem we have, since the streamline is horizontal,

$$\tfrac{1}{2}q_2^2 + \frac{p_2}{\rho} = \tfrac{1}{2}q_1^2 + \frac{p_1}{\rho}.$$

But

$$q_2 A_2 = q_1 A_1$$

by continuity, so the first equation becomes after reduction

$$q_1^2 = \frac{2A_2^2(p_1 - p_2)}{\rho(A_1^2 - A_2^2)}. \tag{3.4,18}$$

Hence the volume of fluid passing in unit time is

$$Q = q_1 A_1 = \sqrt{\left(\frac{2A_1^2 A_2^2(p_1 - p_2)}{\rho(A_1^2 - A_2^2)}\right)}. \tag{3.4,19}$$

Now put $p_1 = g\rho h_1$ and $p_2 = g\rho h_2$. Then (3.4,19) becomes

$$Q = \sqrt{\left(\frac{2gA_1^2 A_2^2(h_1 - h_2)}{(A_1^2 - A_2^2)}\right)}. \tag{3.4,20}$$

It will be seen that for a given instrument the rate of discharge is equal to the square root of the difference of pressure head across the converging cone, multiplied by a constant.

3.5 The Dynamical Equations

We shall now establish the general dynamical equations of an inviscid fluid. In deriving the equations we shall allow the fluid to be compressible since this requires no elaboration of the proof.

In proving Bernoulli's theorem from the Law of Acceleration in § 3.4 we considered the forces on a very slender cylinder of fluid, of length ds and cross-sectional area A. We shall now take the axis ds to be in *any* selected direction at a point in the fluid and at the instant t. Let a_s be the component of acceleration and f_s the component of body force in the direction of ds. Then, by the same argument as before, we deduce (compare equation 3.4,1)

$$a_s = f_s - \frac{1}{\rho}\frac{\partial p}{\partial s}. \qquad (3.5,1)$$

Now let us identify ds in succession with the elements dx, dy, dz, which are parallel to the coordinate axes, and let the corresponding components of body force be X, Y, Z. We now derive from (3.5,1) the three dynamical equations

$$a_x = X - \frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad a_y = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad a_z = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (3.5,2)$$

When we substitute for the accelerations their values as given by equations (2.9,4) to (2.9,6) we obtain

$$\begin{aligned}
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} &= X - \frac{1}{\rho}\frac{\partial p}{\partial x} \\
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (3.5,3) \\
u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z}.
\end{aligned}$$

These are known as Euler's dynamical equations. The reader should note that we have not assumed ρ to be constant, nor have we assumed the motion to be steady or irrotational.

If we now assume that the body forces have the potential χ we have

$$X = -\frac{\partial \chi}{\partial x}, \qquad Y = -\frac{\partial \chi}{\partial y}, \qquad Z = -\frac{\partial \chi}{\partial z}. \qquad (3.5,4)$$

Let us also suppose that during the motion the density is either constant or a function of the pressure only; the fluid is then said to be *barotropic*. Put

$$= \int \frac{dp}{\rho} \qquad (3.5,5)$$

so ϖ is a function of p only. Then

$$d\varpi = \frac{dp}{\rho}$$

and

$$\frac{\partial \varpi}{\partial x} = \frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{\partial \varpi}{\partial y} = \frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad \frac{\partial \varpi}{\partial z} = \frac{1}{\rho}\frac{\partial p}{\partial z}. \qquad (3.5,6)$$

When the fluid is uniform and incompressible we have from (3.5,5)

$$\varpi = \frac{p}{\ }. \tag{3.5,7}$$

Euler's equations can now be written

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} &= -\frac{\partial}{\partial x}(\varpi + \chi) \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} &= -\frac{\partial}{\partial y}(\varpi + \chi) \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} &= -\frac{\partial}{\partial z}(\varpi + \chi). \end{aligned} \tag{3.5,8}$$

The equations (3.5,8) can be integrated whenever the flow is *irrotational* so that a velocity potential ϕ exists (see § 2.12). We now have

$$\begin{gathered} u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z} \\ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \quad \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z}. \end{gathered} \tag{3.5,9}$$

Hence

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} \\ &= \frac{1}{2}\frac{\partial}{\partial x}(u^2 + v^2 + w^2) \\ &= \frac{1}{2}\frac{\partial q^2}{\partial x} \end{aligned} \tag{3.5,10}$$

where q is the resultant velocity. Also

$$\frac{\partial u}{\partial t} = -\frac{\partial^2 \phi}{\partial t\,\partial x} = -\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right). \tag{3.5,11}$$

The first of equations (3.5,8) can accordingly be written

$$\frac{\partial}{\partial x}\left(\tfrac{1}{2}q^2 - \frac{\partial \phi}{\partial t}\right) = -\frac{\partial}{\partial x}(\varpi + \chi)$$

or

$$\frac{\partial}{\partial x}\left(\tfrac{1}{2}q^2 + \varpi + \chi - \frac{\partial \phi}{\partial t}\right) = 0. \tag{3.5,12}$$

In a similar manner we can deduce from the remaining two of equations (3.5,8) that the partial derivatives of the bracketed expression with respect to y and z are also zero.† Hence this expression is independent of the spatial coordinates and is either constant or some function of time alone. Thus, finally,

$$\tfrac{1}{2}q^2 + \varpi + \chi - \frac{\partial \phi}{\partial t} = F(t), \tag{3.5,13}$$

where $F(t)$ is a function of time or constant which must be found from the boundary conditions. This is the general pressure equation for the irrotational flow of a barotropic and inviscid fluid. When the motion is steady $\partial\phi/\partial t$ is zero and the expression on the left of (3.5,13) is independent of time. Hence *for steady irrotational motion* we have

$$\tfrac{1}{2}q^2 + \varpi + \chi = \text{constant}. \tag{3.5,14}$$

The last equation is more general than Bernoulli's theorem in the sense that the quantity on the left is constant throughout the whole fluid and not just on any given streamline, but it is less general in the sense that the motion is restricted to be irrotational.

When the motion is *steady* some results of interest are obtained by considering the components of acceleration in the following mutually perpendicular directions at any point in the fluid:

1. the direction of motion, i.e. along the tangent to a streamline which is also the path of a particle,
2. the principal normal to the streamline,
3. the binormal to the streamline.

The reader may be reminded that the principal normal to a curve is that normal which lies in the osculating plane while the binormal is perpendicular to this plane.‡ Let

n_1 = distance measured along the principal normal from the streamline on its convex side,
n_2 = distance measured along the binormal,

while s is measured along the streamline. In equations (3.5,2) substitute s, n_1 and n_2 for x, y and z respectively. Then we have

$$a_x = a_s = \frac{1}{2}\frac{\partial q^2}{\partial s}, \qquad a_y = a_{n1} = -\frac{q^2}{R_c}, \qquad a_z = a_{n2} = 0 \tag{3.5,15}$$

where R_c is the radius of curvature of the streamline. Accordingly the dynamical equations become

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† This also follows at once by cyclic interchange of the spatial coordinates (see footnote in § 2.11).

‡ For two-dimensional flow the osculating plane is the plane of flow and the principal normal is the normal lying in this plane. The binormal is perpendicular to the plane of flow.

$$\frac{1}{2}\frac{\partial q^2}{\partial s} = -\frac{\partial}{\partial s}(\varpi + \chi), \ -\frac{q^2}{R_c} = -\frac{\partial}{\partial n_1}(\varpi + \chi), 0 = -\frac{\partial}{\partial n_2}(\varpi + \chi). \tag{3.5,16}$$

The first of these equations leads to Bernoulli's theorem, as we have already seen, while the remaining two equations determine the rate of change of pressure in directions perpendicular to the direction of motion. We find from (3.5,14) by differentiation

$$\frac{1}{2}\frac{\partial q^2}{\partial n_1} = -\frac{\partial}{\partial n_1}(\varpi + \chi)$$

and the second of equations (3.5,16) now yields $\frac{1}{2}\frac{\partial q^2}{\partial n_1} = -\frac{q^2}{R_c}$

or

$$\frac{\partial q}{\partial n_1} = -\frac{q}{R_c}. \tag{3.5,17}$$

This equation is valid for *irrotational flow only* since (3.5,14) is so restricted.

The dynamical equations (3.5,8) can be written

$$a_x = -\frac{\partial}{\partial x}(\varpi + \chi), a_y = -\frac{\partial}{\partial y}(\varpi + \chi), a_z = -\frac{\partial}{\partial z}(\varpi + \chi). \tag{3.5,18}$$

Readers acquainted with vector analysis will recognise that these equations can be represented by the single vector equation

$$\mathbf{A} = -\text{grad}\,(\varpi + \chi), \tag{3.5,19}$$

where **A** is the acceleration vector. These equations show that the expression $(\varpi + \chi)$ is a potential function for the acceleration. Some notes on the *acceleration potential* are given in the Appendix to this chapter. It follows from equations (3.5,18) that

$$\frac{\partial a_x}{\partial y} = \frac{\partial a_y}{\partial x}, \quad \frac{\partial a_y}{\partial z} = \frac{\partial a_z}{\partial y}, \quad \frac{\partial a_z}{\partial x} = \frac{\partial a_x}{\partial z} \tag{3.5,20}$$

For *steady motion* each component of acceleration is expressible in terms of the components of velocity and their partial derivatives with respect to the spatial coordinates. Hence it appears that an arbitrary set of velocity components u, v, w will not, in general, be possible in a fluid with body forces possessing a potential.† For an *incompressible and inviscid fluid* the components of velocity must meet the following conditions:

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† But a set of values of $\partial u/\partial t$, $\partial v/\partial t$ and $\partial w/\partial t$ can be found at any one instant to satisfy the equations when the motion is not restricted to be steady.

(a) the equation of continuity (2.3,9) must be satisfied.

(b) the conditions of compatibility (3.5,20) must be satisfied.

(c) the conditions at the boundaries must be satisfied.

It follows from the argument which leads to equation (3.5,13) that the conditions of compatibility are always satisfied when the motion is irrotational. Equations (3.5,20) are not linear in u, v, w so the vector resultant of two possible motions is not, in general, a possible motion. However, since the resultant of two irrotational motions is also irrotational, such motions may always be added. It can be shown that the conditions of compatibility are satisfied by the sum of any irrotational motion and a vortex motion which itself satisfies the conditions of compatibility. When the fluid is compressible there are further conditions to be met, for the density, which appears in the equation of continuity and in the dynamical equations, depends on the pressure.

ILLUSTRATIVE EXAMPLES ON THE DYNAMICAL EQUATIONS

Example 1. *Fluid at rest in equilibrium*

The dynamical equations (3.5,8) are satisfied by a steady state of rest when

$$\varpi + \chi = \text{constant}, \tag{3.5,21}$$

in agreement with equation (1.6,19).

Example 2. *Bernoulli's theorem for a barotropic fluid*

The first of equations (3.5,16) yields

$$\frac{\partial}{\partial s}(\tfrac{1}{2}q^2 + \varpi + \chi) = 0$$

and the expression within the brackets is therefore constant along any streamline when the motion is steady.

Example 3. *Pressure distribution in Hill's spherical vortex*

By Example 3 of § 2.11 the components of velocity are

$$u = A(x^2 + 2y^2 + 2z^2 - a^2),$$
$$v = -Axy, \quad w = -Axz.$$

We take A and a to be constants, so the motion is steady, and we find from (2.9,4) . . . (2.9,6) that

$$a_x = 2A^2x(x^2 - a^2)$$
$$a_y = -A^2y(2y^2 + 2z^2 - a^2)$$
$$a_z = -A^2z(2y^2 + 2z^2 - a^2).$$

The conditions of compatibility (3.5,20) are satisfied and the dynamical equations (3.5,18) are all satisfied by

$$\begin{aligned}\frac{p}{\rho} + \chi &= C + \tfrac{1}{2}A^2a^2(2x^2 - y^2 - z^2) - \tfrac{1}{2}A^2\{x^4 - (y^2 + z^2)^2\} \\ &= C + \tfrac{1}{2}A^2\{a^2(3x^2 - r^2) - r^2(2x^2 - r^2)\}\end{aligned}$$

where C is a constant and $r^2 = x^2 + y^2 + z^2$. On the sphere $r = a$

$$\frac{p}{\rho} + \chi = C + \tfrac{1}{2}A^2a^2x^2.$$

3.6 The Equations of Momentum

Let us consider the fluid which, at a certain instant t, is enclosed by a surface S (see Fig. 3.6,1). For the sake of generality we shall suppose that S also encloses a fixed rigid body B (or several such bodies). The fluid is frictionless but may be compressible. The component of momentum of the enclosed fluid in the direction OX is

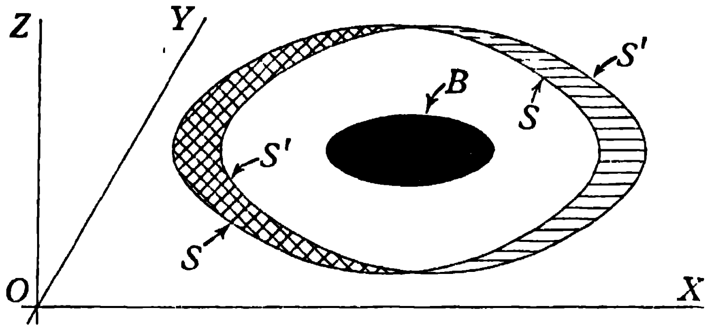

Fig. 3.6,1. Figure to illustrate application of the principle of momentum. (Surface S moves with fluid to S' in time dt.)

$$= \iiint \rho u \, dx \, dy \, dz \tag{3.6,1}$$

where the triple integral extends throughout the volume enclosed by S but excluding that occupied by B. At the later instant $t + dt$ the external surface of the mass of fluid considered is S'. Hence the momentum of this fluid is now

$$I_x + dI_x = \iiint \left[\rho u + \frac{\partial}{\partial t}(\rho u)\, dt\right] dx \, dy \, dz$$

+ integral of the same expression in the shaded region

— integral of the same expression in the cross hatched region.

(3.6,2)

Consider an element dS of the surface S and let the direction cosines of the normal drawn outwards from S be l, m, n. The corresponding part of the shaded border region is a small oblique cylinder whose base is dS and whose slant height is $q\,dt$, where q is the resultant velocity at the element. The volume of the element is

$$\begin{aligned} dV &= dS \times \text{normal height of element} \\ &= dS(lu + mv + nw)\,dt \end{aligned}$$

since the normal height is equal to the normal component of velocity multiplied by dt. This expression can be used for the *whole* border region, for the shaded part is that for which dV is positive while the cross-hatched part is that for which dV is negative. Accordingly the net contribution of the border regions to the increase of momentum is

$$dt \iint (lu + mv + nw)\left[\rho u + \frac{\partial}{\partial t}(\rho u)\,dt\right] dS$$
$$= dt \iint \rho u(lu + mv + nw)\,dS \qquad (3.6,3)$$

when we neglect the term in dt^2; the surface integral covers the entire surface S. When we subtract (3.6,1) from (3.6,2) and use (3.6,3) we get

$$dI_x = dt \iiint \frac{\partial}{\partial t}(\rho u)\,dx\,dy\,dz + dt \iint \rho u(lu + mv + nw)\,dS. \quad (3.6,4)$$

By the Principle of Linear Momentum (see § 3.2) this increase of momentum is equal to the impulse of the forces acting on the fluid in the direction OX during dt. Let F_x be the force in the direction OX which the fluid exerts *on* the body B; then $-F_x$ is the force which B exerts on the fluid. Accordingly the total component of impulse is

$$dt \iiint \rho X\,dx\,dy\,dz - dt \iint lp\,ds - dtF_x \qquad (3.6,5)$$

where the volume integral gives the impulse of the body forces and the surface integral that of the pressures over S, while second order small quantities have been neglected as before. We now get from (3.6,4) and (3.6,5) after rearrangement

$$\iiint \left[\rho X - \frac{\partial}{\partial t}(\rho u)\right] dx\,dy\,dz$$
$$= \iint \rho u(lu + mv + nw)\,dS + \iint lp\,dS + F_x. \quad (3.6,6)$$

Two similar equations can be obtained from the last by cyclic interchange of the symbols (see footnote in § 2.11). When the motion is steady and body forces are absent the equation becomes

$$F_x = -\iint \rho u(lu + mv + nw)\,dS - \iint lp\,dS \tag{3.6,7}$$

and there are two similar equations obtained by cyclic interchange. These results have useful applications in the investigation of the forces on bodies immersed in a stream of fluid. The point of interest and importance is that the force on the body has been expressed in terms of integrals taken over an *arbitrarily chosen* surface enclosing the body and lying entirely within the fluid. As a particular case we may take S to coincide with the surface of the body and then (3.6,6) becomes

$$F_x = -\iint lp\,dS \tag{3.6,8}$$

for

$$lu + mv + nw = 0 \tag{3.6,9}$$

at the surface of a fixed rigid body and the triple integral vanishes since the volume enclosed between S and B is zero. Equation (3.6,8) is obviously true.

Equation (3.6,6) and the others deduced from it remain valid for a viscous fluid, provided only that the viscous stresses over the surface S are negligible; the viscous stresses at the surface of the body B need not be zero and their effect is included in the force F. This fact has very important applications.

We shall now apply the Principle of Angular Momentum to the fluid contained in the surface S and we shall take moments about the axis OX. Let M_x be the resultant moment about OX of the forces which the fluid exerts on the body B, so $-M_x$ is the moment which the body exerts on the fluid. At the instant t the moment of momentum of the fluid within S, about the axis OX, is

$$\iiint \rho(wy - vz)\,dx\,dy\,dz. \tag{3.6,10}$$

Using the same method as before, we find that the time rate of increase of the moment of momentum about OX of the mass of fluid considered is

$$\iiint \frac{\partial}{\partial t}[\rho(wy - vz)]\,dx\,dy\,dz + \iint \rho(wy - vz)(lu + mv + nw)\,dS. \tag{3.6,11}$$

This is equal to the total moment about OX of the forces applied to the mass of fluid considered, namely

$$\iiint \rho(Zy - Yz)\,dx\,dy\,dz - \iint p(ny - mz)\,dS - M_x. \tag{3.6,12}$$

Hence we obtain, after rearrangement,

$$\iiint \left\{ \rho(Zy - Yz) - \frac{\partial}{\partial t}[\rho(wy - vz)] \right\} dx\,dy\,dz$$
$$= \iint \rho(wy - vz)(lu + mv + nw)\,dS + \iint p(ny - mz)\,dS + M_x, \tag{3.6,13}$$

and two similar equations are obtained by cyclic interchange of the symbols. When the motion is steady and body forces are absent we get

$$M_x = -\iint \rho(wy - vz)(lu + mv + nw)\,dS - \iint p(ny - mz)\,dS \tag{3.6,14}$$

and there are two other equations obtained by cyclic interchange.

A closed surface S such as that employed in the foregoing argument is often called a *control surface.*

Equation (3.6,6) can be rewritten

$$\iint \rho u(lu + mv + nw)\,dS + \frac{\partial}{\partial t}\iiint \rho u\,dx\,dy\,dz$$
$$= \iiint \rho X\,dx\,dy\,dz - \iint lp\,dS - F_x.$$

This and the two similar equations obtained by cyclic interchange lead to the following proposition:

The rate of efflux of linear momentum from a closed fixed surface increased by the rate of increase of the momentum of the enclosed fluid is vectorially equal to the resultant of the applied forces, including the thrust of the pressures at the surface.

A similar proposition applies to the angular momentum.

ILLUSTRATIVE EXAMPLES ON THE EQUATIONS OF MOMENTUM

Example 1. *Deduction of the dynamical equations from the equations of momentum*

Choose the surface S so that it does not contain an immersed body B, so F_x is zero. Then, in equation (3.6,6) replace the surface integrals by volume integrals taken throughout the region enclosed by S, by the application of Green's theorem. We then get

$$\iiint \Delta\,dx\,dy\,dz = 0$$

where

$$\Delta = \rho X - \frac{\partial}{\partial t}(\rho u) - \frac{\partial}{\partial x}(\rho u^2) - \frac{\partial}{\partial y}(\rho uv) - \frac{\partial}{\partial z}(\rho uw) - \frac{\partial p}{\partial x}.$$

Since this is true for an arbitrary surface S the integrand must vanish. Hence

$$\frac{\partial}{\partial x}(\rho u^2) + \frac{\partial}{\partial y}(\rho uv) + \frac{\partial}{\partial z}(\rho uw) + \frac{\partial}{\partial t}(\rho u) = \rho X - \frac{\partial p}{\partial x}. \quad (3.6,15)$$

This is equivalent to

$$u\left\{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t}\right\}$$
$$+ \rho\left\{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}\right\} = \rho X - \frac{\partial p}{\partial x}$$

and the coefficient of u vanishes by the equation of continuity (2.3,7). Then, on division by ρ, we obtain the first of the dynamical equations (3.5,3); the remaining two of these follow by cyclic interchange. The equation (3.6,15) is worthy of note.

3.7 Kelvin's Circulation Theorem

The circulation in a circuit which is imagined to be described in a fluid has been defined in § 2.10 and the reader is reminded that this quantity is, by definition, evaluated at one chosen instant. At any later instant the fluid particles which lay on the circuit will lie on a circuit which, in general, is distinct from the original one.† Further, the circulation will, in general, have a value differing from the original value. However, Kelvin showed that the circulation remains constant when the three following conditions are satisfied:

(a) the fluid is inviscid,

(b) the fluid is barotropic, i.e. the density is constant or a function of the pressure alone during the motion,

(c) the body forces, if any, are derivable from a single-valued potential.

Conditions (b) and (c) are satisfied in many important cases; in particular (b) is true of a homogeneous liquid which can be treated as incompressible and for the isentropic flow of a gas while (c) is true of the gravitational field. As regards (a), no real fluid is inviscid, but a complete investigation shows that the circulation in a circuit moving with a viscous fluid only changes through the diffusion of vorticity into or out of the circuit (subject to the satisfaction of (b) and (c)) and in many important and easily distinguishable cases this effect is negligible; such diffusion can only occur when there is vorticity in the fluid *where the circuit lies.* The theorem, as will be shown, leads to conclusions of great importance. It is, however, not useful when the flow *at the circuit* is turbulent, as it is then impracticable to trace the movements of the particles which lay on the circuit at a given instant.

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† In all our discussions we assume that cavitation does not occur. If cavitation occurred between the two instants considered the particles might not lie on a complete circuit at the later instant.

The circulation in the circuit under consideration is (see equation 2.10,11)

$$\Gamma = \oint (u\,dx + v\,dy + w\,dz) \tag{3.7,1}$$

and our object is to calculate the rate of change of Γ with time as the particles composing the circuit move with the fluid. Let PQ be an element of the circuit which has projections dx, dy, dz on the coordinate axes. We wish to calculate the time rate of change of the flow from P to Q and we shall use the expression for the rate of change of any quantity appertaining to the moving fluid given by equation (2.9,3) with the abbreviated notation (2.9,7). Then

$$\frac{D(u\,dx)}{Dt} = dx\frac{Du}{Dt} + u\frac{D\,dx}{Dt} \tag{3.7,2}$$

since D/Dt is a linear operator. Now $D\,dx/Dt$ is the rate at which the length of the projection of PQ on OX changes with time. Hence

$$\frac{D\,dx}{Dt} = u_Q - u_P = du \tag{3.7,3}$$

and (3.7,2) becomes

$$\frac{D(u\,dx)}{Dt} = dx\frac{Du}{Dt} + u\,du = dx\left(X - \frac{1}{\rho}\frac{\partial p}{\partial x}\right) + u\,du$$

by the first of the dynamical equations (3.5,3). *We shall now suppose that the body forces have a single valued potential and that the fluid is barotropic.* Then by (3.5,4) and (3.5,5) the last equation can be written

$$\frac{D(u\,dx)}{Dt} = -dx\frac{\partial}{\partial x}(\chi + \varpi) + u\,du$$

and there are two similar equations obtained by cyclic interchange of the symbols. Hence we get by addition

$$\begin{aligned}\frac{D}{Dt}(u\,dx + v\,dy + w\,dz) &= -dx\frac{\partial}{\partial x}(\chi + \varpi) - dy\frac{\partial}{\partial y}(\chi + \varpi)\\ &\quad - dz\frac{\partial}{\partial z}(\chi + \varpi) + u\,du + v\,dv + w\,dw\\ &= -d(\chi + \varpi) + \tfrac{1}{2}\,dq^2 \end{aligned}\tag{3.7,4}$$

where the increments are all calculated at the initial instant. When we integrate round the circuit we obtain

$$\frac{D\Gamma}{Dt} = \oint \frac{D}{Dt}(u\,dx + v\,dy + w\,dz) = 0 \tag{3.7,5}$$

for the quantities χ, ϖ and q^2 all return to their original values when we pass right round the circuit. Since (3.7,5) is true at every instant it follows that Γ is constant and this is Kelvin's theorem.

We shall now assume that the conditions for the validity of Kelvin's Theorem are satisfied and shall make deductions from it. Suppose that we have a vortex tube of strength Γ. Take any simple circuit on the surface of the tube and embracing it. Then, as we have seen in § 2.11, the circulation in the circuit is Γ and this remains constant as the fluid moves. If, however, we take any circuit on the surface of the tube which does not embrace the tube, the circulation in it is zero and remains so as the fluid moves. These two facts show that the vortices move with the fluid and preserve their strength (Helmholtz's theorem). Thus a vortex is always composed of the same fluid and behaves as a distinct entity and this is true whether or not the vortex is isolated. The reader is reminded that purely kinematic arguments show the strength of a vortex to be constant along its length at a given instant (see § 2.11) but we now find that the strength is also constant in time, subject to the satisfaction of the conditions for the validity of Kelvin's theorem. Let us take a vortex filament of strength Γ and let the normal cross-sectional area be A and the magnitude of the vorticity be Ω at a certain point on it and at a certain instant. Then

$$\Gamma = A\Omega \tag{3.7,6}$$

where Γ has the same value at all points on the filament and at all times. Whenever A changes the vorticity changes so as to maintain the constancy of Γ. We shall consider two cases in detail:

(a) *Two-dimensional flow of an incompressible fluid*

Here A is constant, for the vortex is always composed of the same fluid, which is incompressible. Hence $\Omega = \zeta$ is also constant, i.e., independent of time.

(b) *Symmetric flow in planes through a fixed axis*

Here every vortex line is a circle whose centre is on the axis and whose plane is normal to the axis. Hence a vortex filament takes the form of a circular ring. Let the radius of the ring be r and let the fluid be incompressible so the volume of the ring is constant. On account of the symmetry, A is constant round the ring at any instant, so

$$rA = C \tag{3.7,7}$$

where C is independent of time. Hence, by (3.7,6) we get

$$\Omega = \frac{r\Gamma}{C}. \tag{3.7,8}$$

Thus the vorticity is *not* constant but varies in proportion to the radius of the ring as it moves with the fluid.

Suppose that the motion of the fluid occupying a certain region is irrotational at a certain instant. Then, at this instant, the circulation in any very small circuit is zero and this remains true at any later instant. Hence the motion remains irrotational. In particular, let us think of a fluid which is initially at *rest* and let this be set in motion by moving the boundaries. Then the motion will be irrotational, always subject to the satisfaction of the conditions for the validity of Kelvin's theorem.

Let us now consider how vortices can be generated in a real fluid which was initially at rest. This can happen only when one or more of the conditions for the validity of Kelvin's theorem are unsatisfied. Let us consider these in turn:

(a) *The fluid is viscous.* Suppose we have fluid at rest in a channel with a flat plate, initially at rest, immersed in the fluid. Let the plate now be set moving in its own plane. At the surface of the plate the velocity of the liquid relative to the plate is zero, i.e. the liquid film in contact with the surface shares its velocity. But the liquid at a distance from this surface at first remains at rest and only gradually acquires a small velocity. If, then, we take a rectangular circuit with one side parallel to the direction of motion of the plate and very close to it while the opposite side is at a considerable distance from the surface, there will obviously be a circulation in the circuit and this implies the existence of vortices in the fluid. The vorticity is generated at the surface of the plate and reaches other fluid elements by diffusion. In this instance the vorticity is only appreciable in a thin layer adjacent to the surface of the plate and in the wake left behind the plate. The layer of fluid adjacent to the surface where the vorticity is appreciable is called the *boundary layer* (see further Chapter 6).

(b) *The fluid is not barotropic.* Suppose that we have a gas at rest and in equilibrium in a uniform gravitational field and let the gas be heated locally, say by chemical action. The heated gas expands and the force of buoyancy due to the pressure of the surrounding gas is increased and therefore exceeds the weight of the heated gas. This gas therefore has an upward acceleration and acquires an upward velocity. Take now a rectangular circuit $ABCD$ with AB and CD vertical while BC is horizontal, also let AB pass through the heated region whereas CD is outside it. Then there is circulation in the circuit and vortices therefore pass through it. These will clearly be ring shaped vortices lying horizontally and they have been generated *within* the fluid.

As another example let us suppose that we have a closed rectangular box with one face horizontal and let the box contain two inviscid liquids which do not mix and are of different densities. When the system is at rest and in stable equilibrium in a uniform gravitational field the denser liquid lies at the bottom of the box and the surface of separation of the liquids is a horizontal plane. The fluid as a whole is not barotropic for the density has differing values on the two sides of the surface of separation while the pressure is the same. Now let the box be tilted. The denser liquid flows towards the lower end while the less dense liquid flows in the opposite direction. Hence there is vorticity at the surface of separation and this has been generated *within* the fluid.

(c) *The body force is not derivable from a single-valued potential.* The body forces which we encounter in Nature are derivable from a potential and examples to the contrary are necessarily artificial. However, to illustrate the point at issue, let us suppose that we have a uniform body force parallel to and on one side of a plane and no body force on the other. Then, if the fluid be initially at rest, it will be set in motion by the body forces on one side of the plane but remain at rest on the other. Hence circulation and vorticity are set up.

Next, suppose that the particles of fluid carry single magnetic poles of the same polarity; this is an unrealizable assumption since only dipoles are found in Nature. Set up a magnetic field by passing a steady electric current through a fixed conducting circuit. The magnetic potential is multi-valued and increases by a constant when we pass once round a simple circuit embracing the conductor. Hence the magnetic field, acting on the magnetic poles carried by the fluid particles, will impart motion to the fluid and the circulation in a simple circuit embracing the conductor will increase with time.

3.8 Method of Solution of Problems on the Motion of Perfect Fluids

Very many problems on the flow of inviscid and incompressible uniform fluids fall under one or the other of the following heads:

(a) The fluid is initially at rest and then some or all of its boundaries, possibly including those of totally immersed bodies, are given motions which are compatible with continuity.

(b) A stream which is uniform at infinity flows past an immersed body or bodies which may be at rest or in motion.

The body forces, if any, possess a single-valued potential. It then follows from Kelvin's theorem that in both these cases the motion is irrotational, for in the initial state or in the flow at infinity the circulation is zero in any circuit. One other very important case is where there is irrotational motion everywhere but with specified circulations in certain circuits; here the region occupied by the fluid cannot be simply connected. Hence the typical problem is one of irrotational flow with specified boundary conditions, where we interpret this phrase to include the specification of the flow at infinity and of the circulations, in the case of multiply connected regions (see § 2.12). Since the flow is irrotational a velocity potential ϕ exists and when this has been found the components of velocity can at once be calculated. Finally, the distribution of pressure throughout the fluid can be obtained by use of equation (3.5,13) and the thrusts on immersed surfaces can then be calculated. The essential point is that the solution of the problem is reduced to a purely kinematic question, namely, the determination of the velocity potential which is in conformity with the specified boundary conditions. This method is a powerful and economical one, for the whole question is reduced to the determination of a single function of position. Moreover, this function (the velocity potential) satisfies Laplace's equation (see (2.12,5)). For problems of two-dimensional flow we may use alternatively the stream function (see § 2.4) or the complex potential (see § 2.14) while in axisymmetric motion the Stokes stream function is an alternative to the velocity potential (see Appendix 2 to Chapter 2). Many problems can be solved by adding the motions due to sources, sinks, doublets and isolated vortices and by the combination of these with a uniform stream (see, in particular, § 2.16 for the *method of images*). The systematic solution of Laplace's equation with specified boundary conditions is discussed in books on the theory of potential or spherical harmonics. Reference may also be made to more advanced books on hydrodynamics, such as Lamb's treatise.

EXAMPLES OF PROBLEMS ON THE IRROTATIONAL FLOW OF A PERFECT FLUID

Example 1. *An infinitely long circular cylinder is held fixed in a steady stream which is uniform at infinity and perpendicular to the axis of the cylinder*

The flow is two-dimensional in planes perpendicular to the axis of the cylinder. Take O on the axis and OX in the direction of flow at infinity. The velocity at infinity is U and the pressure there is p_0 while the radius of the cylinder is a. Body forces are supposed absent.

All the kinematic conditions are satisfied by the complex potential

$$w = -U\left(z + \frac{a^2}{z}\right) \tag{3.8,1}$$

which gives

$$\phi = -Ux - \frac{Ua^2x}{x^2 + y^2} \tag{3.8,2}$$

$$\psi = -Uy + \frac{Ua^2y}{x^2 + y^2} \tag{3.8,3}$$

(see Example 6 of § 2.14). The components of velocity are

$$u = U - \frac{Ua^2(x^2 - y^2)}{(x^2 + y^2)^2} \tag{3.8,4}$$

$$v = -\frac{2Ua^2xy}{(x^2 + y^2)^2} \tag{3.8,5}$$

(see Example 1 of § 2.8). The resultant velocity q is then given by

$$q^2 = u^2 + v^2$$

and the pressure is determined by the equation

$$\tfrac{1}{2}q^2 + \frac{p}{\rho} = C$$

where C is a constant, since the flow is steady and body forces are absent. To find C we take the conditions in the uniform stream at infinity where $q = U$ and $p = p_0$. Hence we get

$$p = p_0 + \tfrac{1}{2}\rho U^2 - \tfrac{1}{2}\rho q^2. \tag{3.8,6}$$

At the surface of the cylinder where

$$x = a \cos \theta, \quad y = a \sin \theta$$

we get after reduction

$$u = 2U \sin^2 \theta, \qquad v = -2U \sin \theta \cos \theta, \qquad q = 2U \sin \theta$$

$$p = p_0 + \tfrac{1}{2}\rho U^2(1 - 4 \sin^2 \theta). \tag{3.8,7}$$

The greatest pressure occurs at the two stagnation points where $\theta = 0$ or π. Its value is

$$p_{\max} = p_0 + \tfrac{1}{2}\rho U^2. \tag{3.8,8}$$

This is the stagnation pressure or total pressure† and is the sum of the static pressure p_0 and the dynamic pressure $\tfrac{1}{2}\rho U^2$. The minimum pressure occurs where $\theta = \pm\pi/2$ and its value is

$$p_{\min} = p_0 - \tfrac{3}{2}\rho U^2. \tag{3.8,9}$$

The pressure distribution as given by (3.8,7) is symmetric about both OX and OY and the body itself is also symmetric about these axes. Consequently the resultant thrust of the fluid per unit length of the cylinder is zero. This ceases to be true when there is circulation about the cylinder (see § 11.3).

Example 2. An infinitely long circular cylinder is in steady or accelerated rectilinear motion perpendicular to its axis through a fluid which is at rest at infinity (Body forces absent)

This is a case of unsteady motion (see § 2.2) whether the velocity of the cylinder is constant or variable. Consequently the term $\partial\phi/\partial t$ in the pressure equation (3.5,13) is not, in general, zero. We shall begin with the case where the cylinder moves with constant velocity U in the direction OX and we shall take O to be the centre of the circular section of the cylinder by the plane OXY at the instant considered.

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† In aerodynamics this is sometimes called the total head but this is a misapplication of the term. For the definition of head, see § 3.4.

We can derive the velocity potential from that given in Example 1. First, reverse the sign of ϕ in (3.8,2); we then have a fixed cylinder in a stream with velocity $-U$ parallel to OX. Now impose the velocity $+U$ parallel to OX on the whole system; this does not alter the relative velocities, pressures or thrusts and we have a cylinder moving with velocity U in the direction OX through a fluid which is at rest at infinity. The velocity potential for the imposed uniform velocity is $-Ux$ and the velocity potential for the motion induced by the moving cylinder is accordingly

$$\phi = \frac{Ua^2x}{x^2 + y^2} = \frac{Ua^2 \cos \theta}{r}. \tag{3.8,10}$$

The corresponding stream function is

$$\psi = -\frac{Ua^2y}{x^2 + y^2} = -\frac{Ua^2 \sin \theta}{r} \tag{3.8,11}$$

while the complex potential is

$$w = \frac{Ua^2}{z}. \tag{3.8,12}$$

We have to find $\partial\phi/\partial t$ when U is constant. Now $\partial\phi/\partial t$ is the rate of change of ϕ with time at a point whose spatial coordinates are fixed. The whole field of flow is, in fact, constant in relation to the cylinder, but changes in space because the cylinder moves. During the interval dt a point with constant spatial coordinates x, y acquires the coordinates $x - U\,dt$, y referred to axes having the *displaced* position of the centre as origin. The change in the value of ϕ in the interval dt is accordingly that associated with the increment $-U\,dt$ of x. Hence

$$\frac{\partial\phi}{\partial t} = -U\frac{\partial\phi}{\partial x} \tag{3.8,13}$$

$$= Uu. \tag{3.8,14}$$

Accordingly, since $\varpi = p/\rho$ for an incompressible fluid and since $q^2 = u^2 + v^2$ for two-dimensional flow, equation (3.5,13) becomes

$$\tfrac{1}{2}(u^2 + v^2) + \frac{p}{\rho} - Uu = F(t)$$

or

$$\tfrac{1}{2}\{(u - U)^2 + v^2\} + \frac{p}{\rho} = F(t) + \tfrac{1}{2}U^2. \tag{3.8,15}$$

Now at infinity $u = v = 0$ and $p = p_0$ at all times. Hence

$$F(t) = \frac{p_0}{\rho} \tag{3.8,16}$$

and equation (3.8,15) becomes

$$\tfrac{1}{2}\{(u - U)^2 + v^2\} + \frac{p}{\rho} = \frac{p_0}{\rho} + \tfrac{1}{2}U^2. \tag{3.8,17}$$

Thus the pressure is the same as in a *steady* motion with velocity components $(u - U)$, v. The steady motion is that for the cylinder at rest in a uniform stream of velocity $-U$ in the direction OX at infinity. We have thus confirmed the conclusion already drawn from general considerations about the relativity of motion that the pressures and thrusts are unaffected by imposing the velocity U on the whole system. This result is independent of the sectional form of the moving cylinder, for we have not used any particular expression for ϕ in the course of the investigation.

Next let us suppose that the velocity U of the cylinder is not constant, i.e. the cylinder is in accelerated rectilinear motion through a fluid at rest at infinity. We now have in place of equation (3.8,13)

$$\frac{\partial \phi}{\partial t} = -U\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial U}\frac{dU}{dt} \tag{3.8,18}$$

The additional term in the pressure associated with the acceleration is accordingly, by equation (3.5,13),

$$p_a = \rho\frac{\partial \phi}{\partial U}\frac{dU}{dt} \tag{3.8,19}$$

and this is true for bodies of all kinds. Since ϕ is proportional to U the last equation can be written

$$p_a = \rho\frac{\phi}{U}\frac{dU}{dt} = \rho\phi_1\frac{dU}{dt} \tag{3.8,20}$$

where ϕ_1 is the velocity potential when the cylinder advances with *unit velocity* and ϕ_1 vanishes at infinity.†

For the circular cylinder of radius a we have from (3.8,10)

$$\phi_1 = \frac{a^2\cos\theta}{} . \tag{3.8,21}$$

Hence

$$p_a = \frac{\rho a^2\cos\theta}{r}\frac{dU}{dt} \tag{3.8,22}$$

and the resultant force in the direction OX which the fluid exerts *on unit length of the cylinder* is

$$\begin{aligned} F_x' &= -\int_0^{2\pi} a p_a \cos\theta\, d\theta = -\rho a^2\frac{dU}{dt}\int_0^{2\pi}\cos^2\theta\, d\theta \\ &= -\pi\rho a^2\frac{dU}{dt} \end{aligned} \tag{3.8,23}$$

since
$$p_a = \rho a\cos\theta\frac{dU}{dt}$$

---

† The velocity potential contains an arbitrary constant which is chosen to meet this condition. If this were not met, there would be an additional and compensating term in $F(t)$. (See equation (3.5,13.)

at the surface of the cylinder. It will be seen that there is a resistance to the acceleration which is equal to that of a solid cylinder of radius a and density ρ. This is expressed by saying that the *virtual mass* of the circular cylinder is equal to the mass of the fluid which it displaces (see further § 3.9).

3.9 Kinetic Energy and Virtual Mass

Let us consider a rigid body B moving in a straight line and without rotation in a perfect fluid which is unbounded and at rest, except in-so-far as it is disturbed by the motion of B. Let the velocity of the body be U at the instant considered. Then the velocity of the fluid at any point P which occupies a fixed position relative to the body is *proportional* to U and, if u, v, w are the components of the velocity of the fluid at P, we can write

$$u = Uu_1, \qquad v = Uv_1, \qquad w = Uw_1 \tag{3.9,1}$$

where u_1, v_1, w_1 are the values of u, v, w respectively when $U = 1$ and are functions only of the coordinates of P relative to the body B. A rectangular element $dx\,dy\,dz$ of the fluid at P has kinetic energy

$$\tfrac{1}{2}\rho U^2(u_1^2 + v_1^2 + w_1^2)\,dx\,dy\,dz$$

and the total kinetic energy of the fluid is

$$T = \tfrac{1}{2}\rho U^2 \iiint (u_1^2 + v_1^2 + w_1^2)\,dx\,dy\,dz \tag{3.9,2}$$

where the integral covers the whole region occupied by the fluid. For convenience put

$$k = \iiint (u_1^2 + v_1^2 + w_1^2)\,dx\,dy\,dz \tag{3.9,3}$$

so k is a constant, but its value will depend on the attitude of the body to the direction of its motion (except in the case of a sphere). Equation (3.9,2) now becomes

$$T = \tfrac{1}{2}k\rho U^2. \tag{3.9,4}$$

Next suppose that the rectilinear velocity of the body is variable. The kinetic energy of the fluid now varies with time and, by the Principle of the Conservation of Energy, work must be done upon the fluid. Let X be the component in the direction of motion of the resultant force which B exerts *on* the fluid, so $-X$ is the corresponding component of the force which the fluid exerts on B. By the Principle of Energy the work done while B moves the distance dx is

$$X\,dx = XU\,dt = dT = k\rho U\frac{dU}{dt}\,dt$$

by (3.9,4). Hence

$$X = k\rho\frac{dU}{dt}. \tag{3.9,5}$$

The body experiences a resistance to its acceleration in virtue of the fact that it is immersed in the fluid and this resistance is equal to the product of the acceleration and the quantity $k\rho$, which is the *virtual mass* of the body for translational motion in the direction considered. It is a fact of prime importance that the resistance is zero when the acceleration is zero. This may be expressed in the form: *the drag of a body moving uniformly in a perfect fluid is zero.* It is implied that the fluid is at rest except as disturbed by the body itself and that there are no obstacles or boundaries present. If the fluid had a fixed boundary the quantities u_1, v_1 and w_1 would, in general, depend on the position of B relative to the boundary, so k would not be constant and there would be an additional term in X proportional to U^2 (see Example 4 below).

The foregoing argument demonstiates the existence of resistance to acceleration and of virtual mass in a simple manner but it is incapable of providing complete information about the consequences of acceleration. To obtain complete information we may use the method explained in Example 2 of § 3.8. In this manner we can find the part of the pressure at any point in the fluid which is proportional to the acceleration dU/dt; in particular we can obtain the value of this quantity all over the surface of the accelerated body and then calculate the three components of the resultant force and the three components of the resultant couple. In general, when the acceleration is in the direction of one coordinate axis the force will have components along all three axes and likewise the couple will, in general, have components about all three axes. It can be proved that the cross inertias are equal in pairs; for example, the force in the direction OY associated with unit acceleration in the direction OX is equal to the force in the direction OX associated with unit acceleration in the direction OY.

The kinetic energy of an incompressible and uniform fluid possesses a remarkable and important minimum property which is a particular example of a general dynamical theorem discovered by Kelvin. Kelvin's theorem is as follows: the kinetic energy of a dynamical system set in motion from a state of rest by instantaneous impulses applied at specified points in such a manner that these points acquire prescribed velocities is less than that of any other motion of the system in its given configuration and in which the specified points have their prescribed velocities. Suppose then that the fluid is initially at rest and that it is set in motion by instantaneous impulses applied over its boundaries in such a manner that the boundaries acquire prescribed velocities, which must, of course, be compatible with continuity. Then the kinetic energy in the actual motion is less than in any other motion which is kinematically possible. Since the circulation is zero in any circuit described in the fluid in its initial state of rest, it follows from Kelvin's circulation theorem (see § 3.7) that the motion generated by the impulses is irrotational. The irrotational motion is characterised by having the least possible kinetic energy and this establishes its uniqueness. We may regard this as a principle of maximum possible uniformity of flow. The reader should note that this principle does *not* apply to compressible fluids.

EXAMPLES ON KINETIC ENERGY AND VIRTUAL MASS

Example 1. *Virtual mass of a circular cylinder*

The cylinder, of radius a, moves with velocity U in the direction OX, perpendicular to its axis. Here we have two-dimensional flow in planes normal to the axis of the cylinder and we shall consider a layer of fluid of unit depth. The velocity potential is as given by (3.8,10) and the components of velocity are

$$u = \frac{Ua^2}{r^2}\cos 2\theta\,, \qquad v = \frac{Ua^2}{r^2}\sin 2\theta\,.$$

Hence

$$q = \frac{Ua^2}{r^2} \tag{3.9,6}$$

which could have been derived immediately from (3.8,12) since

$$q = \left|\frac{dw}{dz}\right|$$

where w is now the complex potential. To calculate the kinetic energy we employ polar coordinates, for convenience. The element of area is $r\,d\theta\,dr$ and this is also the element of volume in the layer of unit depth. Hence the energy per unit axial length of the cylinder is

$$\begin{aligned} T' &= \tfrac{1}{2}\rho\int_0^{2\pi}\int_a^{\infty} q^2 r\,d\theta\,dr \\ &= \tfrac{1}{2}\rho U^2a^4\int_0^{2\pi}\int_a^{\infty}\frac{d\theta\,dr}{r^3} = \pi\rho U^2a^4\int_a^{\infty}\frac{dr}{r^3} \\ &= \tfrac{1}{2}\pi\rho a^2U^2. \end{aligned} \tag{3.9,7}$$

Hence the virtual mass per unit length is $\pi\rho a^2$, i.e. equal to the mass of fluid displaced by the cylinder. This result agrees with that obtained by another method in Example 2 of § 3.8. For cylinders of other sectional forms the virtual mass will depend on the direction of motion and will be equal to the mass of fluid displaced, multiplied by a numerical coefficient whose value, in general, differs from unity.

Example 2. *Virtual mass of a sphere*

The velocity potential for a sphere of radius a moving with velocity U in the direction OX is

$$\phi = \frac{Ua^3x}{2r^3} \tag{3.9,8}$$

and the components of velocity are

$$u = \frac{Ua^3(3x^2 - r^2)}{2r^5}, \quad v = \frac{3Ua^3xy}{2r^5}, \quad w = \frac{3Ua^3xz}{2r^5}$$

(see Example 2 of § 2.7 and Appendix 2 to Chapter 2).

Hence $$q^2 = \frac{U^2a^6(3x^2 + r^2)}{4r^8} = \frac{U^2a^6(1 + 3\cos^2\theta)}{4r^6}. \tag{3.9,9}$$

It is convenient to divide the region occupied by the fluid into circular rings whose planes are perpendicular to OX and whose centres lie on OX. The radius of such a ring is $r \sin\theta$ and its volume is $2\pi r^2 \sin\theta \, dr \, d\theta$. Hence

$$T = \tfrac{1}{2}\rho \int_0^\pi \int_a^\infty 2\pi r^2 q^2 \sin\theta \, dr \, d\theta$$

$$= \frac{\pi\rho U^2 a^6}{4} \int_0^\pi \int_a^\infty \frac{(1 + 3\cos^2\theta)\sin\theta \, dr \, d\theta}{r^4}$$

Now $$\int_0^\pi (1 + 3\cos^2\theta)\sin\theta \, d\theta = \Big[-\cos\theta - \cos^3\theta\Big]_0^\pi = 4$$

and $$\int_a^\infty \frac{dr}{r^4} = \frac{1}{3a^3}.$$

Hence $$T = \tfrac{1}{3}\pi\rho a^3 U^2. \tag{3.9,10}$$

The virtual mass is thus $\tfrac{2}{3}\pi\rho a^3$ which is exactly *half* the mass of the fluid displaced by the sphere (see also Example 3 below).

Example 3. *General expression for the virtual mass in rectilinear motion*

The argument given in Example 2 of § 3.8 shows that the pressure associated with rectilinear acceleration is given by the general formula (equation 3.8,20).

$$p_a = \rho\phi_1 \frac{dU}{dt}$$

where ϕ_1 is the velocity potential for $U = 1$. The force corresponding to the pressure which the accelerated body exerts *on* the fluid in the direction OX is

$$X = \iint l p_a \, dS$$

where the surface integral is taken over the surface of the body. Hence

$$\text{virtual mass} = \frac{X}{\dfrac{dU}{dt}} = \rho \iint l\phi_1 \, dS. \tag{3.9,11}$$

As an example, let us apply this to the sphere. We get from (3.9,8) when U is unity

$$\phi_1 = \frac{a^3x}{2r^3}$$

and on the surface of the sphere $r = a$

$$\phi_1 = \tfrac{1}{2}x = \tfrac{1}{2}a\cos\theta.$$

Also $l = \cos\theta$ and we may take a ring of the spherical surface of area $2\pi a^2 \sin\theta\, d\theta$. Hence we get from (3.9,11)

$$\text{virtual mass} = \pi\rho a^3 \int_0^\pi \cos^2\theta \sin\theta\, d\theta = \tfrac{2}{3}\pi\rho a^3.$$

This agrees with the result obtained in Example 2 above.

Example 4. Resistance to the steady rectilinear motion of a body in the neighbourhood of a fixed boundary

When a body moves uniformly in a straight line in the neighbourhood of a fixed boundary the coefficient k given by equation (3.9,3) is not, in general, constant but is some function of the displacement x of the body which we shall write $k(x)$. Accordingly equation (3.9,4) becomes

$$T = \tfrac{1}{2}k(x)\rho U^2.$$

Since U is constant we now have $dT = \dfrac{\partial T}{\partial x}\, dx = X\, dx$

or

$$X = \frac{\partial T}{\partial x} = \tfrac{1}{2}k'(x)\rho U^2. \tag{3.9,12}$$

It will be noted that the force is now proportional to ρU^2 and its sign is determined by that of $k'(x)$. Normally $k'(x)$ will be positive when the moving body is approaching a fixed boundary and then there will be an apparent positive drag, but the apparent drag will be negative during recession from a boundary.

3.10 Elementary Consideration of Viscous Fluids

It is beyond our scope to investigate the general equations of motion of a viscous fluid and in this section we shall content ourselves with discussing some important cases where the distortion of the fluid is of the nature of a simple shearing motion so that the definition of viscosity (see § 1.4) can be applied directly to obtain the shearing stress in the fluid. We shall assume that the viscosity is constant.

As a first example we shall consider the two-dimensional laminar flow of a viscous incompressible and uniform fluid between a pair of parallel planes which are at rest or moving in their own planes and in the same direction (see Fig. 3.10,1). We shall take the axis OX to lie in the lower plane in the direction of motion while OZ is a perpendicular axis in this plane. The equation to the upper plane is then $y = h$, where h is independent

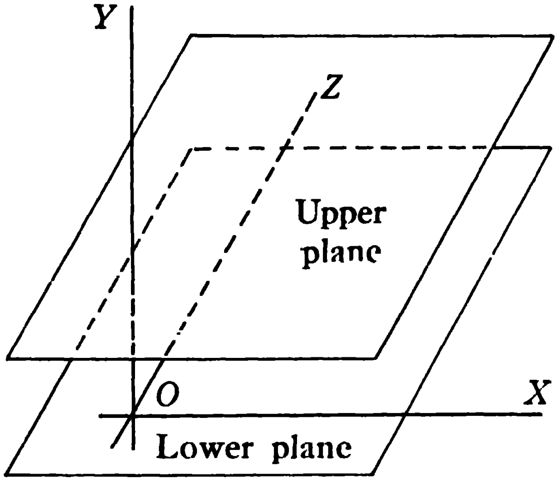

Fig. 3.10,1. Viscous fluid between parallel planes moving in the direction OX.

of time. We shall assume that the components of velocity v and w are always zero. Then the equation of continuity (2.3,9) requires that

$$\frac{\partial u}{\partial x} = 0 \tag{3.10,1}$$

and since we have postulated that the motion is two-dimensional we have

$$\frac{\partial u}{\partial z} = 0. \tag{3.10,2}$$

Thus u is the only non-vanishing component of the velocity and it is, at most, a function of y and of t. The streamlines, which are also the paths of particles, are straight lines parallel to OX and the components of acceleration in the directions OY and OZ are zero. We shall suppose that there is a body force X in the direction of motion while the other components of body force are zero. It follows that the pressure p is independent of y and z, so p is at most a function of x and of t (see also § 3.3).

Let us consider a layer of fluid of unit depth in the direction OZ. The shearing stress on the lower face of the rectangular element $dx\,dy$ is $\mu(\partial u/\partial y)$ and the retarding force on the element is $\mu(\partial u/\partial y)\,dx$. The stress on the upper face is $\mu(\partial u/\partial y) + \mu(\partial^2 u/\partial y^2)\,dy$ and the accelerating force on the element is $[\mu(\partial u/\partial y) + \mu(\partial^2 u/\partial y^2)\,dy]\,dx$. Hence the net accelerating force due to the viscous stresses is $\mu(\partial^2 u/\partial y^2)\,dx\,dy$ while the net retarding force due to the pressures on the ends of the element is $-(\partial p/\partial x)\,dx\,dy$. Finally, the body force contributes the accelerating force $X\rho\,dx\,dy$ since the mass of the element is $\rho\, dx\, dy$. When we equate the net accelerating force to the mass-acceleration and divide by $dx\, dy$ we obtain

$$\rho \frac{\partial u}{\partial t} = \rho X - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \tag{3.10,3}$$

since the expression (2.9,4) for the acceleration a_x here reduces to its last term. Let the velocities of the upper and lower planes be u_b and u_a respectively. Then we have the boundary conditions

$$\left.\begin{array}{lll} u = u_a & \text{when} & y = 0 \\ u = u_b & \text{when} & y = h \end{array}\right\} \tag{3.10,4}$$

since there is no slip at a boundary. The problem consists in finding a solution of the dynamical equation (3.10,3) which is consistent with (3.10,4) and it is to be remembered that p is independent of y. We shall now work out the complete solution in a number of important cases.

Case 1. *Steady motion in the absence of body forces*

Since all quantities are independent of time the dynamical equation (3.10.3) becomes

$$\frac{dp}{dx} = \mu \frac{d^2 u}{dy^2}. \tag{3.10,5}$$

Since p is independent of y we treat dp/dx as a constant when integrating the last equation. We then easily find that the solution which fits the boundary conditions (3.10,4) is

$$u = \frac{1}{2\mu} \frac{dp}{dx} (y^2 - hy) + \frac{y}{h}(u_b - u_a) + u_a. \tag{3.10,6}$$

Also the shearing stress is

$$f = \mu \frac{du}{dy} = \frac{1}{2} \frac{dp}{dx}(2y - h) + \mu\left(\frac{u_b - u_a}{h}\right). \tag{3.10,7}$$

The value of the stress at the lower plane is

$$f_0 = \mu\left(\frac{u_b - u_a}{h}\right) + \tfrac{1}{2}h\left(-\frac{dp}{dx}\right) \tag{3.10,8}$$

and that at the upper plane is

$$f_h = \mu\left(\frac{u_b - u_a}{h}\right) - \tfrac{1}{2}h\left(-\frac{dp}{dx}\right). \tag{3.10,9}$$

It should be remembered that the stress is here the tangential force per unit area which the upper fluid exerts on the lower (see the discussion of stress in § 1.2). Hence f_h is the tangential force per unit area which the upper plane exerts *on* the fluid, so the tangential stress which the fluid exerts on the upper plane is $-f_h$. The *sum* of the stresses exerted *by* the fluid *on* the planes is

$$f_0 - f_h = h\left(-\frac{dp}{dx}\right). \tag{3.10,10}$$

The last equation is clearly in accordance with the requirements of static equilibrium.

We shall now draw attention to some important special cases of the above results. *Suppose that both planes are at rest.* Then the solution (3.10,6) becomes

$$u = \frac{1}{2\mu}(hy - y^2)\left(-\frac{dp}{dx}\right) = \frac{1}{2\mu}\left(-\frac{dp}{dx}\right)\left[\left(\frac{h}{2}\right)^2 - \left(y - \frac{h}{2}\right)^2\right]. \tag{3.10,11}$$

This represents a parabolic distribution of velocity which is symmetric about the plane lying midway between the fixed walls of the channel. It will be noted that the velocity is positive when the pressure gradient is negative; this is as it should be for there is a positive propelling force on the fluid when it flows from a region of higher pressure into one of lower pressure. Let Q' be the volume of fluid discharged per unit of time in a layer of unit depth. Then

$$\begin{aligned} Q' &= \int_0^h u\,dy \\ &= \frac{h^3}{12\mu}\left(-\frac{dp}{dx}\right) \end{aligned} \tag{3.10,12}$$

on substitution from (3.10,11). Since Q' must be independent of x by continuity, it follows that the pressure gradient is constant, an important conclusion. Next, suppose that there is no pressure gradient and that the planes are in motion. Then (3.10,6) yields

$$u = \frac{y}{h}(u_b - u_a) + u_a \tag{3.10,13}$$

so the velocity varies linearly between the planes.† The corresponding expression for the discharge is

$$Q' = \tfrac{1}{2}h(u_a + u_b). \tag{3.10,14}$$

Case 2. Steady motion under the action of a uniform body force and with zero pressure gradient

It follows from equation (3.10,3) that nothing is changed when we substitute ρX for $-(\partial p/\partial x)$. Hence, when both planes are at rest, we derive from equation (3.10,11) the solution

$$\begin{aligned} u &= \frac{\rho X}{2\mu}\left[\left(\frac{h}{2}\right)^2 - \left(y - \frac{h}{2}\right)^2\right] \\ &= \frac{X}{2\nu}\left[\left(\frac{h}{2}\right)^2 - \left(y - \frac{h}{2}\right)^2\right] \end{aligned} \tag{3.10,15}$$

---

† This is sometimes called 'Couette flow' when one plane is at rest.

where ν is the kinematic viscosity. Also the expression (3.10,12) for the rate of discharge becomes

$$Q' = \frac{h^3X}{12\nu}. \tag{3.10,16}$$

If the plates are situated vertically in a uniform gravitational field we have merely to substitute g for X in these formulae.

Case 3. *Unsteady motion with pressure gradient and body force absent*

Here equation (3.10,3) becomes

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \tag{3.10,17}$$

This is *the equation of diffusion or of heat conduction in one dimension* (here the spatial coordinate y). This equation has been studied intensively in relation to the conduction of heat and the solution to a problem of conduction may be reinterpreted in relation to the laminar flow of a viscous fluid. The vorticity is here

$$\zeta = -\frac{\partial u}{\partial y} \tag{3.10,18}$$

and we find by differentiating (3.10,17) with respect to y that

$$\frac{\partial \zeta}{\partial t} = \nu \frac{\partial^2 \zeta}{\partial y^2}. \tag{3.10,19}$$

It follows from equations (3.10,17) and (3.10,19) that the kinematic viscosity is the *diffusivity* of both velocity and vorticity in the present case.

An interesting particular solution of (3.10,17) is

$$u = U\left\{1 - \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right\} \tag{3.10,20}$$

where

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-n^2}\, dn \tag{3.10,21}$$

is known as the 'error function'. We get at once by differentiation that

$$\frac{d}{dx}(\operatorname{erf} x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \tag{3.10,22}$$

and then we obtain from (3.10,20)

$$\frac{\partial u}{\partial y} = -\frac{U}{\sqrt{(\pi\nu t)}} e^{-\frac{y^2}{4\nu t}} \tag{3.10,23}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{Uy}{2\sqrt{(\pi\nu^3 t^3)}} e^{-\frac{y^2}{4\nu t}} \tag{3.10,24}$$

$$\frac{\partial u}{\partial t} = \frac{Uy}{2\sqrt{(\pi\nu t^3)}} e^{-\frac{y^2}{4\nu t}}. \tag{3.10,25}$$

These results show that u satisfies (3.10,17). Moreover, we have

$$\left.\begin{aligned} u &= U \quad \text{when} \quad y = 0 \\ u &= 0 \quad \text{when} \quad y = \infty \end{aligned}\right\} \tag{3.10,26}$$

since $\quad \operatorname{erf} 0 = 0 \quad$ and $\quad \operatorname{erf} \infty = 1.$

Equation (3.10,20) thus represents the motion of an infinitely extended viscous fluid which is at rest when $t = 0$ and is in contact with the plane $y = 0$ which has the constant velocity U in the direction OX. The drag on the moving plane per unit area is

$$\begin{aligned} -\mu\left(\frac{\partial u}{\partial y}\right)_{y=0} &= \frac{\mu U}{\sqrt{(\pi \nu t)}} \\ &= U\rho\sqrt{\left(\frac{\nu}{\pi t}\right)} \end{aligned} \tag{3.10,27}$$

where it should be noted that only the upper surface of the plane is in contact with the fluid. This solution has applications to the theory of the boundary layer (see Chapter 6).

We shall next consider the motion of a viscous fluid contained between a pair of coaxial circular cylinders which rotate about their common axis with constant or variable angular velocities and we shall assume that the fluid particles move in circles lying in planes normal to the axis of the cylinders and with their centres on the axis, the whole motion being two-dimensional. The velocity u_t at the radius r is thus circumferential and is at most a function of r and of t and the same is true of the angular velocity ω which is such that

$$r\omega = u_t. \tag{3.10,28}$$

As a first step in the analysis we must determine the rate of shear in the fluid at a circle of radius r. Now the fluid suffers no distortion when it rotates like a rigid body, i.e. the rate of shear is zero when ω is independent of r or when u_t is proportional to r. The actual velocity at the radius $r + dr$ is $u_t + (\partial u_t/\partial r)\, dr$ and if there were no distortion this would be $u_t(r + dr)/r$. Hence the excess velocity at the radius $(r + dr)$ is $\{(\partial u_t/\partial r) - (u_t/r)\}\, dr$ and this is acquired in the radial distance dr. Therefore the rate of shear is

$$\frac{\partial u_t}{\partial r} - \frac{u_t}{r} = \frac{\partial(r\omega)}{\partial r} - \omega = r\frac{\partial \omega}{\partial r} \tag{3.10,29}$$

and the shearing stress on a cylindrical surface of radius r is

$$f = \mu r \frac{\partial \omega}{\partial r}. \tag{3.10,30}$$

Now let us consider a layer of fluid of unit axial depth. The moment about the axis of the cylinder of the shearing stresses on a cylinder of radius r and unit length is

$$N' = 2\pi r^2 f = 2\pi\mu r^3 \frac{\partial\omega}{\partial r}. \tag{3.10,31}$$

Hence the net accelerating moment on an annulus of radial thickness dr and unit axial length is

$$\frac{\partial N'}{\partial r} dr = 2\pi\mu \frac{\partial}{\partial r}\left(r^3 \frac{\partial\omega}{\partial r}\right) dr.$$

This annulus has mass $2\pi\rho r\, dr$ and moment of inertia about the axis equal to $2\pi\rho r^3\, dr$. This, when multiplied by the angular acceleration $\partial\omega/\partial t$, is equal to the net accelerating moment and we obtain the relation

$$\rho r^3 \frac{\partial\omega}{\partial t} = \mu \frac{\partial}{\partial r}\left(r^3 \frac{\partial\omega}{\partial r}\right)$$

or

$$\frac{\partial\omega}{\partial t} = \frac{\nu}{r^3} \frac{\partial}{\partial r}\left(r^3 \frac{\partial\omega}{\partial r}\right). \tag{3.10,32}$$

An equivalent equation in terms of the vorticity is given and discussed below.

First let us consider the case of steady motion. Equation (3.10,32) then becomes

$$\frac{d}{dr}\left(r^3 \frac{d\omega}{dr}\right) = 0 \quad \text{or} \quad r^3 \frac{d\omega}{dr} = A$$

where A is a constant of integration. Hence $\dfrac{d\omega}{dr} = \dfrac{A}{r^3}$

and

$$\omega = -\frac{A}{2r^2} + B \tag{3.10,33}$$

where B is a second constant of integration. Now let the inner cylinder of radius a have the angular velocity ω_a while the outer cylinder of radius b has the angular velocity ω_b. By fitting (3.10,33) to these conditions we derive the equation

$$\omega = \frac{b^2\omega_b - a^2\omega_a}{b^2 - a^2} - \frac{(\omega_b - \omega_a)a^2b^2}{r^2(b^2 - a^2)}. \tag{3.10,34}$$

The moment per unit length has the constant value

$$\begin{aligned} N' &= 2\pi\mu A \\ &= \frac{4\pi\mu a^2b^2(\omega_b - \omega_a)}{(b^2 - a^2)}. \end{aligned} \tag{3.10,35}$$

This is the accelerating moment per unit axial length on the inner cylinder and the retarding moment per unit axial length on the outer cylinder.

In discussing unsteady motion it is convenient to introduce the vorticity ζ which can be related to ω as follows. The circulation in a circle of radius r is

$$\begin{aligned} \Gamma &= \text{circumference} \times \text{velocity} \\ &= 2\pi r^2\omega. \end{aligned} \tag{3.10,36}$$

Hence the increase in the circulation in passing from a circle of radius r to one of radius $r + dr$ is

$$d\Gamma = 2\pi\, d(r^2\omega).$$

But this is equal to ζ multiplied by the area of the annulus, namely $2\pi r\, dr$. Therefore

$$\zeta = \frac{1}{r}\frac{\partial(r^2\omega)}{\partial r} = r\frac{\partial\omega}{\partial r} + 2\omega \tag{3.10,37}$$

and

$$\begin{aligned} \frac{\partial\zeta}{\partial t} &= r\frac{\partial^2\omega}{\partial t\,\partial r} + 2\frac{\partial\omega}{\partial t} = r\frac{\partial}{\partial r}\left(\frac{\partial\omega}{\partial t}\right) + 2\frac{\partial\omega}{\partial t} \\ &= \nu\left\{r\frac{\partial^3\omega}{\partial r^3} + 5\frac{\partial^2\omega}{\partial r^2} + \frac{3}{r}\frac{\partial\omega}{\partial r}\right\} \end{aligned}$$

on substitution from (3.10,32). By use of (3.10,37) the last equation can be reduced to

$$\frac{\partial\zeta}{\partial t} = \nu\left\{\frac{\partial^2\zeta}{\partial r^2} + \frac{1}{r}\frac{\partial\zeta}{\partial r}\right\}. \tag{3.10,38}$$

This is the equation of diffusion or of heat conduction for the case of symmetric radial conduction in two-dimensions. Since the total quantity of heat is unaffected by conduction except in so far as it is lost or gained at boundaries, it follows that the total integral of the vorticity remains constant except for gains or losses at the surfaces of immersed bodies. Once again we may adapt known solutions of the problem of heat conduction to obtain solutions of problems of viscous flow. An interesting solution of (3.10,38) is

$$\zeta = \frac{\Gamma_0}{4\pi\nu t}\, e^{-\frac{r^2}{4\nu t}}. \tag{3.10,39}$$

On account of the general relation between vorticity and circulation it follows that, when the fluid fills the space from the axis outward, the circulation in a circle of radius r at time t is†

$$\begin{aligned} \Gamma(r, t) &= \int_0^r 2\pi r\zeta\, dr \\ &= \Gamma_0\left(1 - e^{-\frac{r^2}{4\nu t}}\right) \end{aligned} \tag{3.10,40}$$

---

† The integral can be obtained at once by the substitution $s = r^2/4\nu t$.

and the velocity is accordingly

$$u_t = \frac{\Gamma(r, t)}{2\pi r} = \frac{\Gamma_0}{2\pi r}\left(1 - e^{-\frac{r^2}{4\nu t}}\right). \tag{3.10,41}$$

When $r \to \infty$, $\Gamma \to \Gamma_0$ in accordance with the total vorticity remaining constant. This solution represents the diffusion of an infinitely long straight line vortex of strength Γ_0 outwards into the surrounding fluid. The expression 'decay of a vortex' is sometimes applied to this phenomenon of diffusion but is rather misleading since the vortex spreads outward without changing its total strength. The total vorticity can only be influenced by happenings at boundaries; vorticities of opposite signs can be destroyed by interdiffusion, just as adjacent hot and cold regions in a body disappear by conduction of heat.

The particular problems of viscous flow discussed here are of very special simplicity. In general it is true to say that problems of viscous flow are intractable by "exact" methods and that we are driven to adopt approximations. On this, see in particular Chapter 6, where the 'boundary layer' is discussed.

The stability of the steady flow of a viscous fluid between coaxial rotating cylinders has been investigated by Taylor[1] who discovered very remarkable phenomena. When ω_b/ω_a was positive and greater than 1 the motion was stable up to the highest speeds used in the tests. With

$$\frac{\omega_b}{\omega_a} < \frac{a^2}{b^2}$$

it was found that there was a critical rate of rotation such that ring vortices developed, the rings embracing the inner cylinder and lying one above the other axially. Thus there was a vortex motion in planes *through* the common axis of the cylinders; this motion occurred in cells, corresponding to the individual vortex rings, with contrary directions of rotation in adjacent cells. The measured critical speeds agreed well with predictions based on Taylor's theory. For speeds appreciably above the critical the motion became turbulent. The theory indicates that for given ratios b/a and ω_b/ω_a the critical state is associated with a constant value of $\omega_a a^2/\nu$ which is, in effect, a Reynolds number (see § 4.10).

The laminar motion of viscous fluids in tubes is discussed in § 7.6.

Mention will be made of Stokes' solution of the problem of the motion of a sphere with uniform rectilinear velocity and no rotation, through a viscous fluid which is unbounded and at rest, except as disturbed by the sphere, but the reader should consult Chapter XI of Lamb's *Hydrodynamics* for the mathematical details. Stokes' solution is for the case where the Reynolds number (see § 4.10) is so small that the inertia forces are negligible in comparison with the forces due to viscosity. The drag for a sphere of radius a moving with speed U is

$$D = 6\pi\mu a U \tag{3.10,42}$$

and the value of the drag coefficient, based on the area S of a diametral plane, is

$$C_D = \frac{12\mu}{\rho a U} = \frac{24}{R} \tag{3.10,43}$$

where R is the Reynolds number based on the diameter $2a$ as the typical length. Stokes' formula (3.10,42) is extensively applied to find the diameter of a small droplet from its observed rate of fall through a gas under gravity. It is not valid when R is greater than about 0·5.†

3.11 Flow in a Rotating Channel

We shall now consider the flow of an inviscid fluid in a rigid channel which rotates about the fixed axis OZ with *constant* angular velocity ω. We shall suppose that the body force, if any, is of intensity Z in the direction OZ, the other components being zero.‡

Before we attempt to write down the equations of motion we must define the kinematics of the problem. We shall take a fixed origin O and axes OX, OY forming with OZ a rectangular system *fixed relative to the channel*; thus the axes OX and OY rotate about OZ with the constant angular velocity ω. Let x, y, z be the coordinates of a particle referred to these rotating axes. Then the components of its relative velocity are defined to be

$$u' = \frac{dx}{dt}, \qquad v' = \frac{dy}{dt}, \qquad w' = \frac{dz}{dt}. \tag{3.11,1}$$

The total components of velocity or components of the velocity in fixed space are obtained by adding the contributions due to the angular velocity about OZ. Thus

$$u = u' - \omega y, \qquad v = v' + \omega x, \qquad w = w', \tag{3.11,2}$$

where these components are referred to the instantaneous positions of the axes OX, OY, OZ. Now consider the state of affairs after the lapse of time dt. The components of relative velocity are now $u' + du'$, $v' + dv'$ and $w' + dw'$ and the coordinates of the particle are $x + dx$, $y + dy$, $z + dz$, where all these quantities are referred to the axes as situated after the interval

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† See also *Modern Developments in Fluid Dynamics*, Chapter XI.

‡ When the fluid is incompressible *any* conservative body force influences the pressure but not necessarily the motion. The restriction on the body force adopted here allows us to cover the steady motion of a compressible fluid.

dt; thus OX and OY have rotated through the angle $\omega\, dt$. Now (see Fig. 3.11,1) let us resolve the velocity in the directions of the axes as they were at the beginning of the interval dt. The component in the direction OX is

$$[(u' + du') - \omega(y + dy)] \cos \omega\, dt - [(v' + dv') + \omega(x + dx)] \sin \omega\, dt$$

and when we neglect small quantities of the second order this becomes

$$(u' + du') - \omega(y + dy) - (v' + \omega x)\omega\, dt$$
$$= (u' - \omega y) + du' - 2\omega v'\, dt - \omega^2 x\, dt$$

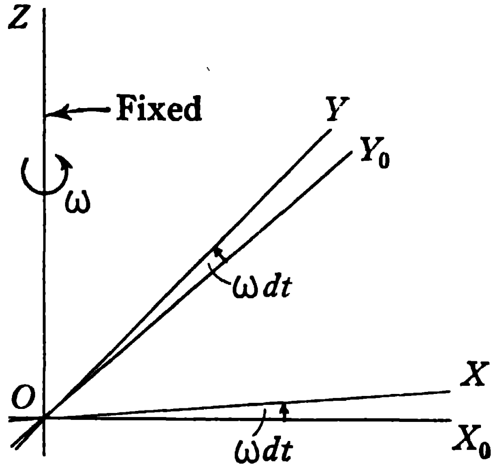

Fig. 3.11,1. Diagram to illustrate the use of rotating axes.

since $dy = v'\, dt$. Hence the increment of the component of velocity in the direction OX during the interval dt is

$$du' - 2\omega v'\, dt - \omega^2 x\, dt$$

and therefore the component of acceleration in the direction OX is

$$a_x = \frac{du'}{dt} - 2\omega v' - \omega^2 x. \tag{3.11,3}$$

The component of velocity in the direction OY at the end of the interval dt is

$$[(v' + dv') + \omega(x + dx)] \cos \omega\, dt + [(u + du) - \omega(y + dy)] \sin \omega\, dt.$$

When we neglect small quantities of the second order we find that the increment of this component is

$$dv' + 2\omega u'\, dt - \omega^2 y\, dt$$

since $dx = u'\, dt$ and the corresponding component of acceleration is

$$a_y = \frac{dv'}{dt} + 2\omega u' \quad \omega^2 y. \tag{3.11,4}$$

Since the axis OZ has a fixed direction the component of acceleration in this direction is simply

$$a_z = \frac{dw'}{dt}. \tag{3.11,5}$$

Now when we adopt the Eulerian method (see § 2.2) according to which u', v', w' are supposed to be given in terms of x, y, z and t we get, when we follow a particle of fluid,

$$\frac{du'}{dt} = u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z} + \frac{\partial u'}{\partial t}$$

as shown in § 2.9. Hence (3.11,3) becomes

$$a_x = u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z} + \frac{\partial u'}{\partial t} - 2\omega v' - \omega^2 x. \tag{3.11,6}$$

Similarly (3.11,4) and (3.11,5) become

$$a_y = u'\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y} + w'\frac{\partial v'}{\partial z} + \frac{\partial v'}{\partial t} + 2\omega u' - \omega^2 y \tag{3.11,7}$$

and

$$a_z = u'\frac{\partial w'}{\partial x} + v'\frac{\partial w'}{\partial y} + w'\frac{\partial w'}{\partial z} + \frac{\partial w'}{\partial t}. \tag{3.11,8}$$

We have next to consider the boundary conditions and continuity. Here we need only consider boundaries which are fixed with reference to the axes OX, OY, OZ. Hence the kinematic condition is simply that the resultant of u', v' and w' is tangential to the boundary.† So far as continuity is concerned the rotation of the axes is irrelevant so we have merely to substitute u', v', w' for u, v, w in equation (2.3,7). Hence the general equation of continuity is

$$\frac{\partial(\rho u')}{\partial x} + \frac{\partial(\rho v')}{\partial y} + \frac{\partial(\rho w')}{\partial z} + \frac{\partial \rho}{\partial t} = 0. \tag{3.11,9}$$

We can define streamlines and streamtubes for the flow relative to the rotating axes in exactly the same manner as for fixed axes. The flow relative to the rotating axes is *steady* when u', v', w' are functions of x, y, z but independent of t. For such steady motion the equation of continuity can be put in the convenient form (see equation 2.3,1)

$$\rho q' A = \text{constant} \tag{3.11,10}$$

where q' is the resultant velocity relative to the rotating axes and A is the cross-sectional area of a slender streamtube of the relative flow.

The dynamical equations are obtained at once from equations (3.5,2) on substitution of the values of the components of acceleration given by equations (3.11,6) to (3.11,8) and it is not necessary to write these at length.

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† A fluid particle at the boundary moves normally to it in the rotating frame in time dt by a distance ultimately proportional to dt^2 or some higher power of dt.

We shall now consider in detail the case where the relative motion is steady and we shall obtain a generalization of Bernoulli's theorem. The fluid is supposed to be barotropic and we shall employ the function ϖ (see equations (3.5,5) and (3.5,6)). As already stated, the components X, Y of the body force are zero and we shall suppose that Z is derivable from a potential χ which is here a function of z only. We shall use the equations

$$x = \frac{1}{2}\frac{\partial r^2}{\partial x}, \qquad y = \frac{1}{2}\frac{\partial r^2}{\partial y}, \qquad 0 = \frac{1}{2}\frac{\partial r^2}{\partial z} \tag{3.11,11}$$

where

$$r^2 = x^2 + y^2. \tag{3.11,12}$$

Since the relative motion is steady we have

$$\frac{\partial u'}{\partial t} = \frac{\partial v'}{\partial t} = \frac{\partial w'}{\partial t} = 0. \tag{3.11,13}$$

Let ds be an element of arc of a streamline of the relative motion. By the definition of a streamline we have

$$\frac{dx}{ds} = \frac{u'}{q'}, \qquad \frac{dy}{ds} = \frac{v'}{q'}, \qquad \frac{dz}{ds} = \frac{w'}{q'} \tag{3.11,14}$$

and by cross-multiplication we get the relations

$$v'\frac{dx}{ds} = u'\frac{dy}{ds}, \qquad w'\frac{dy}{ds} = v'\frac{dz}{ds}, \qquad u'\frac{dz}{ds} = w'\frac{dx}{ds} \tag{3.11,15}$$

which will be used later. It will now be found that, when use is made of (3.11,6) to (3.11,8) together with (3.11,11), (3.11,13) and the functions ϖ and χ, the three dynamical equations (3.5,2) can be written

$$u'\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z} - 2\omega v' = \frac{\partial}{\partial x}\left(\tfrac{1}{2}\omega^2 r^2 - \chi - \varpi\right) \tag{3.11,16}$$

$$u'\frac{\partial v'}{\partial x} + v'\frac{\partial v'}{\partial y} + w'\frac{\partial v'}{\partial z} + 2\omega u' = \frac{\partial}{\partial y}\left(\tfrac{1}{2}\omega^2 r^2 - \chi - \varpi\right) \tag{3.11,17}$$

$$u'\frac{\partial w'}{\partial x} + v'\frac{\partial w'}{\partial y} + w'\frac{\partial w'}{\partial z} = \frac{\partial}{\partial z}\left(\tfrac{1}{2}\omega^2 r^2 - \chi - \varpi\right). \tag{3.11,18}$$

Multiply these equations by dx/ds, dy/ds and dz/ds respectively, add and use (3.11,15). Equation (3.11,16) when multiplied by dx/ds can be written

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\tfrac{1}{2}\omega^2 r^2 - \chi - \varpi\right)\frac{dx}{ds} &= u'\frac{\partial u'}{\partial x}\frac{dx}{ds} + v'\frac{\partial u'}{\partial y}\frac{dx}{ds} + w'\frac{\partial u'}{\partial z}\frac{dx}{ds} - 2\omega v'\frac{dx}{ds} \\
&= u'\left(\frac{\partial u'}{\partial x}\frac{dx}{ds} + \frac{\partial u'}{\partial y}\frac{dy}{ds} + \frac{\partial u'}{\partial z}\frac{dz}{ds}\right) - \frac{2\omega u'v'}{q'} \\
&= u'\frac{du'}{ds} - \frac{2\omega u'v'}{q'} = \frac{1}{2}\frac{du'^2}{ds} - \frac{2\omega u'v'}{q'}.
\end{aligned} \tag{3.11,19}$$

Similarly equations (3.11,17) and (3.11,18) can be written

$$\frac{\partial}{\partial y}(\tfrac{1}{2}\omega^2 r^2 - \chi - \varpi)\frac{dy}{ds} = \frac{1}{2}\frac{dv'^2}{ds} + \frac{2\omega u'v'}{q'} \tag{3.11,20}$$

$$\frac{\partial}{\partial z}(\tfrac{1}{2}\omega^2 r^2 - \chi - \varpi)\frac{dz}{ds} = \frac{1}{2}\frac{dw'^2}{ds}. \tag{3.11,21}$$

The sum of equations (3.11,19) to (3.11,21) is

$$\frac{d}{ds}(\tfrac{1}{2}\omega^2 r^2 - \chi - \varpi) = \frac{1}{2}\frac{d}{ds}(u'^2 + v'^2 + w'^2) = \frac{1}{2}\frac{dq'^2}{ds}.$$

Therefore

$$\frac{d}{ds}(\tfrac{1}{2}q'^2 + \varpi + \chi - \tfrac{1}{2}\omega^2 r^2) = 0$$

or

$$\tfrac{1}{2}q'^2 + \varpi + \chi - \tfrac{1}{2}\omega^2 r^2 = C \tag{3.11,22}$$

where C is constant on a streamline of the relative motion. This is the generalization of Bernoulli's theorem and it will be seen that it is of the same form as for stationary axes, except that the potential is modified by the addition of the term $-\tfrac{1}{2}\omega^2 r^2$, which we shall call the *centrifugal potential.* Resisted flow in a rotating channel is briefly considered in § 12.4.

3.12 **Cavitation**

It is found by experiment that a change of state, as from vapour to liquid or from solution to crystal, is influenced by *nuclei* whose presence can initiate a change which, in otherwise unchanged circumstances, would not occur in their absence. A substance in a given state is said to be *metastable* if it does not change its state in the absence of the appropriate nuclei but does change when such nuclei are present. Metastability is thus an intermediate condition between complete stability and instability; a substance in an unstable state will change its state even in the absence of nuclei. The precise nature of the nuclei is not always known, but techniques have been developed in various cases for the removal of nuclei. For example, ordinary air contains abundant nuclei for the condensation of water when the temperature falls below the dewpoint. These can be got rid of by expanding the air adiabatically so that its temperature falls sufficiently for a cloud of droplets to form; when the droplets have fallen under gravity, the air which is left is cleared of nuclei and can be cooled many degrees below its ordinary dewpoint without any condensation occurring. As is very well known, these facts are usefully applied in the 'cloud chamber' invented by C. T. R. Wilson for showing the paths of particles which cause ionisation, for the ions are effective nuclei for condensation.

Experiment has shown that liquids, and water in particular, are metastable or unstable when the pressure is negative. Specially treated liquids have, however, been found to withstand appreciable tensile stresses without rupture and the appearance of cavities. It can, nevertheless, be accepted that bubbles will, in practice, form in a liquid when the local pressure falls below the vapour pressure of the liquid, as it exists locally. This vapour pressure is influenced by many factors but temperature is the dominant one. It is found that the vapour pressure of any liquid rises rapidly with increase of temperature and becomes equal to the ambient atmospheric pressure at the boiling point. The vapour pressure is influenced to some extent by the presence of substances in solution, whether solid, liquid or gaseous. Cavitation is to be looked upon as a local boiling of the liquid which occurs where its pressure is sufficiently low, and this will happen where the local velocity is *high*, as follows from Bernoulli's theorem or equation (3.5,14). Let p_0 be the pressure in a certain region when the liquid is at rest and let p' be the total vapour pressure of the liquid as it exists locally. Then the resultant velocity q_c of the fluid when cavitation is about to begin is such that

$$\tfrac{1}{2}q_c^2 + \frac{p'}{\rho} = \frac{p_0}{\rho}$$

or

$$q_c^2 = \frac{p_0 - p'}{\tfrac{1}{2}\rho}, \tag{3.12,1}$$

where we have assumed the flow to be steady. Cavitation will start when the greatest velocity of the liquid reaches this critical value.†

Next, let us consider a body B advancing steadily with speed V and in a fixed attitude into a liquid at rest at infinity where the pressure is p_1 (body forces supposed absent). The velocity at any point P of the surface will, in fact, be inclined backwards where the relative velocity is highest. Let the velocity at P be cV inclined at the angle $\pi - \theta$ to the direction of motion, where c and θ will depend on the choice of P. To reduce the problem to one of steady motion we superpose the velocity $-V$ on the whole system. The resultant velocity q at P as given by the parallelogram of velocities is then such that

$$q^2 = V^2(1 + 2c\cos\theta + c^2) \tag{3.12,2}$$

and we get by Bernoulli's theorem

$$\tfrac{1}{2}q^2 + \frac{p}{\rho} = \tfrac{1}{2}V^2 + \frac{p_1}{\rho}$$

or, by (3.12,2),

$$\frac{p_1 - p}{\tfrac{1}{2}\rho V^2} = \left(\frac{q}{V}\right)^2 - 1$$

$$= 2c\cos\theta + c^2. \tag{3.12,3}$$

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† In a more general relation p' would be replaced by a critical pressure p_c depending on the nature of the nuclei and on the concentration of the substances in solution, including, in particular, gases.

Now let

$$\lambda = \frac{p_1 - p'}{\frac{1}{2}\rho V^2}. \tag{3.12,4}$$

This is called the *cavitation number* and it is non-dimensional (see also § 12.12). Also let λ_m be the maximum value of $(2c \cos \theta + c^2)$ for any point P on the surface of B and for the given attitude of B to the direction of motion. Then, in the light of what has been said, we expect cavitation to begin when

$$\lambda = \lambda_m \tag{3.12,5}$$

and cavitation should be absent so long as $\lambda > \lambda_m$. For an elliptic cylinder moving at right angles to its axis and in the direction of the semi-axis a of its elliptic section we have

$$\theta = 0, \qquad c = \frac{b}{a}$$

where b is the semi-axis perpendicular to the direction of motion and the maximum velocity occurs at the ends of this transverse axis. Hence by (3.12,3)

$$\lambda_m = \left(\frac{b}{a}\right)\left(2 + \frac{b}{a}\right). \tag{3.12,6}$$

When the ellipse is slender and the motion occurs in the direction of the major axis we have the approximation

$$\lambda_m = 2\Big(\frac{b}{\ }$$

$$= \text{twice thickness/chord ratio.}$$

A *small* value of the critical cavitation number is advantageous from the point of view of avoiding cavitation. For slender sections λ_m rises steeply when the flow becomes inclined to the chord (see further § 11.5).

When cavitation first begins, only very small bubbles are formed but these may have serious effects. Thus, when a bubble is formed and later collapses, a very high transient pressure is generated where the fluid elements meet. If this collapse occurs on the surface of an immersed body it will receive a blow and in persistent cavitation such blows will be applied with great frequency and perhaps for long periods. This hammering of the surface, perhaps working in conjunction with corrosion, is believed to be the cause of the pitting often observed to occur on propeller blades and parts of hydraulic machinery.

If the velocity be increased after cavitation has begun, large spaces void of fluid and attached to the moving body may be formed. The whole flow pattern is then altered significantly and the simple approach to the problem sketched above becomes inadequate. We then have to satisfy the condition that the pressure in a cavity has the constant value p', which may for most purposes be taken as zero, and the velocity of the liquid at the surface of the cavity is therefore constant (when the body forces are absent or neglected). When the flow is two-dimensional this problem can be treated by the 'hodograph method'.[†]

APPENDIX. CHAPTER 3

THE ACCELERATION POTENTIAL

We suppose here that the fluid is inviscid and barotropic and that the body forces, if any, can be derived from a potential. Accordingly the three components of the acceleration of a particle of fluid are given by equations (3.5,18) which may be written

$$a_x = -\frac{\partial \Phi}{\partial x}, \qquad a_y = -\frac{\partial \Phi}{\partial y}, \qquad a_z = -\frac{\partial \Phi}{\partial z} \tag{1}$$

where

$$\Phi = \varpi + \chi \tag{2}$$

and is called the *acceleration potential.* This function exists under the conditions stated, whether the motion is steady or unsteady, irrotational or rotational. Since the pressure and the potential of the body forces are continuous, the acceleration potential is continuous even when the velocity is discontinuous, e.g. at a vortex sheet.

In general Φ does not satisfy any particularly simple differential equation but when the flow is nearly uniform and the fluid incompressible Φ is harmonic, as shown below.

Let **A** be the acceleration vector. Then we have by the definition of the divergence of a vector

$$\operatorname{div} \mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \tag{3}$$

and when we substitute from equations (2.9,4) . . . (2.9,6) we obtain

$$\operatorname{div} \mathbf{A} = \frac{D\Delta}{Dt} + E \tag{4}$$

[†] See Lamb's *Hydrodynamics*, Chapter IV, under 'free streamlines' or Milne-Thomson's *Theoretical Hydrodynamics*, Chapter XII.

where
$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{5}$$
is the 'expansion,' D/Dt is the operator defined in equation (2.9,7) and
$$E = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial u}{\partial z} \cdots$$
$$\cdots + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial v}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial w}{\partial y}. \tag{6}$$
When the fluid is *incompressible* $\Delta = 0$ by the equation of continuity (2.3,9) and we obtain from (1), (3) and (4)
$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = -E. \tag{7}$$
Further, when the flow is irrotational, so that $\partial w/\partial y = \partial v/\partial z$ etc., (6) becomes
$$E = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \cdots$$
$$\cdots + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2. \tag{8}$$
Hence E is now necessarily *positive* and Φ, which satisfies (7), is called a *convex function.*

Lastly, let the motion be nearly uniform so that we can put
$$u = U + u', \qquad v = v', \qquad w = w' \tag{9}$$
where U is constant and u', v', w' and their differential coefficients with respect to x, y and z are small quantities of the first order. By (6) E is then a small quantity of the second order which may be neglected. Consequently (7) becomes
$$\nabla^2\Phi = 0. \tag{10}$$
Finally, Φ is harmonic when the fluid is incompressible and the flow nearly uniform.

The equations (9) are often used in technical applications of hydrodynamics and the quantities u', v', w' are called the *velocity perturbations.* The process of neglecting the squares and products of the perturbations and their spatial differential coefficients is an example of *linearization.*

EXERCISES. CHAPTER 3

1. A locomotive picks up water from a horizontal trough by means of a scoop with a forward-facing entry. The speed of the locomotive is V and the water in the trough is stationary while the vertical height of the discharge pipe above the free surface of the water in the trough is h. On the assumptions that frictional resistance to the flow is negligible and that a steady state has been reached, find the velocity of discharge and the minimum speed for any discharge.

   (Answer. Vel. of discharge $= \sqrt{(V^2 - 2gh)}$. $V_{\min} = \sqrt{(2gh)}$.)

2. *Simplified theory of the Giffard injector.* This apparatus works as follows. Steam from a boiler is expanded down to atmospheric pressure in a nozzle and thereafter mixes with cold water, is condensed and imparts its momentum to the mixture. The velocity of this is, in normal operation, great enough to overcome the boiler pressure, so the mixture can be fed into the boiler through a check valve. Regard the steam as an incompressible fluid of small density σ which, however, condenses when it meets the water (density ρ). Neglect frictional losses and assume momentum conserved in the mixing. Find the condition that feeding may occur if the masses of steam and cold water passing in unit time are m_s, m_w. (Treat the water as at rest before the mixing and neglect gravity.)

   (Answer. $m_s^2\rho > (m_s + m_w)^2\sigma$.)

3. The cross-sectional area A of a straight and slender vertical tube of length l varies linearly from A_1 at the upper to A_2 at the lower end. Discharge occurs at the constant rate Q from the lower end where the pressure is p_a (constant). Find the pressure at the distance s from the upper end. (Treat the fluid as perfect.)

   $$\left(\text{Answer. } p = p_a - g\rho(l - s) + \frac{\rho Q^2(A_1 - A_2)(l - s)[A_1(l - s) + A_2(l + s)]}{2A_2^2[(l - s)A_1 + sA_2]^2}.\right)$$

4. The problem is the same as in Exercise 3 but the pressure at the upper end of the tube varies in such a manner that the discharge varies at the rate dQ/dt. Find the addition to the pressure.

   $$\left(\text{Answer. } \frac{\rho l \dfrac{dQ}{dt}}{(A_2 - A_1)} \ln\left[\frac{lA_2}{lA_1 + s(A_2 - A_1)}\right].\right)$$

5. The stream function for the steady motion of a perfect fluid is

   $$\psi = -Uy + ke^{-ry} \sin rx$$

   and the body force has components $X = 0$, $Y = -g$ where U, k, r and g are all constant. Verify that the flow is irrotational and find the pressure, given that it is zero where $y = h$ (and it may be assumed that e^{-rh} is negligible).

   $$\left(\text{Answer. } p = g\rho(h - y) - rk\rho U\left[\sin rx + \frac{rk}{2U}e^{-ry}\right]e^{-ry}.\right)$$

6. In the last Exercise show that, when rk/U is very small, the streamline $\psi = 0$ is $y = (k/U) \sin rx$ and find the pressure on this line.

   $$\left(\text{Answer. } p = g\rho h - \frac{g\rho k}{U}\left(1 + \frac{rU^2}{g}\right) \sin rx.\right)$$

7. A perfect gas in steady flow through a tube of small cross-sectional area A behaves barotropically so that $pv^n = \text{constant}$. Body forces are absent and frictional losses negligible. Obtain the expression for the area ratio A/A_0 in terms of the pressure ratio p/p_0, given that the velocity is q_0 and density ρ_0 at the section where the area is A_0 and pressure p_0.

   $$\left(\text{Answer. } \left(\frac{A_0}{A}\right)^2 = \left(\frac{p_0}{p}\right)^{2/n}\left\{1 + \frac{2np_0}{(n - 1)\rho_0 q_0^2}\left[1 - \left(\frac{p}{p_0}\right)^{(n-1)/n}\right]\right\}.\right)$$

8. A perfect fluid is in two-dimensional motion with Cartesian velocity components

   $$u = -V + \frac{Va^2 \cos 2\theta}{r^2} - \frac{\Gamma \sin \theta}{2\pi r}, \qquad v = \frac{Va^2 \sin 2\theta}{r^2} + \frac{\Gamma \cos \theta}{2\pi r}.$$

Find the outward momentum flux in a layer of unit depth from the annular region between the circles $r = a$, $r = R > a$.

$$\left(\text{Answer. } X \text{ component zero. } Y \text{ component } -\tfrac{1}{2}\rho U\Gamma\left(1 - \frac{a^2}{R^2}\right).\right)$$

9. With the data as in Exercise 8 find the pressure thrust on the circle $r = R$ in a layer of unit depth (body forces absent). Combine with the former result to obtain the components of thrust per unit length on the cylinder $r = a$. (*Note.* The motion is that about a fixed cylinder $r = a$ in a uniform stream of velocity V and with circulation Γ.)

(Answer. X component zero. Y component $\rho U\Gamma$.)

10. With data as in Exercises 8 and 9 find the pressure thrust per unit length of the cylinder $r = a$ and verify that this agrees with the result obtained by application of the equations of momentum.
11. An inviscid fluid is in steady irrotational motion and body forces are absent. Show that the tangent to an isobar lying in the osculating plane of a streamline cuts the streamline at an angle given by

$$\tan\theta = -R\frac{\partial q}{\partial s}\bigg/ q$$

where s is the arc of the streamline and R is its radius of curvature. (*Note.* An *isobar* is a curve on which the pressure is constant.)

12. A barotropic and inviscid fluid is in steady motion. Show that $(\varpi + \chi)$ is constant on a curve which is such that its tangent at any point P is the binormal to the streamline at P. (*Note.* When body forces are absent the curve is an isobar.)
13. Show that the results of the last two Exercises serve to define the tangent plane to an isobaric surface (when body forces are absent).
14. A perfect fluid flows two-dimensionally in a layer of unit depth towards a sink of strength m situated at O. A straight radial wall extends from $r = a$ to $r = b$ $(b > a)$ and the pressure is zero where $r = b$ while body forces are absent. Find the resultant thrust on one face of the wall and the situation of the centre of pressure.

$$\left(\text{Answer. Thrust } \frac{\rho m^2(b-a)^2}{8\pi^2 ab^2} \text{ towards the fluid.}\right.$$

Distance from O to centre of pressure

$$\left.\frac{ab^2}{(b-a)^2}\ln\left(\frac{b}{a}\right) - \frac{a(b+a)}{2(b-a)}.\right)$$

15. A column of perfect fluid of length l lies within a cylinder and in contact with a piston which moves with velocity at (a constant). The pressure at the free end of the column is always zero and body forces are absent. Find the velocity potential and the pressure at the distance x from the piston.

(Answer. $\phi = -atx$. $p = \rho a(l - x)$.)

16. A cylinder whose normal section is the ellipse $x^2/a^2 + y^2/b^2 = 1$ is situated in a uniform stream of perfect fluid of speed U parallel to the major axis (two-dimensional steady flow, body forces absent). The pressure in the stream at infinity is p_0. Find the pressure at the surface and its maximum and minimum values. (Use the result of Exercise 50 of Chapter 2.)

$$\left(\text{Answer. } p = p_0 + \tfrac{1}{2}\rho U^2 - \tfrac{1}{2}\rho U^2\frac{(a+b)^2\sin^2\eta}{a^2\sin^2\eta + b^2\cos^2\eta}\right.$$

$$\left.\text{Max. } p = p_0 + \tfrac{1}{2}\rho U^2 \qquad \text{Min. } p = p_0 - \tfrac{1}{2}\rho U^2(2ab + b^2)/a^2.\right)$$

17. A perfect fluid is in steady motion and the velocity potential is

$$\phi = k \tan^{-1}\left(\frac{y}{x}\right).$$

If the potential of the body forces is $\chi = gz$ and the pressure is p_0 where $x = r_0, y = z = 0$, find the general expression for the pressure.

$$\left(\text{Answer. } p = p_0 - g\rho z + \frac{\rho k^2(x^2 + y^2 - r_0^2)}{2r_0^2(x^2 + y^2)}.\right)$$

18. A perfect fluid is in steady motion in contact with the fixed plane wall $z = 0$ and body forces are absent. The velocity potential is $\phi = -\frac{1}{2}a(x^2 + y^2 - 2z^2)$ and the pressure is p_0 at O. Find the radius of a circle with centre O lying in the face of the wall such that the resultant fluid thrust upon it is zero. (Answer. $r = 2\sqrt{(p_0/\rho a^2)}$.)

19. Liquid flows out of a sharp-edged circular orifice in the bottom of a large tank. A circular disk is held at a height h above the plane of the orifice. Show that the total downward force on the disk is

$$\pi\rho g\left\{(H - h)r_1^2 + \frac{Q^2}{4\pi^2 g h^2}\ln\left(\frac{r_2}{r_1}\right)\right\}$$

where r_1, r_2 are the radii of the orifice and disk respectively, H is the depth of liquid in the tank and Q is the rate of discharge (volume per unit time). (Neglect any force on the under side of the disk directly above the orifice.)

20. An infinite straight vortex of strength Γ is fixed at a distance a from a fixed and rigid plane wall to which it is parallel. The fluid is perfect, body forces are absent and the pressure is zero at infinity in the fluid and behind the wall. Find the thrust T on a strip of the wall lying perpendicular to the vortex and of unit width.

$$\left(\text{Answer. } T = \frac{\rho\Gamma^2}{4\pi a} \text{ towards the vortex.}\right)$$

21. An infinite straight and uniform line source of strength m is fixed at a distance a from a fixed and rigid plane wall to which it is parallel. The fluid is perfect, body forces are absent and the pressure is zero at infinity in the fluid and behind the wall. Find the thrust T on a strip of the wall lying perpendicular to the source and of unit width.

$$\left(\text{Answer. } T = \frac{\rho m^2}{4\pi a} \text{ towards the source.}\right)$$

22. A fixed three-dimensional source of strength σ is situated in a perfect fluid at the normal distance a from a fixed rigid plane wall. Body forces are absent and the pressure is zero in the fluid at infinity and behind the plane. Find the total thrust T on the plane.

$$\left(\text{Answer. } T = \frac{\rho\sigma^2}{16\pi a^2} \text{ towards the source.}\right)$$

23. A fixed three-dimensional doublet of strength μ is situated in a perfect fluid at the normal distance a from a fixed rigid plane wall and its axis is normal to the wall. Body forces are absent and the pressure is zero in the fluid at infinity and behind the plane. Find the total thrust T on the plane.

$$\left(\text{Answer. } T = \frac{3\rho\mu^2}{16\pi a^4} \text{ towards the doublet.}\right)$$

24. A perfect fluid is initially at rest in a tube which divides into two branches a and b at J_1 and the branches reunite at J_2; after J_2 the tube is single. The

fluid in the system is set in motion by a pressure difference applied across its ends. Discuss the division of the flux between the two branches.

(Answer. It follows from Kelvin's theorem that the "flow" from J_1 to J_2 is the same along a and b.)

25. A fixed ring-shaped rigid solid S is situated in a perfect fluid. The velocity potentials ϕ_1 and ϕ_2 are both such that the kinematic condition at the surface of S is satisfied. The circulation in a simple circuit C linked with the ring is Γ_1 for ϕ_1 and Γ_2 for ϕ_2. If the fluid is initially at rest and is set in motion by pressures applied at a boundary, show that the only possible velocity potential which depends linearly on ϕ_1 and ϕ_2 is a constant multiple of $(\Gamma_2\phi_1 - \Gamma_1\phi_2)$.

26. The Stokes' stream function for a steady motion of a perfect fluid is $\psi = \frac{1}{2}U\{y^2 - 2a(r - x)\}$ where $r^2 = x^2 + y^2$. Show that the streamline $\psi = 0$ consists of OX and the parabola $y^2 = 4a(a - x)$; show also that the components of velocity may be written

$$u = -U\left(1 - \frac{a}{r}\right), \qquad v = \frac{Ua(r - x)}{ry}$$

If body forces are absent and the pressure when x is $+\infty$ is p_0, find the pressure on the parabola $\psi = 0$. (*Note.* The motion is that about a fixed paraboloid of revolution placed in a uniform stream of velocity U parallel to its axis.)

$$\left(\text{Answer. } (p - p_0)/\rho = \frac{aU^2}{4a - 2x}.\right)$$

27. A layer of viscous liquid of depth h flows under the influence of gravity down a plane inclined at θ to the horizontal. On the assumptions that h is constant, all conditions steady and the pressure at the free surface constant, find the velocity as a function of the normal distance y from the plane, also the rate of discharge Q' per unit width of plate.

$$\left(\text{Answer. } u = \frac{g\sin\theta}{\nu}\left(hy - \tfrac{1}{2}y^2\right)\right.$$

$$\left. Q' = \frac{gh^3\sin\theta}{3\nu}.\right)$$

28. A cylindrical shell whose inner and outer radii are b and c is free to rotate and is situated between concentric cylinders of the same length and of radii a and d $(a < b < c < d)$. The spaces between the shell and the cylinders are filled with liquid of viscosity μ. If the angular velocities of the inner and outer cylinders are ω_a, ω_d respectively, find the angular velocity ω of the shell. (Assume motion two-dimensional and steady.)

(Answer. $\omega[b^2d^2(a^2 + c^2) - a^2c^2(b^2 + d^2)]$
$= \omega_a a^2b^2(d^2 - c^2) + \omega_d c^2d^2(b^2 - a^2)$.)

29. A cylindrical drum of radius r and breadth b rotates with angular velocity ω inside a concentric fixed casing and the constant gap h is very small. Liquid of viscosity μ escapes through the gap under a constant pressure difference p across the drum. Find the total power loss and the width of the gap for which it is a minimum. (Assume that the leakage may be calculated as for parallel planes and that the longitudinal and circumferential motions of the liquid are independent. Retain only the lowest powers of h.)

$$\left(\text{Answer. Total power loss } \frac{\pi h^3 r p^2}{6\mu b} + \frac{2\pi\mu b r^3\omega^2}{h}.\right.$$

This is a minimum when $h = \sqrt{(2\mu b r\omega/p)}$.)

30. The investigation which leads to equation (3.10,12) remains valid when h is a function of x provided that the rate of taper of the channel is very small and the motion slow enough for the acceleration of the fluid to be negligibly small. In such circumstances, find the expression for the pressure drop p over a length l of the channel and give the result when the initial and final widths are h_1, h_2 with both walls plane.

$$\left(\text{Answer. } p = 12\mu Q'\int_0^l \frac{dx}{h^3}\cdot p = \frac{6\mu Q' l(h_1 + h_2)}{h_1^2 h_2^2}.\right)$$

31. A viscous incompressible fluid is in unsteady two-dimensional motion in circles about the origin and the tangential velocity is

$$u_t = \frac{1}{r} f\left(\frac{r^2}{\nu t}\right).$$

Find the vorticity ζ and show that it satisfies the equation of diffusion (3.10,38) provided that

$$f(x) = a + be^{-x/4}$$

where a and b are arbitrary constants.

$$\left(\text{Answer. } \zeta = \frac{2}{\nu t} f'\left(\frac{r^2}{\nu t}\right).\right)$$

CHAPTER 4

SIMILARITY AND THE APPLICATION OF DIMENSIONAL ANALYSIS

4.1 Introduction

Let us begin by considering an infinitely long circular cylinder of radius a moving at right angles to its axis with constant speed U in the fixed direction OX through a *perfect fluid* which is at rest, except as disturbed by the cylinder itself. The velocity potential for the motion of the fluid is given by equation (3.8,10) and the corresponding rectangular Cartesian components of velocity are given by

$$\left.\begin{aligned}\frac{u}{U} &= \left(\frac{a}{r}\right)^2 \cos 2\theta \\ \frac{v}{U} &= \left(\frac{a}{r}\right)^2 \sin 2\theta,\end{aligned}\right\} \tag{4.1,1}$$

where the origin of the polar coordinates is the centre of the circular section of the cylinder and the initial line is the direction of motion OX. It follows from these equations that both u and v are fixed fractions of U when the angle θ and the ratio $r{:}a$ are constant. This exemplifies a general law of similarity for the flow of a perfect fluid: a body of given shape whose typical linear dimension is l advances steadily and in a given fixed attitude with velocity U through a perfect fluid. Then the velocity components for a point in the fluid whose coordinates relative to the body bear fixed ratios to l are in fixed ratios to U.

Further, let p be that part of the fluid pressure which depends on the constant velocity U and let ρ be the constant density of the fluid. Then $p/\rho U^2$ is constant for a point whose coordinates relative to the body bear fixed ratios to l. It readily follows from this that the component F of the force which the fluid exerts on the body in any direction fixed relative to the body and its direction of motion, and which depends on its constant velocity, is proportional to $\rho U^2 l^2$, i.e. the dynamic force on the body is $k\rho U^2 l^2$, where the value of k depends only on the shape of the body, its attitude to the direction of motion and on the particular component of force considered. These propositions regarding the dynamics of perfect fluids can be further generalized, as explained below.

Suppose next that the fluid, while still uniform and incompressible, is viscous and has constant viscosity μ. When the fluid flows and is subject to distortion, forces come into play which would be absent if the fluid were inviscid. Hence, as will be shown in detail later, similarity of flow will, in general, be maintained only when a certain new condition is satisfied, in addition to those which are requisite for a perfect fluid. The new condition is that the quantity

$$R = \frac{\rho U l}{\mu} = \frac{U l}{\nu} \tag{4.1,2}$$

shall have the same numerical value for the bodies considered. This quantity is non-dimensional (see §§ 4.7 and 7.3) and is called the *Reynolds number*. When the fluid is compressible there are still further conditions to be satisfied in order that there may be similarity of flow (see § 4.12).

In this chapter we give a brief discussion of the similarity of physical systems, with special reference to the mechanics of fluids, and explain the application of the technique of dimensional analysis in obtaining the quantitative conditions for similarity of behaviour.†

4.2 Similar Physical Systems

The statement that two physical systems are similar implies that they are geometrically similar but signifies much more than this. The further conditions for similarity are concerned with:

(a) the internal constitution of the systems

(b) the conditions at their boundaries.

As regards (a), all *relevant* physical properties must be similarly distributed in the two systems. Suppose that p_1 and p_2 are the measures of the same scalar property for systems 1 and 2 respectively. Then these are similarly distributed when

$$p_2 = k p_1 \tag{4.2,1}$$

where k is a constant and p_2 and p_1 are measured at geometrically corresponding points in the two systems. Such points will have coordinates, referred to corresponding axes in the two systems, which are related by the equations

$$x_2 = \lambda x_1, \qquad y_2 = \lambda y_1, \qquad z_2 = \lambda z_1 \tag{4.2,2}$$

where λ is the constant linear scale factor or ratio. The last equations are equivalent to the single equation

$$\mathbf{R}_2 = \lambda \mathbf{R}_1 \tag{4.2,3}$$

between the position vectors of the corresponding points, referred to corresponding origins and axes in the systems. Examples of physical quantities whose distributions may be related as here described are density, thermal conductivity, specific heat, elastic modulus. It follows from the foregoing that the surfaces or curves on which p_1 and p_2 have corresponding values in accordance with (4.2,1) are geometrically similar. As regards (b), we may take the kinematic boundary conditions as examples. The boundaries themselves must be similar and correspond as required by equations (4.2,2) and the components of velocity at geometrically corresponding points must be related by the equations

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† For a more thorough and detailed discussion of these topics the reader may consult *Physical Similarity and Dimensional Analysis* by W. J. Duncan (Edward Arnold, 1953).

$$u_2 = \nu u_1, \qquad v_2 = \nu v_1, \qquad w_2 = \nu w_1, \tag{4.2,4}$$

where ν is the constant velocity ratio. Similar equations apply to other vector quantities, e.g. accelerations, if any, which may be specified at boundaries. Scalar quantities whose values are specified at boundaries must be related at corresponding points of the boundaries by equations of the type (4.2,1). It will be noted that there will, in general, be several independent constants of proportionality for the various relevant quantities. For example, in fluid systems we shall have factors for linear scale, velocity, density and possibly for other quantities. Under boundary conditions we include *initial conditions* which are conditions to be satisfied at a particular instant of time which is taken as the origin for the measurement of time. The initial conditions are thus conditions to be satisfied at the initial *temporal boundary* and they must be specified throughout the whole system and not merely at its spatial boundaries.

4.3 Similarity of Behaviour

Two systems are said to behave similarly when corresponding scalar quantities, measured at corresponding points as related by equations (4.2,2), are *always* related by an equation of the type (4.2,1), while corresponding vector quantities are related by equations of the type (4.2,4), where ν is now to be replaced by a general constant of proportionality. We may emphasise that these relations must be satisfied *throughout* the two systems considered and *at all times* subsequent to the initial instant.

When two systems are behaving similarly, knowledge of the behaviour of one will enable us to deduce what the behaviour of the other must be. For example, knowledge of the behaviour of a model will enable us to deduce the corresponding behaviour of its full-scale prototype, always provided that the two systems are behaving similarly. It will be shown below that two physically similar systems will behave similarly when certain relations between the various relevant scale factors are satisfied, but not otherwise, in general. These sufficient numerical conditions can be obtained in many ways. Some useful methods of obtaining them, including the application of dimensional analysis, are very briefly discussed below.

4.4 Dynamical Similarity

We shall now consider two dynamical systems which are strictly similar and enquire into the further conditions which are sufficient to ensure that their behaviours shall be strictly similar. Call the systems 1 and 2. Then corresponding points of the systems are such that their position vectors, referred to sets of fixed axes in the respective systems, satisfy the equation

$$\mathbf{R}_2 = \lambda \mathbf{R}_1 \tag{4.4,1}$$

where λ is the constant linear scale factor. We shall suppose that the body forces, or forces per unit mass f_1, f_2 *measured at corresponding points* are related by the vector equation

$$\mathbf{f}_2 = \mu \mathbf{f}_1 \tag{4.4,2}$$

where μ is constant. We shall further suppose that the *initial velocity vectors* $\mathbf{v}_1$, $\mathbf{v}_2$, measured at corresponding points, are related by the equation

$$\mathbf{v}_2 = \nu \mathbf{v}_1 \tag{4.4,3}$$

where ν is a constant.

Take first the case where the "system" is a mere particle. Let the particles considered be at the corresponding points P_1, P_2 at the initial instants, also let Q_1, Q_2 be points on the paths such that

$$P_2Q_2 = \lambda P_1Q_1$$

where P_1Q_1 is very small. Let these intervals be described in times dt_1, dt_2 respectively. Then, on account of (4.4,3), we have

$$\frac{dt_2}{dt_1} = \frac{P_2Q_2}{\mathbf{v}_2} \bigg/ \frac{P_1Q_1}{\mathbf{v}_1} = \frac{\lambda}{\nu}. \tag{4.4,4}$$

The body forces have the same directions relative to the local axes in the two systems and are equal to the corresponding accelerations. Hence the velocity increments have corresponding directions and

$$\frac{\text{increment of velocity for particle 2}}{\text{increment of velocity for particle 1}} = \frac{\mathbf{f}_2\, dt_2}{\mathbf{f}_1\, dt_1} = \frac{\mu\lambda}{\nu}. \tag{4.4,5}$$

The velocity vectors at Q_2 and Q_1 will still satisfy (4.4,3) provided that the velocity increments are in the same ratio as the initial velocities. By (4.4,3) and (4.4,5) the condition for this is

$$\frac{\mu\lambda}{\nu} = \nu$$

or

$$\frac{\nu^2}{\lambda\mu} = 1. \tag{4.4,6}$$

Let this equation be satisfied. Then, when the particles move from Q_1, Q_2 to corresponding points R_1, R_2 respectively, the velocity increments and therefore also the resultant velocities will again be in corresponding directions and be in proportion to the initial velocities. Thus we see that the particles will describe geometrically similar paths and their velocities at points which correspond in accordance with (4.4,1) will always satisfy (4.4,3). The two motions will thus be strictly similar.

Let l_1 and l_2 be corresponding linear dimensions of the paths of the particles, and v_1 and v_2 velocities measured at corresponding points of their paths and likewise f_1 and f_2 body forces measured at corresponding points. Then, in accordance with the foregoing, we have

$$l_2 = \lambda l_1, \qquad f_2 = \mu f_1, \qquad v_2 = \nu v_1. \tag{4.4,7}$$

Hence

$$\frac{{v_2}^2}{l_2 f_2} = \left(\frac{\nu^2}{\lambda\mu}\right)\frac{{v_1}^2}{l_1 f_1} = \frac{{v_1}^2}{l_1 f_1} \tag{4.4,8}$$

on account of (4.4,6). The quantity $\left(\dfrac{v^2}{lf}\right)$ is non-dimensional (see § 4.7) and is called the *Froude number*.† We see, therefore, that equality of the Froude numbers (measured at corresponding points) is a numerical condition for similarity of the motions of two particles. In the foregoing discussion and hereafter we use the expression "body force" as equivalent to the *force per unit mass* and the origin or nature of the force is irrelevant; it may, for example, arise from contact with another particle.

The mass of the particle does not appear explicitly in the above investigation but the masses must satisfy a condition when the dynamical systems considered contain several particles which exert forces on each other. Since, by the Third Law of Motion, action and reaction are equal and opposite, it follows that the accelerations induced by the mutual action of a pair of particles will remain in a fixed ratio only so long as their masses are in a fixed ratio. Hence it is, in general, necessary for similarity of behaviour that corresponding masses of the two systems shall be in a fixed ratio. Since the corresponding linear dimensions are in a fixed ratio, this amounts to saying that the *densities at corresponding points of the systems are in a fixed ratio*. This we have already recognised as a condition of similarity of the internal structures of the systems.

The expression for the Froude number F can be written in various ways. Let P be a typical force and m a typical mass of the system considered. Then the typical "body force" is

$$f = \frac{P}{m} \tag{4.4,9}$$

and the expression for the Froude number is

$$F = \frac{v^2}{lf} = \frac{mv^2}{lP}. \tag{4.4,10}$$

When forces of several different kinds influence the motion of a system there will be a Froude number for each of these and, for similarity of behaviour, it will be necessary, in general, for the values of all the corresponding Froude numbers to agree. Suppose that P and P' are forces of different kinds, e.g., weights and frictional forces. Then there will be two Froude numbers

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† Some writers apply the name to the square root of this quantity.

$$F = \frac{mv^2}{lP} \quad \text{and} \quad F' = \frac{mv^2}{lP'} \tag{4.4,11}$$

which must be kept constant in order that similarity of behaviour may be maintained. This amounts to saying the F and P'/P must be kept constant.

Equation (4.4,8), which expresses the equality of the Froude numbers of two systems, may be called *the law of corresponding speeds*, for the equation shows how the speeds must be related in order that the systems shall behave similarly. Now the times t_1 and t_2 occupied by corresponding motions will be such that

$$\frac{t_1 v_1}{l_1} = \frac{t_2 v_2}{l_2}. \tag{4.4,12}$$

When we square this and use (4.4,8) we derive

$$\frac{t_1^2 f_1}{l_1} = \frac{t_2^2 f_2}{l_2} \tag{4.4,13}$$

and on division by (4.4,12) this yields

$$\frac{t_1 f_1}{v_1} = \frac{t_2 f_2}{v_2}. \tag{4.4,14}$$

Equations (4.4,12) to (4.4,14) are various expressions of *the law of corresponding times*. These apply, for instance, to periodic times in oscillatory phenomena.

The Froude number has numerical significance for a system only after the typical length l, typical velocity v and typical body force f have been *defined* and there is always an element of arbitrariness in this. The choice of these quantities should be guided by convenience.

It is important for the reader to appreciate that equality of all the corresponding Froude numbers of two completely similar systems is a *sufficient* condition for similarity of behaviour but not always a necessary condition for similarity of motion. To illustrate this, let us consider a body moving in a perfect fluid which is enclosed by a rigid outer boundary and let the system be in a uniform gravitational field of intensity g. Then we know that there will be similarity of flow in similar systems irrespective of the value of the Froude number v^2/lg. However, corresponding total pressures will only be in a fixed ratio when the Froude numbers of the systems are the same.

The foregoing investigation of dynamical similarity makes no pretence to be complete but it should suffice to introduce the reader to the essential ideas.

ILLUSTRATIVE EXAMPLES OF DYNAMICAL SIMILARITY

Example 1. *Projectile in uniform gravitational field*

We shall suppose the projectile to move in a vacuum and to be projected with speed V at the angle α to the horizontal. Then it can easily be shown that the greatest height h of the path above the point of projection is given by

$$h = \frac{V^2 \sin^2 \alpha}{2g}.$$

Let us take h as the typical length, the initial velocity V as the typical velocity and g as the typical body force. Then the Froude number is

$$F = \frac{V^2}{gh} = 2 \operatorname{cosec}^2 \alpha.$$

This is constant so long as α is constant, as required for similarity of the initial conditions.

Example 2. *Body moving uniformly and parallel to the free surface in a perfect fluid and subject to a uniform gravitational field*†

This covers the cases of a surface ship and of a submerged submarine. For geometric similarity the draught of the ship must be a fixed fraction of its length at the water line while the depth of immersion of the submarine must bear a fixed ratio to its length or diameter. Take the length of the submarine or length of the ship at the water line as the typical linear dimension l. Also let the constant speed of the vessel be the typical speed V and let g be the typical body force. Then the Froude number is

$$F = \frac{V^2}{lg}.$$

Under the conditions postulated the motion of the vessel will be resisted since kinetic energy is constantly supplied to the fluid to maintain the system of surface waves set up by the vessel. When the Froude number is constant the pressures at corresponding points of the hulls of the similar vessels considered will be proportional to ρV^2 and the resistance will be given by

$$R = k\rho V^2 l^2$$

where k will depend on the Froude number. But $V^2 = lg\,F$ and

$$R = kF\rho g l^3,$$

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† Here the effects of a gaseous phase adjacent the free surface are assumed negligible.

where kF is constant so long as F is constant. Thus for similar vessels and for equal values of the Froude number

$$R = \text{const. } \rho g l^3.$$

Now let a geometrically similar model of a ship be tested in a tank where ρ and g have the same values as for the ship itself. The speed V' for the model of length l' which corresponds to the ship speed V is, since F is to be unchanged, given by

$$V' = V\sqrt{\left(\frac{l'}{l}\right)}$$

and the resistance R' of the model will be given by

$$R' = R\left(\frac{l'}{l}\right)^3.$$

These equations embody *Froude's Law of Comparison* which can be stated thus: when the speed of a similar model of a ship is equal to that of the ship multiplied by the square root of the length ratio the resistance is equal to that of the ship multiplied by the displacement ratio. It is to be understood that this statement applies to the wave-making resistance only since we have supposed the fluid to be perfect. The quantity $V/\sqrt{l}$ which must have the same value for the ship and its model is called the 'reduced speed'.

4.5 **Similarity of Flow**

We shall now apply the principles explained in § 4.4 to *steady* fluid motion and we shall begin by considering a fluid which is *uniform and incompressible but viscous*, the constant viscosity being μ. In outline the method which we shall use is as follows: we take two fluid systems which are strictly similar in the sense explained in § 4.2 and we shall *assume* that their motions are similar. We can then find how the corresponding forces of various kinds compare in the two systems; in accordance with § 4.4 the ratio of any two selected forces of different kinds must, for similarity, be the same for the two systems. In this way we obtain the quantitative conditions for similarity of behaviour.

First let us consider the 'inertia forces' or reversed mass-accelerations of small elements of the systems which correspond geometrically in accordance with equations (4.2,2). The volume of an element is *proportional* to l^3, where l is the representative length of the particular system to which the element belongs, and the mass is proportional to ρl^3. Corresponding components of acceleration at corresponding elements are, since we are assuming the motions to be steady and similar, proportional to v^2/l. Thus, let V be the resultant velocity at the point considered and r the radius of curvature of the path. Then the normal component of acceleration is

V^2/r and this is proportional to v^2/l. Again, the tangential component of acceleration is

$$\frac{dV}{dt} = \frac{dV}{ds}\frac{ds}{dt} = V\frac{dV}{ds}$$

where s is the arc of the path of the element. Now dV/ds is proportional to v/l and therefore the tangential component of acceleration is proportional to v^2/l. The inertial force on the element is (mass of element) × (reversed acceleration of element) and this is proportional to

$$\rho l^3 \times v^2/l = \rho v^2 l^2. \tag{4.5,1}$$

The surface area of the element is proportional to l^2 and the viscous stress is proportional to μ times a velocity gradient. The velocity gradient is proportional to v/l, so the stress is proportional to $\mu v/l$ and the force on the element is proportional to μvl. We saw in § 4.4 that, if P and P' are forces of different kinds, the ratio P/P' must remain constant for similarity of behaviour. Hence the ratio (inertia force on element):(viscous force on element) must remain constant for similarity of behaviour. This requires that

$$\frac{\rho v^2 l^2}{\mu vl} = \frac{\rho vl}{\mu} = \frac{vl}{\nu} = R \tag{4.5,2}$$

shall be constant. This quantity is again called the Reynolds number and it is non-dimensional (see § 4.7), being the ratio of two forces. Another way of looking at the Reynolds number is that it is, in effect, a Froude number for the viscous forces. We have seen that the force on an element due to the viscous stresses is proportional to μvl while the mass of the element is proportional to ρl^3. Hence the corresponding body force f is proportional to $\mu v/\rho l^2$ and the Froude number v^2/lf is proportional to

$$\frac{v^2}{\mu v/\rho l} = \frac{\rho vl}{\mu}.$$

Like the Froude number, the Reynolds number only has numerical significance when the representative length and velocity have been defined.

For a viscous incompressible uniform fluid with a free surface, in a uniform gravitational field, there are two numerical conditions for the similarity of behaviour of two similar systems in steady motion, namely the equality of the Froude numbers v^2/lg for the systems and the equality of their Reynolds numbers vl/ν. When the fluid is contained within a fixed boundary gravity does not affect the flow and the only numerical condition for similarity of flow is equality of the Reynolds numbers (see the end of § 4.4).

Next, let us consider a compressible fluid in steady motion and let its bulk modulus of elasticity be K. The stress for similar elements in systems behaving similarly will be proportional to K, for the volume strains at such elements will be the same.† Hence an elastic force will be proportional to Kl^2 and the corresponding body force will be proportional to

$$\frac{Kl^2}{\rho l^3} = \frac{K}{\rho l}.$$

Hence the Froude number for the elastic forces will be proportional to

$$\frac{v^2}{K/\rho} = \frac{\rho v^2}{K}. \tag{4.5,3}$$

Now (see § 10.3) the velocity of sound a in the fluid is given by Newton's formula

$$a = \sqrt{\left(\frac{K}{\rho}\right)}. \tag{4.5,4}$$

Hence

$$\frac{\rho v^2}{K} = \left(\frac{v}{a}\right)^2. \tag{4.5,5}$$

The condition that $\rho v^2/K$ shall be constant, as required for similarity of behaviour, is thus equivalent to the condition that v/a shall be constant. This is called the *Mach number* and represented by the symbol M; it is non-dimensional, being the ratio of two velocities. If the fluid is inviscid and gravity is absent (or the conditions such that it is unimportant), equality of the Mach numbers is the single numerical condition for similarity of the behaviours of similar systems. There is, however, a restriction implicit in the assumption that there is a unique modulus of elasticity K. This is further discussed in § 4.11. If the fluid is viscous and compressible, equality of the Reynolds numbers will be an additional condition for similarity of behaviour. Finally, if gravity is influential, the Froude number v^2/lg must have the same value for the systems considered.

For similar systems and provided that F, R and M are constant, the pressure is proportional to ρv^2 or $p/\rho v^2$ is constant. However, the value of $p/\rho v^2$ will, in general change when F, R and M change. This amounts to saying that

$$\frac{p}{\rho v^2} = f(F, R, M) \tag{4.5,6}$$

where the form of the function cannot be found by general arguments based on the principle of similarity alone. The quantity on the left of the last equation is non-dimensional and is called a non-dimensional pressure coefficient. Similarly, if P is a force on an immersed body, we deduce from (4.5,1) that for similar systems

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† The reader may be reminded that volume strain = (increase of volume)/(original volume) and that stress = strain × elastic modulus.

$$\frac{P}{\rho v^2 l^2} = \phi(F, R, M) \tag{4.5,7}$$

where the expression on the left is a non-dimensional force coefficient. Again, if N be a moment on an immersed body, we have for similar systems

$$\frac{N}{\rho v^2 l^3} = \psi(F, R, M). \tag{4.5,8}$$

These are precisely the conclusions drawn from the application of dimensional analysis (see below). When gravity is absent or ineffective, F may be removed from these formulae and M is to be removed when the fluid is incompressible.

4.6 Units and Measures

As a first step towards an understanding of dimensional analysis we must consider the relation between a *physical quantity*, its *measure* and the *unit of measurement*. First of all, the unit *must* be a physical quantity of the *same nature* as the quantity to be measured and it is constant. The measure of the physical quantity is its ratio to the unit of like physical nature and it is therefore a *number*. This relation between the physical quantity Q, its measure q and the unit U can be represented symbolically by the equation

$$Q \equiv qU \tag{4.6,1}$$

where the symbol $\equiv$ indicates physical equality. The symbols representing physical quantities which appear in the mathematical equations and formulae related to a physical phenomenon therefore stand for the product of the numbers which are the measures of the physical quantities in question and the corresponding selected units. For the sake of brevity these units are frequently omitted where they are assumed to be understood. Once the unit has been chosen the measure of a given physical quantity becomes a *unique* number.

Suppose now that we change the unit from U to U' and let

$$U' \equiv kU \tag{4.6,2}$$

where k is the measure of U' with U as unit. Also let

$$Q \equiv q'U' \tag{4.6,3}$$

where q' is the measure of Q with U' as unit. When we use (4.6,2) the last equation becomes

$$Q \equiv q'kU$$

and on comparison with (4.6,1) we see that

$$q'k = q \tag{4.6,4}$$

since the measure of Q with U as unit is unique. The last equation can be written

$$q' = \frac{q}{k} \tag{4.6,5}$$

so the measure of a given physical quantity is changed in the ratio $1:k$ when the unit of measurement is altered in the ratio $k:1$.

Units are either *fundamental* or *derived*. Any physical science has a structure of laws and hypotheses such that once a certain small number of units of basic quantities are specified (fundamental units) the units of any other quantity can be described in terms of those fundamental units (the former are then *consistent* or *coherent* derived units) or in terms of units which are simply proportional to the fundamental units (non-consistent derived units). In mechanics the fundamental units are three in number and are conveniently taken as those of length, mass and time. Thus, if the unit of length is 1 m and of time is 1 s, the consistent unit of velocity is 1 m/s. The great advantage of consistent units is that the equations require no change by way of numerical conversion factors if the fundamental units are changed.

4.7 Measure or Dimensional Formulae

Suppose the fundamental units of length, mass, time, etc., are changed by numerical factors L, M, T, etc, respectively. Then a consistent derived unit for a quantity would in consequence change by a factor which could be expressed as a function of (L, M, T, etc.). This expression is here called the measure formula or dimensions of that quantity. For example, the new consistent unit of area is a square whose side is the new unit of length or L of the original units of length. Hence the square contains L^2 of the original consistent units of area. Consequently the measure formula of the derived unit of area is L^2. Similarly the measure formula of the derived unit of volume is L^3. Again, the new consistent unit of velocity is such that L of the original units of length are traversed in T of the original units of time. Consequently the new consistent unit of velocity contains L/T of the original consistent units of velocity and the measure formula of velocity is accordingly LT^{-1}. The new consistent unit of acceleration is such that the new consistent unit of velocity is acquired in the new unit of time. Therefore the measure formula of acceleration is LT^{-2}. The new consistent unit of force imparts acceleration equal to LT^{-2} of the original consistent units to a mass equal to M of the original units and its measure in terms of the original consistent unit of force is accordingly MLT^{-2}; this is the measure formula

TABLE 4.7,1.

Measure Formulae or Physical Dimensions of Quantities Occurring in Mechanics.

Based on mass, length and time as fundamental units.

Quantity	Measure formula	Quantity	Measure formula
Mass	M	Mass per unit area	ML^{-2}
Length	L	Mass moment	ML
Time	T	Moment of inertia and product of inertia	ML^2
Speed or velocity	LT^{-1}	Stress and pressure	$ML^{-1}T^{-2}$
Acceleration	LT^{-2}	Strain	$M^0L^0T^0$
Momentum and impulse	MLT^{-1}	Elastic modulus	$ML^{-1}T^{-2}$
Force	MLT^{-2}	Flexural rigidity of a beam EI	ML^3T^{-2}
Energy and work	ML^2T^{-2}	Torsional rigidity of a shaft GJ	ML^3T^{-2}
Power	ML^2T^{-3}	Linear stiffness (force per unit displacement)	MT^{-2}
Moment of force	ML^2T^{-2}	Angular stiffness (moment per radian)	ML^2T^{-2}
Angular momentum or moment of momentum	ML^2T^{-1}	Linear flexibility or receptance (displacement per unit force)	$M^{-1}T^2$
Angle	$M^0L^0T^0$	Vorticity	T^{-1}
Angular velocity	T^{-1}	Circulation (hydrodynamics)	L^2T^{-1}
Angular acceleration	T^{-2}	Viscosity	$ML^{-1}T^{-1}$
Area	L^2	Kinematic viscosity	L^2T^{-1}
Volume and first moment of area	L^3	Diffusivity of any quantity	L^2T^{-1}
Second moment of area	L^4	Coefficient of solid friction	$M^0L^0T^0$
Density	ML^{-3}	Coefficient of restitution	$M^0L^0T^0$

of force. Proceeding in this way we can obtain the measure formula of the unit of any quantity arising in mechanics. Such formulae are summarised in Table 4.7,1.

In view of the discussion in § 4.6 we see that the new measure of a given physical quantity resulting from changes of the sizes of the fundamental units is obtained by *dividing* the original measure by the measure formula of the quantity, with the particular values of L, M, T substituted, where these are the measures of the new fundamental units in terms of the old.

It is convenient to have a special symbol to indicate equality as regards measure formulae and we shall adopt the symbol $\boxminus$.† Thus we have, for example,

$$\text{Force} \boxminus MLT^{-2}$$

$$\text{Kinematic viscosity} \boxminus L^2T^{-1}$$

$$\boxminus \text{circulation.}$$

Also, if
$$M^aL^bT^c \boxminus M^\alpha L^\beta T^\gamma$$
then
$$a = \alpha \qquad b = \beta \qquad c = \gamma.$$

We must emphasise that the symbol $\boxminus$ indicates *nothing more* than the identity of the measure formulae of the quantities connected by the symbol. Thus the symbol does *not* indicate numerical equality of the quantities nor does it imply that the equated quantities are of the same physical nature.

The measure formula of a *non-dimensional quantity* has all its indices zero. Consequently the measure of a non-dimensional quantity is independent of the sizes of the fundamental units, always provided that consistent units are employed. We have already in § 4.5 drawn attention to the facts that the Froude, Reynolds and Mach numbers are non-dimensional and have defined non-dimensional coefficients for pressures, forces and moments.

4.8 **Dimensional Analysis**

Dimensional analysis‡ is a particular technique for obtaining the quantitative conditions for similarity of behaviour of strictly similar physical systems and, once understood, it is easy to apply. The basis of dimensional analysis is the *principle of dimensional homogeneity* which can be stated thus:

Every term in a complete physical equation has the same measure formula. An equation is *complete* when every essential factor is explicitly represented. Incomplete equations are sometimes used by engineers because a certain quantity, which in reality plays an essential part, has a constant measure in ordinary practice. Examples are "g" and the density of fresh water. However, in the systematic development of theory it is necessary to use complete equations.

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† This was introduced in 'A Review of Dimensional Analysis' by W. J. Duncan, *Engineering*, June 1949, pp. 533, 556 and used consistently in the book *Physical Similarity and Dimensional Analysis.*

‡ This was at one time known as 'the method of dimensions.'

The principle of dimensional homogeneity enables us to deduce useful information about the dependence of one physical quantity on the other relevant physical quantities. In the simplest cases the form of the relationship is completely determined but it contains a numerical constant whose value cannot be found by dimensional analysis. In general, the relation yielded by dimensional analysis contains an undetermined function of one or more variables. It will be clear from the above principle that by dividing a physical equation by any of the terms involved it becomes a relation between non-dimensional quantities. We can however go further and specify the least number of independent non-dimensional quantities that can describe the relationship. The general theorem of dimensional analysis goes by the name of the "Pi Theorem" and can be stated as follows:

Any complete physical relationship can be represented as one subsisting between a set of independent non-dimensional product combinations of the physical quantities concerned.

Moreover, the relationship *necessarily* takes this form when it is so written that the number of distinct variables is a minimum. The *least possible* number of independent non-dimensional quantities which appear in the relationship is equal to the number of the related physical quantities *less* the number of the fundamental units and in non-singular cases the number is equal to this minimum.

The conclusions drawn from dimensional analysis can always be derived by general arguments about similarity or from the differential equations of the phenomenon under consideration. However, the technique of dimensional analysis enables the required results to be obtained with great ease.

In the following sections we shall apply dimensional analysis to questions of the mechanics of fluids.

4.9 Perfect Fluids

Let us consider the force on a body of given form placed in a steady stream of uniform perfect fluid and held in a fixed attitude. Let P be the component of the dynamic force† in a fixed direction, l the typical linear dimension of the body, v the undisturbed velocity of the stream relative to the body and ρ the constant density of the fluid. In accordance with the procedure of dimensional analysis we assume that

$$P = kl^{\alpha}v^{\beta}\rho^{\gamma} \tag{4.9,1}$$

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† That is, the part of the force which depends on the velocity. Any force of buoyancy is excluded.

where k, α, β, γ are constants. By the principle of dimensional homogeneity this yields the dimensional relation

$$MLT^{-2} \mathrel{\stackrel{\frown}{=}} L^{\alpha}(LT^{-1})^{\beta}(ML^{-3})^{\gamma}$$

where we have used the measure formulae listed in Table 4.7,1. The last equation is equivalent to

$$MLT^{-2} \mathrel{\stackrel{\frown}{=}} M^{\gamma}L^{\alpha+\beta-3\gamma}T^{-\beta}$$

and when we equate corresponding indices we get the following *indicial equations*

$$\begin{aligned} &(M) \quad && 1 = \gamma \\ &(L) \quad && 1 = \alpha + \beta - 3\gamma \\ &(T) \quad && -2 = -\beta. \end{aligned}$$

The first and third equations give $\gamma = 1$, $\beta = 2$ and then the second gives $\alpha = 2$. Hence (4.9,1) becomes

$$P = k\rho l^2 v^2. \tag{4.9,2}$$

This can be written alternatively

$$\frac{P}{\rho v^2 l^2} = k. \tag{4.9,3}$$

The expression on the left is non-dimensional for both numerator and denominator have the measure formula MLT^{-2}. We have already called this a non-dimensional force coefficient in § 4.5 and it will be seen that the result of the dimensional analysis agrees with that obtained from considerations of dynamical similarity. In the same manner we find that, if N is a component of the dynamic couple on the body, then

$$\frac{N}{\rho v^2 l^3} = k' \tag{4.9,4}$$

where k' is a constant and the expression on the left is a non-dimensional moment coefficient.† Similarly, let p be the dynamic pressure at points occupying corresponding positions on or in the neighbourhood of bodies of the geometrically similar family under consideration.

Then

$$\frac{p}{\rho v^2} = k'' \tag{4.9,5}$$

where k'' is a constant and the expression on the left is a non-dimensional pressure coefficient.

Next, let the body be accelerated along a straight line and in a fixed attitude through a fluid at rest, except as disturbed by the body, the constant measure of the acceleration being a. We assume that the force P_2 on the body and which depends on the acceleration is given by

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† It is superfluous to go through the argument again since the measure formulae of both numerator and denominator contain the additional factor L.

$$P_a = kl^\alpha a^\beta \rho^\gamma \tag{4.9,6}$$

where k, α, β, γ are constants. This yields the relation between measure formulae

$$MLT^{-2} \stackrel{\frown}{=} L^\alpha(LT^{-2})^\beta(ML^{-3})^\gamma$$

and the indicial equations are

$$\begin{aligned} &(M) \quad & 1 &= \gamma \\ &(L) \quad & 1 &= \alpha + \beta - 3\gamma \\ &(T) \quad & -2 &= --2\beta. \end{aligned}$$

Hence $\beta = \gamma = 1$ and $\alpha = 3$, so (4.9,6) yields

$$P_a = a(k\rho l^3). \tag{4.9,7}$$

The force is the same as that on a mass $k\rho l^3$ when the acceleration is a. Hence $k\rho l^3$ is the *virtual* mass of the body for linear acceleration in the given attitude (see also § 3.9).

So far we have supposed that the fluid is infinitely extended or contained within a fixed outer boundary.† Now let the fluid be situated in a uniform gravitational field and let it have a free surface, which will necessarily be plane when the fluid is undisturbed.‡ The forces will now, in general, be influenced by g and we must substitute for equation (4.9,1)

$$P = kl^\alpha v^\beta \rho^\gamma g^\delta. \tag{4.9,8}$$

This leads to the dimensional equation

$$MLT^{-2} \stackrel{\frown}{=} L^\alpha(LT^{-1})^\beta(ML^{-3})^\gamma(LT^{-2})^\delta$$

and the indicial equations are

$$\begin{aligned} &(M) \quad & 1 &= \gamma \\ &(L) \quad & 1 &= \alpha + \beta - 3\gamma + \delta \\ &(T) \quad & -2 &= -\beta - 2\delta. \end{aligned}$$

These *three* equations are linearly independent and they contain *four* unknowns, so there is an indeterminacy, except as regards γ. For convenience put $\delta = -n$. Then we find $\beta = 2 + 2n$ and $\alpha = 2 - n$. Consequently (4.9,8) becomes

$$P = kl^2v^2\rho\left(\frac{v^2}{lg}\right)^n. \tag{4.9,9}$$

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† For similarity the dimensions of any such boundary must be proportional to the representative length l.

‡ We here neglect the effects of any gas above the free surface.

This expression for P is dimensionally correct for all values of n and of k. Hence

$$P = l^2v^2\rho\left[k_1\left(\frac{v^2}{lg}\right)^{n_1} + k_2\left(\frac{v^2}{lg}\right)^{n_2} + \text{etc.}\right]$$

$$= l^2v^2\rho \sum k_r\left(\frac{v^2}{lg}\right)^{n_r} \tag{4.9,10}$$

is also dimensionally correct. Now the indices n_r and coefficients k_r are completely arbitrary, also the number of terms in the summation, which may be infinite. This implies that the summation is just an *arbitrary function* of the variable v^2/lg, which we recognise as the Froude number F. Accordingly, let us put

$$\frac{v^2}{lg} = F \tag{4.9,11}$$

and

$$\sum k_rF^{n_r} = \phi(F). \tag{4.9,12}$$

Then (4.9,10) becomes

$$P = \rho v^2l^2\phi(F) \tag{4.9,13}$$

or

$$\frac{P}{\rho v^2l^2} = \phi(F). \tag{4.9,14}$$

The form of the function $\phi(F)$ cannot be determined by dimensional analysis, but equation (4.9,14) gives us very valuable information about the dependence of the force P on other factors. Thus, so long as the measurements are made in such circumstances that F is constant, P will be proportional to ρv^2l^2. This is in exact agreement with the conclusions already drawn from considerations of dynamical similarity (see § 4.4).

4.10 Viscous Incompressible Fluid

We shall suppose first that gravity is absent or the conditions such that it does not appreciably influence the flow. We now assume that the dynamic force is given by

$$P = kl^\alpha v^\beta \rho^\gamma \mu^\delta \tag{4.10,1}$$

where μ is the constant viscosity of the fluid. This leads to the following relation between measure formulae

$$MLT^{-2} \boxminus L^\alpha(LT^{-1})^\beta(ML^{-3})^\gamma(ML^{-1}T^{-1})^\delta$$

and the indicial equations are

$$\begin{aligned} &(M) \quad 1 = \gamma + \delta \\ &(L) \quad 1 = \alpha + \beta - 3\gamma - \delta \\ &(T) \quad -2 = -\beta - \delta. \end{aligned}$$

These three equations are linearly independent and they contain four unknowns; consequently there is one degree of indeterminacy in the solution. For convenience put $\delta = -n$. Then we obtain

$$\beta = 2 + n, \qquad \gamma = 1 + n, \qquad \alpha = 1 + 3\gamma + \delta - \beta = 2 + n.$$

Equation (4.10,1) becomes

$$P = kl^2v^2\rho\left(\frac{vl\rho}{\mu}\right)^n. \tag{4.10,2}$$

Here n and k are completely arbitrary and the argument used in § 4.9 leads to the conclusion that

$$P = l^2v^2\rho f(R) \tag{4.10,3}$$

where

$$R = \frac{\rho vl}{\mu} = \frac{vl}{\nu} \tag{4.10,4}$$

is the Reynolds number and the function $f(R)$ cannot be determined by dimensional analysis. Equation (4.10,3) can be written alternatively

$$\frac{P}{\rho v^2 l^2} = f(R), \tag{4.10,5}$$

so the non-dimensional force coefficient is some function of the Reynolds number.

Next, suppose that the conditions are such that gravity influences the flow. We now assume that

$$P = kl^\alpha v^\beta \rho^\gamma \mu^\delta g^\varepsilon \tag{4.10,6}$$

which leads to the dimensional relation

$$MLT^{-2} \mathrel{\ominus} L^\alpha(LT^{-1})^\beta(ML^{-3})^\gamma(ML^{-1}T^{-1})^\delta(LT^{-2})^\varepsilon.$$

Accordingly the indicial equations are

$$\begin{aligned}
&(M) \quad & 1 &= \gamma + \delta \\
&(L) \quad & 1 &= \alpha + \beta - 3\gamma - \delta + \varepsilon \\
&(T) \quad & -2 &= -\beta - \delta - 2\varepsilon.
\end{aligned}$$

There are now two degrees of indeterminacy and it is convenient to put $\delta = -n$, $\varepsilon = -m$. Accordingly we obtain

$$\alpha = 2 + n - m, \qquad \beta = 2 + n + 2m, \qquad \gamma = 1 + n$$

and (4.10,6) becomes

$$P = kl^2v^2\rho\left(\frac{lv\rho}{\mu}\right)^n\left(\frac{v^2}{lg}\right)^m$$

Hence
$$P = l^2v^2\rho \sum_n \sum_m k_{nm}\left(\frac{lv\rho}{\mu}\right)^n \left(\frac{v^2}{lg}\right)^m$$

is dimensionally correct for all values of k_{nm}, n, m and for any number of terms in the double summation. This means that

$$\frac{P}{\rho v^2 l^2} = f(R,F) \tag{4.10,7}$$

where the form of the function $f(R, F)$ cannot be determined by dimensional analysis. We have here an illustration of the Pi theorem (see § 4.8). Thus we are concerned with six physical quantities P, l, v, ρ, μ and g. There are three fundamental units and $3 = 6 - 3$ independent non-dimensional combinations of the physical quantities which, as we have seen, may be taken as $P/\rho v^2l^2$, $\rho vl/\mu$ and v^2/lg. These quantities are certainly independent, for the first is the only one to contain P, the second is the only one to contain μ and the third is the only one to contain g. This is a normal or *non-singular* case, so the number of independent non-dimensional quantities is *equal* to the number of the physical variables *less* the number of the fundamental units.†

It must not be supposed that the non-dimensional quantities which we have obtained are unique; on the contrary, there is an immense choice, for any product combination of the non-dimensional quantities is also non-dimensional. First, we might, for example, take $P/\mu vl$ as the non-dimensional force coefficient, for

$$\frac{P}{\mu vl} = \frac{P}{\rho v^2 l^2} \times \frac{\rho vl}{\mu}, \tag{4.10,8}$$

and it follows that (4.10,7) can be written alternatively

$$\frac{P}{\mu vl} = f'(R,F) \tag{4.10,9}$$

where $f'(R,F)$ is a new function of R and F. To show that (4.10,9) is *exactly equivalent* to (4.10,7), let us put

$$f'(R,F) = Rf(R,F) \tag{4.10,10}$$

which is always legitimate, for the product of R and a function of R and F is also a function of R and F. Accordingly, when we multiply both sides of equation (4.10,7) by R, we obtain equation (4.10,9) on account of (4.10,8). Again let

$$\alpha = FR = \frac{\rho v^3}{\mu g} \tag{4.10,11}$$

and
$$\beta = \frac{R^2}{F} = \frac{g\rho^2 l^3}{\mu^2}, \tag{4.10,12}$$

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† If the density of the gas (ρ_g) is considered significant then (4.10,7) becomes $P/\rho v^2l^2 = f(R,F,\rho_g/\rho)$.

so α and β are non-dimensional. Then

$$R = \sqrt[3]{\alpha\beta} \tag{4.10,13}$$

and

$$F = \sqrt[3]{\frac{\alpha^2}{\beta}}. \tag{4.10,14}$$

Accordingly R and F are fixed when α and β are given and it follows that (4.10,7) can be written in the equivalent form

$$\frac{P}{\rho v^2 l^2} = \phi(\alpha,\beta) = \phi\left(\frac{\rho v^3}{\mu g}, \frac{g\rho^2 l^3}{\mu^2}\right) \tag{4.10,15}$$

while (4.10,9) yields the further equivalent relation

$$\frac{P}{\mu v l} = \psi\left(\frac{\rho v^3}{\mu g}, \frac{g\rho^2 l^3}{\mu^2}\right). \tag{4.10,16}$$

The foregoing are merely a few specimens of the infinite variety of possible ways of writing the relationship.

Since we have an immense variety of mathematically equivalent expressions to choose from it is clear that our choice among these should be governed mainly by convenience; it is also advisable to adopt a traditional form unless there is some good reason for departing from it. Now it would be highly convenient if the non-dimensional force coefficient were *constant*, at least for the range of the variables with which we happened to be concerned. When the phenomena are not appreciably influenced by gravity, so that the Froude number v^2/lg may be omitted from (4.10,7) and (4.10,8), it is found by experiment that $P/\rho v^2 l^2$ varies only slightly when R is large whereas $P/\mu v l$ is nearly constant when R is very small. Accordingly it is convenient to use the coefficient $P/\rho v^2 l^2$ when R is large and $P/\mu v l$ when R is small. Since R necessarily arises in relation to the flow of an incompressible viscous fluid in the absence of gravitational effects and F necessarily arises in relation to the flow of an inviscid fluid when gravity is effective, it is certainly convenient, as a rule, to use R and F rather than other combinations of them such as those defined in equations (4.10,11) and (4.10,12).

Example 1. *Resistance of a ship*

We have considered ship resistance and tests of ship models in Example 2 of § 4.4 for the case of an inviscid fluid; here the sole requirement for similarity of behaviour of geometrically similar systems is equality of the Froude numbers. When the fluid is viscous we have the additional requirement of equality of the Reynolds numbers. When the prototype and the model are tested in the same medium and with the same value of g, the two requirements cannot be met, or, to be precise, they can only be met when the "model" is equal to the prototype and moves at the same speed. For, when both R and F have the same values for the model and for the prototype, α and β must also have the same values and then l and v are fixed (see (4.10,11) and (4.10,12)). This awkward situation is met in practice by making the *assumption* that the total resistance can be split into three parts: (a) the wave-making resistance (b) the skin frictional resistance and (c) the eddy-making resistance.† It is then further assumed that (a) is independent of the Reynolds number, so that its value can be deduced from the results of a test on a model by substracting items (b) and (c) from the total measured resistance of the model. Since (c) cannot be readily estimated and is usually a small fraction of the whole resistance, it is commonly included with the wave-making resistance. It is customary to estimate the skin frictional resistances of model and prototype from the measured resistances of flat plates set at zero incidence to the stream. The radical way out of these difficulties would be to test the model in another medium and perhaps with another value of g. In order that both v and l should be small for the model it would be necessary to use a medium of very small kinematic viscosity, e.g. water at a temperature much above ordinary room temperatures.

4.11 Compressible Inviscid Fluid

We begin by making the assumption that the force P depends only on v, ρ, l and the bulk modulus K of the fluid. Accordingly we put

$$P = kl^{\alpha}v^{\beta}\rho^{\gamma}K^{\delta} \tag{4.11,1}$$

and the corresponding relation between measure formulae is

$$MLT^{-2} \stackrel{\frown}{=} L^{\alpha}(LT^{-1})^{\beta}(ML^{-3})^{\gamma}(ML^{-1}T^{-2})^{\delta}.$$

The indicial equations are therefore

$$\begin{aligned}
&(M) \quad 1 = \gamma + \delta \\
&(L) \quad 1 = \alpha + \beta - 3\gamma - \delta \\
&(T) \; -2 = -\beta - 2\delta.
\end{aligned}$$

There is one degree of indeterminacy and it is convenient to put $\delta = -n$. Then we obtain $\alpha = 2$, $\beta = 2 + 2n$, $\gamma = 1 + n$ and (4.11,1) becomes

$$P = k\rho v^2 l^2 \left(\frac{\rho v^2}{K}\right)^n.$$

Following the argument already exemplified in § 4.9 and later, we conclude that

$$\frac{P}{\rho v^2 l^2} = f\left(\frac{\rho v^2}{K}\right). \tag{4.11,2}$$

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† This is referred to in Chapter 6 as the boundary layer pressure drag or form drag.

We have already seen in § 4.5 that constancy of $\rho v^2/K$ is a condition for dynamical similarity and that this ratio is equal to the square of the Mach number v/a. Thus (4.11,2) is equivalent[†] to

$$\frac{P}{\rho v^2 l^2} = f(M). \tag{4.11,3}$$

We have postulated that K is *the* bulk modulus of the fluid but when the ratio of the maximum pressure to the minimum pressure in the flowing fluid differs greatly from unity there will not be a unique bulk modulus and it is necessary to take account of other properties of the fluid. For a perfect gas the value of

$$\gamma = \frac{c_p}{c_v} \tag{4.11,4}$$

influences the behaviour in such circumstances and constancy of γ becomes a further condition for similarity of behaviour.[‡] More generally, there are conditions for the similarity of distinct fluids and these are further considered in § 4.14.

4.12 Compressible Viscous Fluid

We shall suppose at first that gravity is absent or ineffective and we shall assume that the force P on the body depends on l, v, ρ, μ and K. We can obtain the required relation immediately by application of the Pi theorem (see § 4.8) because we have already established the existence of three independent non-dimensional combinations of the quantities, namely, the non-dimensional force coefficient and the Reynolds and Mach numbers. Accordingly we have

$$\frac{P}{\rho v^2 l^2} = f(R,M). \tag{4.12,1}$$

By an obvious further application of the Pi theorem the relation becomes

$$\frac{P}{\rho v^2 l^2} = f(R,M,F) \tag{4.12,2}$$

when g affects the phenomena; F here stands for the Froude number.

When a viscous fluid flows, heat is generated within it and this will result in a change of density when the fluid is compressible. The phenomena will also be influenced by the rate of conduction of the heat generated by the fluid friction. Now the *thermometric conductivity* of a medium, which is the *diffusivity of temperature* within it, is given by

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[†] We use the functional symbol $f(\ )$ to represent *any* undetermined function. Obviously the functional symbol represents different functions in (4.11,2) and (4.11,3).

[‡] It can be proved by thermodynamic reasoning that, for any fluid, the ratio of the isentropic to the isothermal bulk modulus is equal to c_p/c_v.

$$\kappa = \frac{k}{\rho c_p} \qquad (4.12,3)$$

where k is the conductivity for heat. We have seen in § 3.10 that the diffusivity of vorticity is ν, so ν/κ is non-dimensional, being the ratio of two diffusivities. This is called the *Prandtl number*[†]

$$\sigma = \frac{\nu}{\kappa} = \frac{\mu c_p}{k}. \qquad (4.12,4)$$

Equality of the Prandtl numbers is a further condition for similarity of behaviour whenever thermal and viscous effects within the fluid are important as, for example, in the boundary layer for high values of the Mach number.[‡] Again, the viscosity of a fluid in general depends on the temperature and this dependence becomes influential when the range of temperature within the fluid is considerable. If we assume that μ varies in proportion to T^n, where T is the absolute temperature, then the index n is a non-dimensional quantity which will influence the motion in the conditions already mentioned (see also § 4.14).

4.13 Convection

Consider a fluid at rest and in equilibrium in a gravitational field. Then the pressures on the surface of any element of the fluid exactly balance the weight of the element. Now suppose that the temperature of the element of fluid under consideration is raised while that of the surrounding fluid remains unaltered. The heated element expands[¶] and therefore the upward thrust on its surface is increased while its weight is unaltered. Hence upward acceleration of the element ensues until the upward velocity is such that the buoyancy is balanced by the viscous drag. The motion produced by localised heating is called a *free convection current* and this is a very effective means for the conveyance of heat in the fluid.

Under conditions of similar behaviour the velocity of a free convection current will be such that the Reynolds number is constant. Thus the representative velocity v will be proportional to $\mu/\rho l$. Let θ be the representative temperature difference and a the coefficient of thermal expansion of the fluid. Then the *volume strain* will be proportional to θa, and this is a non-dimensional quantity. The force of buoyancy per unit mass is therefore proportional to θag and the Froude number for this force is proportional to

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[†] The Stanton number is the reciprocal of σ.

[‡] See Howarth *Modern Developments in Fluid Dynamics. High-Speed Flow,* Oxford, 1953.

[¶] For water in the temperature range 0°C to 4°C there would be contraction.

$$\frac{v^2}{l\theta ag} \text{ or to } \frac{\mu^2}{(\rho l)^2 l\theta ag} = \frac{\mu^2}{\rho^2 l^3 \theta ag}.$$

This quantity must be constant for similarity of behaviour and its reciprocal is called the *Grashof number* G; thus

$$G = \frac{\rho^2 l^3 \theta ag}{\mu^2} = \frac{l^3 \theta ag}{\nu^2} \tag{4.13,1}$$

must be constant in order that similar systems may behave similarly. A further numerical condition for similarity of behaviour is constancy of the Prandtl number σ (see § 4.12).

The non-dimensional heat transmission coefficient, often known as the Nusselt number N, is defined by the equation (applicable to a steady state)

$$N = \frac{hl}{k\theta} \tag{4.13,2}$$

where h = measure of the quantity of heat transferred per unit of time per unit of area (here of heated surface)
l = representative linear dimension of the similar systems considered
k = thermal conductivity (here of the fluid)
θ = difference of temperature (here between the heated surface and the unheated fluid).

It follows from what has been said that N is some function of G and σ, i.e.

$$N = f(G,\sigma). \tag{4.13,3}$$

For laminar (non-turbulent) flow it appears that this can be replaced by the simpler relation

$$N = f(G\sigma) \tag{4.13,4}$$

where

$$G\sigma = \frac{l^3 \theta ag}{\nu\kappa} \tag{4.13,5}$$

is called the *Rayleigh number* and is non-dimensional. The velocity of the convection current will, in general, be obtained from the relation between the Reynolds number, G and σ, say

$$R = \phi(G,\sigma) \tag{4.13,6}$$

or

$$V = \frac{\nu}{l}\,\phi(G,\sigma), \tag{4.13,7}$$

where V is the representative velocity. The last equation may be written in a more convenient equivalent form as

$$V = \frac{l^2 \theta ag}{\nu} F(G,\sigma). \tag{4.13,8}$$

In place of the Reynolds number we may use the *Péclet number*

$$P_e = R\sigma = \frac{Vl}{\kappa}. \qquad (4.13,9)$$

In *forced convection* motion is impressed on the fluid by some external agency and in fully forced convection the influence of buoyancy on the motion is negligible; accordingly the Grashof number is not significant in fully forced convection. Thus the relation deducible from dimensional analysis is here

$$N = f(R, \sigma) \qquad (4.13,10)$$

for the Reynolds number of the flow is now one of the data of the problem. Very often the relation (4.13,10) can be adequately represented by a power law

$$N = cR^n\sigma^m \qquad (4.13,11)$$

where c, n and m are constants.

4.14 Similar Fluids

We have already seen that it is not always adequate to regard the mechanical properties of a fluid as completely specified by constant values of ρ, μ and K; in other words, the numerical conditions for similarity of flow are not completely covered by constancy of the Reynolds and Mach numbers.† This will be so when the ranges of temperature and pressure actually occurring in the flow are such that the precise relation between temperature, pressure and density, as summarised in the *equation of state*, is important; also when the dependence of viscosity on temperature cannot be ignored and when heat conduction within the fluid is influential (influence of the Prandtl number and of the variability of the Prandtl number). We may, in the light of the foregoing, frame the following definition‡ of *similar fluids*:

A pair of fluids is similar when equality of the corresponding values of the Reynolds and Mach numbers is sufficient to ensure complete similarity of the motions, the boundaries being geometrically similar non-conductors of heat and the kinematic boundary conditions being similar. Non-conducting boundaries are specified because the flow would, in general, be influenced by the properties of the materials of the boundaries if these were conductors of heat.

It is not certain that any two fluids are strictly similar in accordance with this definition. There are, however, important cases where fluids can legitimately be regarded as similar:

(a) The fluids are liquids or gases, the Mach number small and thermal effects negligible.

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† We suppose here that gravity is absent or ineffective.

‡ Given on p. 83 of *Physical Similarity and Dimensional Analysis.*

(b) The fluids are gases existing in such circumstances that the equation of state of a perfect gas is practically valid. Further, the values of γ, σ and the index of temperature in the power law for μ have the same values for the two fluids.

4.15 Periodic Phenomena in Fluids

Periodic phenomena occur in fluids in the following circumstances:

(a) when an immersed body has a periodic motion,
(b) when eddies (vortices) are detached periodically from bodies held at rest in a stream of fluid,
(c) in wave motion.

We shall consider these in turn.

Let us suppose that a body immersed in a uniform stream of fluid of speed V has a simple-harmonic motion of some definite kind, with displacements proportional to $\sin \omega t$. Then $\omega l/V$ is non-dimensional, for ω has the measure formula T^{-1}. This is called the non-dimensional frequency parameter and it must remain constant for similarity (in general). Let f be the frequency of the oscillation and $\mathcal{T}$ the periodic time. Then

$$\omega = 2\pi f = \frac{2\pi}{\mathcal{T}}. \tag{4.15,1}$$

Accordingly

$$\frac{\omega l}{V} = \frac{2\pi l}{V\mathcal{T}}. \tag{4.15,2}$$

Now $V\mathcal{T}$ is the distance moved by the fluid (where undisturbed by the body) relative to the body in one complete period of the oscillation and the frequency parameter is 2π times the ratio of the typical linear dimension l of the body to the distance $V\mathcal{T}$ moved in one period. When the body is a plate or aerofoil of rectangular form we may identify l with the chord c and the frequency parameter is equal to 2π divided by the number of chord lengths moved through in one complete period.

When a flat plate whose length:breadth ratio is large is held with its surface perpendicular to a uniform stream of fluid it is found that eddies are detached with nearly constant frequency from the two long edges of the plate. The two sets of eddies form what is called a *vortex street* and they are placed *alternately*, so that an eddy in one row is opposite the middle of the space between adjacent eddies in the other row. Let f be the frequency of detachment of the eddies in *one* row; then the whole phenomenon is periodic with frequency f. Let c be the chord (or width) of the plate. Then we expect, on the basis of dimensional analysis, that fc/V will be a function of R and M. It is found by experiment that, when M is small (say less than 0.2) and R not too small, fc/V has the constant value 0.146. Similar phenomena are observed when any "bluff" body, such as a circular cylinder or an aerofoil or plate set at an angle of incidence above the stall (see § 6.2), is placed in a uniform stream. The non-dimensional quantity fl/V, where l is the representative length such as the diameter of a circular cylinder, is called the *Strouhal number* and it is equal to the already mentioned frequency parameter, divided by 2π. The periodic detachment of eddies is responsible for the singing of telegraph wires and for aeolian tones generally. In the part of the wake not near the body the flow is confused and turbulent; there is then no clearly predominant frequency.

There are two important kinds of wave motion in fluids, namely, propagation of waves of compression and rarefaction in a compressible fluid and propagation of waves in a liquid with a free surface, under the action of gravity or surface tension or both. First let us consider the velocity of propagation of waves of compression and rarefaction (sound waves) and let us assume that the velocity of propagation a is given by

$$a = k\lambda^{\alpha}\rho^{\beta}K^{\gamma} \tag{4.15,3}$$

where k is a constant, λ is the wave length, ρ the density and K the bulk modulus of the fluid. The last equation yields the dimensional relation

$$LT^{-1} \stackrel{\frown}{=} L^{\alpha}(ML^{-3})^{\beta}(ML^{-1}T^{-2})^{\gamma}$$

so the indicial equations are

$$(M)\quad 0 = \beta + \gamma, \qquad (L)\quad 1 = \alpha - 3\beta - \gamma, \qquad (T)\quad -1 = -2\gamma.$$

Hence $\gamma = \frac{1}{2}$, $\beta = -\gamma = -\frac{1}{2}$ and $\alpha = 0$ and (4.15,3) becomes

$$a = k\surd(K/\rho). \tag{4.15,4}$$

Detailed analysis (see § 10.3) confirms this and shows that k has the value unity; the modulus K is that appropriate to isentropic changes of volume. It is worthy of comment that the velocity of propagation is independent of the wave length, so the propagation is *not* dispersive (see § 10.4).

Let us now consider waves in a very deep uniform liquid with a free surface, situated in a uniform gravitational field of strength g. Let us assume that the velocity of propagation V is given by

$$V = k\lambda^{\alpha}\rho^{\beta}g^{\gamma} \tag{4.15,5}$$

which yields the dimensional relation

$$LT^{-1} \stackrel{\frown}{=} L^{\alpha}(ML^{-3})^{\beta}(LT^{-2})^{\gamma}.$$

The indicial equations are

$$(M)\quad 0 = \beta, \qquad (L)\quad 1 = \alpha - 3\beta + \gamma, \qquad (T)\quad -1 = -2\gamma,$$

so $\alpha = \frac{1}{2}$, $\beta = 0$, $\gamma = \frac{1}{2}$. Hence (4.15,5) becomes

$$V = k\surd(g\lambda). \tag{4.15,6}$$

Detailed mathematical analysis (see § 10.5) confirms this and shows that the value of k is $1/\sqrt{2\pi}$. The last equation can be written

$$\frac{V^2}{g\lambda} = k^2. \tag{4.15,7}$$

Thus the Froude number of the waves is constant, where the wave length is adopted as the characteristic length. Here the velocity of the waves is proportional to the square root of the wave length and the propagation is dispersive. Next, let us consider the propagation of waves under the action of surface tension (of measure σ) with gravity absent. We assume that

$$V = k\lambda^\alpha \rho^\beta \sigma^\gamma \tag{4.15,8}$$

and the corresponding dimensional relation is

$$LT^{-1} \triangleq L^\alpha (ML^{-3})^\beta (MT^{-2})^\gamma.$$

Hence the indicial equations are

$$(M) \quad 0 = \beta + \gamma, \qquad (L) \quad 1 = \alpha - 3\beta, \qquad (T) \quad -1 = -2\gamma$$

and their solution is $\alpha = -\frac{1}{2}$, $\beta = -\frac{1}{2}$, $\gamma = \frac{1}{2}$. Consequently (4.15,8) becomes

$$V = k\sqrt{\left(\frac{\sigma}{\lambda\rho}\right)} . \tag{4.15,9}$$

The propagation is again dispersive but the wave velocity is now *inversely proportional* to the square root of the wave length. Complete analysis shows that $k = \sqrt{2\pi}$. When both gravity and surface tension are effective we shall have V dependent on the four variables λ, ρ, g and σ. By a quite straightforward application of dimensional analysis we can show that

$$V = \sqrt{(g\lambda)}\, f\left(\frac{\sigma}{\lambda^2 g\rho}\right) \tag{4.15,10}$$

where the function f cannot be determined by dimensional analysis and the argument of the function is a non-dimensional quantity which has been called the Weber number. We can see that this number arises here without going through the details of a fresh analysis. It follows from (4.15,9) that $\sigma/\lambda\rho$ has the dimensions of (velocity)^2 and, by (4.15,6), the same is true of $g\lambda$. The ratio of these, namely $\sigma/\lambda^2 g\rho$, is therefore non-dimensional and it follows from the Pi theorem that the non-dimensional quantity $V^2/\lambda g$ is some function of the non-dimensional quantity $\sigma/\lambda^2 g\rho$. This statement is equivalent to equation (4.15,10). Hitherto we have supposed the liquid to be so deep that the presence of the bottom does not influence the wave motion appreciably. Now let the depth (assumed constant) be h. Then h/λ is a non-dimensional quantity and it follows from the Pi theorem that

$$\frac{V^2}{\lambda g} = F\left(\frac{\sigma}{\lambda^2 g\rho}, \frac{h}{\lambda}\right) . \tag{4.15,11}$$

Detailed analysis (see § 10.5) shows that, provided the waves are of very small amplitude in relation to the wave length,

$$\frac{V^2}{\lambda g} = \frac{1}{2\pi}\left[1 + 4\pi^2\left(\frac{\sigma}{\lambda^2 g\rho}\right)\right] \tanh\left(\frac{2\pi h}{\lambda}\right) \qquad (4.15,12)$$

which accords with (4.15,11).

4.16 Non-Dimensional Force and Moment Coefficients

We have already come across non-dimensional coefficients for forces, moments and pressures in §§ 4.5, 4.9 and we shall now explain the system of such coefficients which has become standard in aerodynamics. As a preliminary, the method of specifying the components of force and moment will be explained.

In aerodynamics we are mainly concerned with forces on a wing or complete aircraft moving uniformly through a compressible fluid; this is exactly equivalent to holding the body fixed in a uniform stream of the fluid and it is often convenient to regard the system in this manner. The wing or aircraft has a plane of symmetry and its right and left halves are mirror images in this plane. The *root chord* line is a straight line in the plane of symmetry which serves as a datum line for the wing sections. The wing area S is defined to be the area of the projection of the wing† on a plane through the root chord line and perpendicular to the plane of symmetry. When the wind velocity vector lies in the plane of symmetry the aircraft is said to be unyawed (or to have zero velocity of sideslip) and its attitude to the stream is then completely determined by the *angle of incidence* α which is the inclination of the datum chord line to the direction of motion, taken positive when points on this chord line lying forward are raised relative to the line of flight. In such symmetric flight the resultant aerodynamic force on the aircraft lies in the plane of symmetry. Let O be a convenient reference point in the plane of symmetry. Then, by the principles of statics, the resultant force is equivalent to a force acting at O and a moment M about O, called the *pitching moment*. This is taken as positive when it tends to raise the forward parts of the aircraft relative to the rearward parts, i.e. when it is a "nose-up" moment. The magnitude of M depends on the choice of the reference point and M vanishes when O lies on the line of action of the resultant force. On the other hand, the force at O is independent of the choice of O. This force is resolved into two components:

The drag. This is the component of the resultant aerodynamic force along the direction of motion and is positive when the force opposes the motion. The drag D is thus the total force of aerodynamic resistance to the motion.

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† By convention the wing is supposed to be prolonged through the region occupied by the body or fuselage.

The lift. This is the component of the resultant aerodynamic force perpendicular to the direction of motion. The lift L is taken positive upwards.[†]

We have already shown that $P/\rho v^2 l^2$ is a non-dimensional force coefficient, where P is a force. In the system used in aerodynamics l^2 is replaced by the wing area S which has the same measure formula. Then the *lift coefficient*, according to the British system in use up to 1937, is

$$k_L = \frac{L}{\rho V^2 S} \tag{4.16,1}$$

where V is the velocity of the aircraft relative to the undisturbed air and ρ is the density of the undisturbed air in the neighbourhood of the aircraft. Similarly the drag coefficient is

$$k_D = \frac{D}{\rho V^2 S} \tag{4.16,2}$$

and the pitching moment coefficient is

$$k_M = \frac{M}{\rho V^2 S \bar{c}} \tag{4.16,3}$$

where $\bar{c}$ is the mean chord given by

$$\bar{c} = \frac{S}{b} \tag{4.16,4}$$

and b is the total span from wing tip to wing tip. In the present system, which is almost universally adopted, the coefficients are

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 S} \tag{4.16,5}$$

$$C_D = \frac{D}{\frac{1}{2}\rho V^2 S} \tag{4.16,6}$$

$$C_M = \frac{M}{\frac{1}{2}\rho V^2 S \bar{c}} . \tag{4.16,7}$$

The factor $\frac{1}{2}$ in the denominator is entirely superfluous but it was introduced, presumably, because the dynamic pressure is $\frac{1}{2}\rho V^2$ when the air is regarded as incompressible, in accordance with Bernoulli's theorem. For geometrically similar aircraft the non-dimensional coefficients are functions of the angle of incidence and of the Reynolds and Mach numbers. When the aircraft is yawed, the angle of yaw becomes an additional variable and there is then, in general, a cross-wind force while the aerodynamic moment has components about all three axes.

[†] We suppose here that the aircraft is in its normal attitude. In inverted flight the lift will be positive when downward.

EXERCISES. CHAPTER 4.

1. A perfect fluid issues under gravity from a large tank through a small nozzle whose axis is inclined upward at θ to the horizontal. The free surface in the tank is at height h above the orifice of the nozzle. Find the Froude number F for the fluid at exit from the nozzle with h as the characteristic length. Find also the Froude number when the horizontal range of the jet is taken as characteristic length (air resistance neglected).
(Answer. $F = 2$, $F = \operatorname{cosec} 2\theta$)

2. Show that Bernoulli's theorem for a perfect fluid in a uniform gravitational field can be written in the non-dimensional form

$$\tfrac{1}{2}F + \frac{p}{g\rho z_0} + \frac{z}{z_0} = \text{Const.},$$

where F is a variable Froude number and z_0 is a reference height which is adopted as characteristic length.

3. Investigate the dependence of the moments of inertia of physically similar rigid bodies (about geometrically corresponding axes) on the typical length l and typical density σ. (Answer. $I = k\sigma l^5$)

4. A spherical drop of liquid, diameter d, oscillates under the influence of its surface tension. Investigate the frequency of oscillation in the fundamental (or other given) mode. (Answer. $f = k\sqrt{(\sigma/\rho d^3)}$)

5. Given that the mean free path λ of the molecules of a gas depends only on the gas density ρ, molecular diameter l and molecular mass m, obtain the most general expression for λ. $\left(\text{Answer. } \lambda = lf\left(\frac{\rho l^3}{m}\right)\right)$

6. The solution of a problem concerning a viscous incompressible fluid contains a term proportional to $\cos pt$ where t is the measure of time. Given that p can depend only on a linear dimension l, the density ρ and viscosity μ, find the most general expression for p.
$\left(\text{Answer. } p = \frac{k\mu}{\rho l^2} \text{ with } k \text{ a numerical constant}\right)$

7. The force on any member of a family of similar and similarly situated bodies in a stream of incompressible viscous fluid is

$$F = \rho V^2 l^2 f(R)$$

where R is the Reynolds number $\rho Vl/\mu$. Show that the following expressions for the force are exactly equivalent to the above:—

$$F = \mu Vl\phi(R)$$

$$F = \frac{\mu^2}{\rho}\,\psi(R).$$

Give the relations between the functions f, ϕ and ψ.
$\left(\text{Answer. } f(x) = \frac{\phi(x)}{x} = \frac{\psi(x)}{x^2}\right)$

8. The rate of discharge Q (volume of liquid per unit time) in steady laminar flow of a viscous liquid through a cylindrical pipe of given shape of bore is given as depending only on the following quantities:
(1) Typical linear dimension d of the bore.

(2) Pressure gradient $(p_1 - p_2)/l$, where l is the length of pipe considered and p_1, p_2 are the pressures at the upstream and downstream ends, respectively.

(3) Viscosity μ of the liquid.

Obtain the expression for Q. $\left(\text{Answer. } Q = \dfrac{k(p_1 - p_2)d^4}{\mu l}\right)$

9. Verify that $\left(\frac{1}{2}q^2 + \varpi + \chi - \dfrac{\partial \phi}{\partial t}\right)$ is dimensionally homogeneous (see § 3.5).
10. Verify that equation (3.10,3) is dimensionally homogeneous.
11. A circular disk is set spinning in an incompressible viscous fluid with angular velocity ω. Investigate the time t for the angular velocity to fall to half its initial value on the assumption that this depends only on ω, typical linear dimension l, density of the disk σ, density of the fluid ρ and viscosity μ.

    (The answer can be written in various equivalent forms such as

    $$t = \frac{1}{\omega} f\left(\frac{\rho\omega l^2}{\mu}, \frac{\sigma}{\rho}\right) \quad \text{and} \quad t = \frac{\sigma l^2}{\mu}\phi\left(\frac{\rho\omega l^2}{\mu}, \frac{\sigma}{\rho}\right)\Bigg)$$
12. If heat quantity and temperature are regarded as independent physical quantities, find the measure formulae of specific heat c and of thermal conductivity k. (Use Q for heat quantity and Θ for temperature in the measure formulae.) (Answer. $c \boxminus Q\Theta^{-1}M^{-1} \quad k \boxminus Q\Theta^{-1}L^{-1}T^{-1}$)
13. The Prandtl number of a fluid is $\sigma = \mu c_p/k$ where μ is the viscosity. By use of the results of the last exercise show that σ is non-dimensional.
14. Give the measure formula of the aircraft derivative X_u. (This is the rate of change of the longitudinal force X with the increment u of the longitudinal velocity.) (Answer. $X_u \boxminus MT^{-1}$)
15. Obtain the measure formula of the aircraft derivative L_p. (This is the rate of change of the rolling moment L with angular velocity in roll p.)

    (Answer. $L_p \boxminus ML^2T^{-1}$)
16. A model of an aeroplane is tested in a compressed air wind tunnel at n times atmospheric density but at atmospheric temperature. The linear scale of the model is such that the Reynolds number of the model test is equal to that for the full-scale prototype (assume that viscosity of air depends only on temperature) and the Mach numbers are also identical. Find the ratio of an aerodynamic force on the model to the corresponding force for the prototype. (Answer. Ratio is $1/n$)
17. Find a complete set of independent non-dimensional quantities when the following quantities are concerned:

    Power P
    Typical length l
    Number of revolutions n in unit time
    Linear velocity V
    Density of fluid ρ
    Viscosity of fluid μ.

    $\left(\text{Answer. } \dfrac{P}{\rho n^3 l^5}, \dfrac{V}{nl}, \dfrac{\rho V l}{\mu}\right.$

    There are infinitely many other solutions which are all obtainable from this by multiplication, powering, etc.)
18. Obtain a complete set of independent non-dimensional quantities when the following quantities are involved:

    Moment of force M
    Typical length l
    Linear velocity V
    Intensity of gravity g
    Density of fluid ρ
    Velocity of sound in fluid a.

    $\left(\text{Answer. } \dfrac{M}{\rho V^2 l^3}, \dfrac{V}{a}, \dfrac{V^2}{lg}\right)$

19. A viscous incompressible fluid rotates about a fixed axis in such a manner that the circumferential velocity u_θ is a function only of the radial distance r from the axis and the time t. Apply dimensional analysis to investigate the velocity u_θ.

(Answer. The relation can be written in various equivalent forms such as

$$u_\theta = \frac{r}{t} f\left(\frac{\nu t}{r^2}\right) \quad \text{and} \quad u_\theta = \frac{\nu}{r}\phi\left(\frac{\nu t}{r^2}\right)\Bigg)$$

20. Obtain the most general expression for the viscosity μ of a homogeneous gas, given that it depends only on the following quantities:

l = molecular diameter
C = root mean square velocity of the molecules
m = mass of one molecule
n = number of molecules in unit volume of gas.

What does the expression become when μ is independent of n, as suggested by the kinetic theory of gases?

$$\left(\text{Answer.}\ \mu = \frac{mC}{l^2} f(nl^3)\right.$$

$$\left.\mu = \frac{kmC}{l^2}\right)$$

CHAPTER 5

EXPERIMENTAL TECHNIQUES

5.1 Introduction

A comprehensive and thorough exposition of the experimental techniques employed in the mechanics of fluids would require a large treatise devoted to this subject alone. All that will be attempted here is to give, in outline, the principles of some of the methods in current use and to describe a few of the more important pieces of apparatus.

Almost always we are concerned with a physical system which contains solid bodies as well as a fluid or fluids and these are, in general, in relative motion. There are two specially important cases:

(a) The fluid is at rest, except as disturbed by the bodies moving through it, while the motion of these is prescribed.

(b) The bodies are at rest and situated in a stream of fluid which is uniform except in so far as it is disturbed by the bodies themselves.

It follows from the general Principle of Relativity that the phenomena are uninfluenced by the superposition of any uniform velocity on the whole system (see § 3.2, Principle (e)). Hence the following systems are equivalent:

(1) A body moving uniformly and without rotation through a fluid at rest.

(2) The same body at rest in a uniform stream whose velocity is that of the body in case (1) but reversed in direction.

In practice there are often circumstances which complicate the relationships, such as the presence of the boundaries of the tanks or channels used for tests. Thus a test of a model of a ship in a lake (whose boundaries may legitimately be regarded as infinitely remote from the model) is not exactly equivalent to a test of the same model in a 'ship tank' or 'model basin' and this again is not exactly equivalent to a test in a flume of the same sectional area as the tank, for the fluid in contact with the walls of the flume is brought to rest by viscosity and thus has zero velocity relative to the body under test. Such complicating circumstances as these may, however, be allowed for with the help of special techniques of experiment or calculation. It should particularly be noted that a body rotating in a fluid at rest is *not* equivalent to a body at rest in a fluid rotating about it in the reversed sense, for the distributions of acceleration in the fluid are very different.

The topics to be discussed in this chapter fall under the following headings:

(A) The means for setting up the required relative velocity of body and fluid.

(B) Measurement of the relative velocity and, in particular, determination of the velocity at any point in a fluid relative to the earth or other body. We may include here measurement of discharge (volume or mass).

(C) Measurement of the forces and moments on bodies wholly or partly immersed in a fluid at rest or in motion.

(D) Measurement of fluid pressure and head.

(E) Measurement of temperature.

(F) Means for making flow visible.

(G) Special techniques concerned, for example, with turbulent flow.

5.2 Towing, Free Flight and Tank Tests

One very obvious method of obtaining a relative velocity consists in towing a body through a fluid and this can be done on the full-scale or with models in a tank. A celebrated towing test was made by William Froude with H.M.S. *Greyhound*,[1] a vessel of length 172 ft. 6 in. between perpendiculars. This was towed by H.M.S. *Active* at speeds up to 12½ knots. Some interesting features of the tests were as follows:

(a) The tow rope passed from the bow of the *Active* over the end of an outrigger of 45 ft. overhang on the starboard side amidships so that the *Greyhound* was towed to starboard of the wake from the *Active*. To reduce interference the bow of the *Greyhound* was situated about 190 ft. astern of the *Active*.

(b) The tension in the tow rope was measured by a dynamometer and the resistance of the water to the motion of the hull was obtained by deducting the estimated wind resistance. This was assumed to be proportional to the square of the wind speed relative to the vessel. The constant in the formula for the wind resistance was obtained by allowing the hull to drift before the wind without other propulsion, the speed of drift relative to the water being measured; the water resistance at this speed was known with sufficient accuracy (the correction for wind resistance is normally a small fraction of the total resistance).

We may also include a reference here to the important tests of the 'Lucy Ashton'[2] although towing was not employed. The hull of the vessel was not provided with any propulsive agency acting on the water and the propulsive thrust was obtained from jet engines mounted above the deck.

Free flight testing of aircraft and missile models is a well established technique.† For low speed tests they may be initially towed or carried by aircraft or helicopters and then cast off, for high speed tests they may be rocket propelled to the required flight condition and they then coast on in free flight. Their characteristics may be determined by sensors telemetering to the ground and photography may also be used. Tests of gliders towed by powered aircraft have also been made on various occasions.

Towing tests of scale models of ships in 'ship tanks' or 'model basins' are now a matter of routine. The tank consists of a long horizontal channel, usually of rectangular cross-section, containing fresh water which is at rest except when disturbed by the model. This is towed from a wheeled carriage situated above the water and running on accurately horizontal and smoothly machined rails.‡ The towing force is measured and recorded by a dynamometer on the carriage while arrangements may be made to record trim changes and oscillations of the model. Many ship tanks are provided with wave generators so that tests can be carried out in a seaway. In making a test, the carriage is quickly accelerated until the required speed is attained and then the speed is kept constant during the measurement. Recording is started after sufficient time has elapsed for a steady regime to be established and ceases before braking of the carriage at the end of the run begins. The model is then towed back to the starting point but the next test must not be started until the motion of the water set up by the model has had time to disappear.¶ In order to damp out waves quickly the tank may be provided with gently sloping 'beaches' along its sides and ends.

It is beyond our scope to enter into the details of the construction of dynamometers but we shall briefly mention the principles of the methods used to measure the torque in the shaft driving the propeller of a model in a 'powered' test. Three possible methods are as follows:

(1) The driving motor within the model is mounted on pivots so that it can rock about the centre line of the propeller shaft. The moment required for equilibrium is measured by some form of dynamometer.

(2) The propeller shaft is broken by the insertion of a gear which reverses the direction of rotation without changing its speed. The torque reaction on the housing of the gear is then *twice* the torque in the shaft and can be measured by a suitable dynamometer. (This method has been used in the William Froude Tank at the National Physical Laboratory, Teddington).

(3) The torque can be determined by a measurement of the strain in a suitable flexible element.

In all cases allowance must be made for the frictional resistance of the stern bearing.

In order to ensure that the boundary layer of a model under test shall be turbulent, 'trip wires' are sometimes fitted near the bow (see also § 6.2).

5.3 **Flumes**

A flume is a fixed open-topped channel in which there is a current of water or other liquid; this current must be as uniform as possible. Thus a flume is the counterpart of a ship tank, for the force on a body held fixed in a flume is, apart from secondary effects, the same as for a body towed in a tank when the relative velocity is the same.

A Venturi flume is a means for providing a current of relatively high speed;[1] here the liquid flows through an open channel with a throat and the liquid beyond the throat can discharge freely. The working section is situated downstream from the throat and here the velocity is above the critical, i.e., it is *superundal* (see § 8.1). In accordance with an analogy pointed out by Riabouchinsky[2] the flow in such a channel may be used to obtain information about the supersonic flow of a gas. The depth of the liquid is the analogue of the density of the gas and for strict quantitative correspondence of the phenomena it would be necessary that the ratio of the specific heats γ for the gas should have the value 2. This is in fact unattainable since, even for a monatomic gas where γ is a maximum, the value is only 1.67.

5.4 **Whirling Arms**

A whirling arm consists of a long single or double cantilever beam which can be caused to rotate about an axis (usually vertical) and capable of carrying a model or other body for test near its outer end. The model is thus moved through the air, but it also shares the rotational velocity of the arm; a velocity of translation without any rotation cannot be obtained with a whirling arm. Usually the arm is contained in a cylindrical room concentric with the axis of rotation and provided with evenly spaced baffle plates to reduce the swirl set up by the arm. The arm when rotating acts to some extent as a centrifugal pump, so there is an outward radial component of velocity at the model unless means are provided to neutralize this. One method of controlling the radial velocity consists in mounting aerofoils on the arm with their spans parallel to the axis of rotation and with the model placed between them; the radial velocity can then be varied by adjusting the incidence of these aerofoils. Even the best faired whirling arm sets up some swirl and it is therefore necessary to measure this, either directly with an anemometer or indirectly by measuring the relative air speed in the region of the model. The centrifugal forces associated with the rotation must be balanced out or measured and, as they may be large in comparison with the aerodynamic forces on the model, they are apt to be troublesome.

A whirling arm can also be used for experiments in water, which is contained in a circular pond or ring-shaped channel; beaches for the damping of waves may be provided, as in ship tanks. The whirling arm itself may with great advantage be situated in the air above the water while the model is suspended from the arm by well faired members which disturb the water only slightly. With this arrangement the swirl is small and may be further reduced by means of fixed baffle plates.

5.5 Low Speed Wind Tunnels

In 'low speed' wind tunnels the velocity of the gas relative to the fixed model is decidedly less than the velocity of sound in the gas (almost always air) flowing in the tunnel. The Mach number (see § 4.5) of the flow will be less than say 0·5 and is sometimes very much smaller. Throughout this section it will be understood that "wind tunnel" means a low speed wind tunnel.

Wind tunnels are of two types, called 'closed' and 'open', according to whether the working section of the tunnel is or is not enclosed by a fixed wall. Except as regards ease of access and of viewing, the advantages lie with the closed tunnel and this type only will be described here.† Tunnels may again be classified as 'straight through' and 'return flow'. In the latter type the same air circulates continually in a closed circuit, apart from a slow interchange with the surrounding atmosphere through leaks etc.

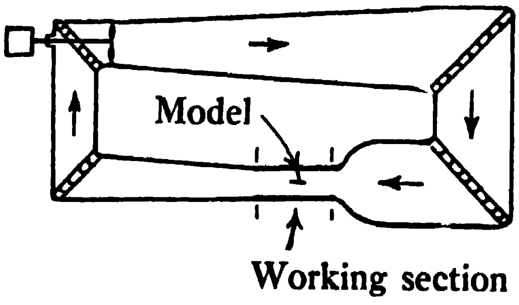

Fig. 5.5,1. Return flow wind tunnel with closed working section.

A diagram of a typical return flow tunnel with closed working section is shown in Fig. 5.5,1. The air is propelled round the circuit by a power driven fan or propeller whose speed of rotation can be accurately controlled.

After passing the fan, the air travels along a gently expanding duct until it reaches a cascade of fixed vanes which deflect it through a right angle. It then flows through another gently expanding passage to a second set of vanes at the next corner. It may now pass through a honeycomb whose function is to smooth the flow by breaking up large eddies or, preferably, through one or more screens of specially uniform wire gauze. The air is now in the 'settling chamber' from which it passes to the working section *via* the 'contraction'. In the process of passing through the contraction the air is greatly speeded up while the flow is made more smooth and regular. The working section has parallel (or nearly parallel) walls and the velocity of the air here reaches its maximum. At the downstream end of the working section there are vents or slots of small area in the walls to allow the pressure inside the tunnel to be atmospheric. It follows from Bernoulli's theorem (see § 3.4) that the pressure in all other parts of the tunnel is above atmospheric, so any leakage is outward. The walls of the tunnel must be strong enough to withstand safely the greatest excess pressure, which occurs when the tunnel is running at top speed, and be stiff enough not to deflect appreciably.† The working section has a hinged door for access to the model; this door has a plane glass window and there may be other windows in the tunnel walls while adequate illumination must be provided. After passing the working section the air passes to the *diffuser*, i.e. the gently expanding passage which conducts the air back to the fan. In the tunnel shown in the diagram there are two right angled bends, each provided with a cascade of vanes, in this return duct before the fan, but the fan may be placed before or after the first bend. Fixed vanes are usually placed before and after the fan and so arranged that the air leaving the fan section of the tunnel has little or no swirl about the axis of the return duct.

The balances for measuring the forces and moments on the model are situated outside the working section and usually above or below it. In many cases the balance is arranged to measure only lift, drag and pitching moment (see § 4.16) but in the larger tunnels there are usually balances for measuring, in addition, rolling and yawing moments and cross-wind force while the model is mounted on a turntable so that the angle of yaw can be varied. The model may be mounted on slender struts attached to the balance; these struts are shielded from the airstream by hollow streamlined guards fixed to the floor or roof of the tunnel and ending close to the model. Alternatively the model may be suspended by piano wires from a frame attached to the balance or on a 'sting' at its rear end. Whatever method of support is adopted, the measured forces on the model must be corrected to allow for the forces on the exposed supports.

It is usual in routine work to measure the speed of flow at the working section indirectly by determining the pressure difference between tappings in the settling chamber and working section. A smooth metal plate is inset flush with the inner surface of the tunnel wall and there is a smooth-lipped hole in the centre of the plate connecting through leak-proof tubing to one side of a manometer. The functional relation between the pressure difference and the value of ρV^2 at the working section is determined by calibration tests in which ρV^2 is found accurately by means of a Pitot-static tube (see § 5.9). The calibration curve usually takes the form of a graph of V against the pressure difference. The pressure tapping in the working section should be upstream from the model and as far from it as possible so as to minimise interference effects. A complete calibration of the tunnel includes an exploration of the flow throughout the working section for speed, direction and steadiness.

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† High transient pressures may occur if the fan is used for dynamic "braking."

Let the cross-sectional area of the working section of the tunnel be A and let V be the air speed there while ρ is the corresponding air density. Then the mass of air passing in unit time is ρAV and the kinetic energy of this mass is $\frac{1}{2}\rho V^3 A$. If all this energy were dissipated before the air returned to the working section *via* the return duct and if the fan had an efficiency of 100 per cent, the power P supplied by the motor would be exactly equal to $\frac{1}{2}\rho AV^3$. Consequently the non-dimensional power factor

$$\lambda = \frac{P}{\frac{1}{2}\rho AV^3} \tag{5.5,1}$$

would have the value unity; the lower the value of this factor the more economical is the tunnel as regards its power demand.† In a straight through tunnel the dissipation of kinetic energy is complete and, since the efficiency of the fan must be less than 100 per cent, the value of λ must exceed 1·0 and may be as high as 2·0. On the other hand, values as low as 0·1 have been attained in well designed and *large* closed return flow tunnels working without gauze screens. Open jet return flow tunnels are less efficient than those with closed working sections, other things being the same.

In order to raise the Reynolds number (see § 4.10) of a test on a model a 'compressed air tunnel' may be used. Here the channel is enclosed in a gas-tight shell within which the air pressure can be raised by power driven compressors. The viscosity is uninfluenced by the rise of pressure, so the Reynolds number is increased in proportion to the air density.

5.6 High Speed Wind Tunnels

Wind tunnels working at high subsonic speeds may differ little from low speed tunnels, except that, for a given cross-sectional area of the working section, the power required is increased approximately in proportion to the cube of the speed. The high rate of dissipation of energy in return flow high speed tunnels would lead to a rapid rise in the temperature of the circulating air if steps were not taken to remove the heat generated. This may be done by making some of the vanes of the cascades at the corners or the straightening vanes of the fan of hollow metal and circulating cooled fluid through these. In order to reduce the power required to drive the tunnel at top speed, the whole channel may be contained within a gas-tight shell which is partially evacuated by air pumps.†

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† The reciprocal of λ is sometimes used. For further information the reader may consult Pankhurst and Holder (*loc. cit.*).

The supersonic wind tunnel differs from the subsonic one in necessarily having a throat, where the flow is locally sonic, upstream from the working section (see § 9.1). Fig. 5.6,1 shows part of a supersonic wind tunnel in section and it will be seen that nozzle liners are fitted into the channel to provide a throat and suitably proportioned expansion between the throat and the working section. The liners must be shaped so as to give uniform flow of the desired Mach number at the working section, their shapes may be adjusted by jacks or a set of liners must be provided for each Mach number which is to be used. The air may be propelled round the tunnel circuit by a turbo-compressor, which is usually of the axial flow type. Alternatively, induced flow may be adopted, the high pressure air being obtained from reservoirs or the exhaust of a jet engine.[1] In another system the tunnel draws air from the atmosphere or a reservoir and exhausts into an evacuated tank, but only intermittent operation is then possible. With a high Mach number tunnel it is advisable to provide a variable second throat downstream of the working section so as to control the shock wave position in the diffuser and thus to ensure minimum diffuser losses. The ranges of temperature and pressure in supersonic wind tunnels are relatively large and if moist air is used there may be a precipitation of droplets and 'condensation shocks'. Hence it is standard practice to use dried air which may be obtained by passage over beds of silica gel.‡

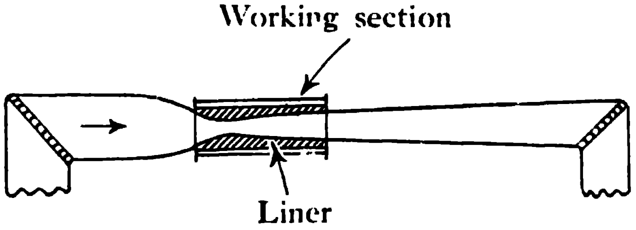

Fig. 5.6,1. Supersonic wind tunnel (with liners adapted to Mach number).

5.7 Water and Cavitation Tunnels

A water tunnel is, in general design and layout, similar to a return flow subsonic wind tunnel but uses water instead of air as the working fluid.

On account of the low kinematic viscosity of water, a given Reynolds number can be obtained with a given size of model at a comparatively low speed of flow. This is an advantage when it is desired to make the flow pattern visible and water tunnels are very suitable for this purpose. The technique consists in adding finely divided but highly reflective particles to the water which is strongly illuminated in the region where the flow is to be explored and it is advantageous to confine the illumination to a narrow slab-shaped region whose position can be changed as required. By the application of intermittent illumination and by taking photographs in two perpendicular directions it is possible to determine the velocities over the illuminated region. Mention may also be made here of the tank developed in the Department of Aeronautical Engineering at Cambridge for the study of the impulsive generation of fluid motion.[1] Here the reflective particles were very small droplets of an oil whose density was adjusted to agree with that of the water in the tank.

A cavitation tunnel is a water tunnel specially adapted to the study of the phenomena of cavitation (see § 3.12). Means are provided for varying the pressure at the working section as desired throughout a range extending from some degree of vacuum to pressures above atmospheric. In order to make sure that the water approaching the working section shall not contain minute bubbles in suspension, the water is passed to a *resorber* after leaving the working section. This consists of a wide vertical U tube with its lower end sunk 50 ft. or more below the working section. The water passes *slowly* through the resorber under high pressure and any small bubbles of gas are redissolved in the water.†

5.8 Tunnel Corrections

The flow in the neighbourhood of a body under test in a wind or water tunnel is not exactly the same as it would be in a completely open or unconstrained stream and measurements made in tunnels accordingly require some correction. In general, the correction is approximately proportional to some positive power of the ratio (significant linear dimension of model): (depth of working section of tunnel) and accordingly tends to become very small when the linear dimensions of the model are small in relation to those of the working section. There are three principal sources of the discrepancies between the results of tests in tunnels and in unconstrained streams:

(1) The constraint on the flow imposed by the tunnel walls causes the effective velocity at the model to be raised slightly. This is commonly called the *blockage effect.* The effect is present irrespective of the existence of any

lift or other aerodynamic force normal to the stream and is fully effective when a symmetrical body is placed symmetrically in the stream. For this reason the effect is also known as 'tunnel interference on a symmetrical body.'†

(2) The effective angle of incidence of a lifting surface is modified. For a closed working section this may be regarded as resulting from the influence of the images in the tunnel walls of the vortices which are associated with the lift. It appears that the angle of incidence is effectively increased by the amount

$$\Delta\alpha = \delta\left(\frac{S}{C}\right)C_L \tag{5.8,1}$$

where S is the plan area of the lifting surface, C is the cross-sectional area of the working section, C_L is the lift coefficient and δ is a numerical coefficient, depending on various factors, such as the shape of the surface and of the cross-section of the tunnel. For similar models in a given tunnel the correction is thus proportional to the square of the linear scale of the model. Since the effective angle of incidence is increased, the lift has a component which reduces the measured drag. Accordingly the measured drag coefficient must be *increased* by the amount

$$\Delta C_D = \delta\left(\frac{S}{C}\right)C_L^2 \tag{5.8,2}$$

In applying the corrections, we regard the angle of incidence as the measured angle plus $\Delta\alpha$, the lift is treated as correct since the correction is proportional to the square of $\Delta\alpha$ and the measured drag coefficient is increased in accordance with (5.8,2). There are also corrections to measured angles of 'downwash' and 'tail setting' (see the references given below).

(3) The turbulence of the stream in the tunnel may differ from that for the full-scale phenomenon. In association with the usual discrepancy of the model and full-scale Reynolds numbers, this will influence the measured forces. The quantities particularly affected are drag and *maximum* lift.

Another correction to be mentioned is that for 'horizontal buoyancy'. The measured drag will be in error when there is a gradient of static pressure in the direction of flow at the working section. In calculating the correction to the drag the volume of the model is to be increased by the amount corresponding to its virtual mass for longitudinal acceleration (see § 3.9). A pressure gradient can arise due to the boundary layers (see Chapter 6) on the working section walls but it can be removed by suitably varying the cross-sectional area, as by fitting corner fillets of properly adjusted width.

Many of the corrections to wind tunnel measurements are of opposite signs for open and closed working sections. It has been established that the corrections can be made very small by using a working section whose walls are provided with longitudinal slots of suitable width; the working section is then 'neutral' or half way between open and closed. This system is originally due to G. I. Taylor. This principle is widely adopted for tests at transonic speeds.

It is beyond our scope to enter into further details about the theory or practice of tunnel corrections. For further information the reader may consult the references.[1]

5.9 The Measurement of Velocity

The particles of a fluid are not individually visible and direct observations of velocity are therefore not possible. However, direct observation becomes possible when particular fluid elements are marked in some way, as, for instance, by colour or radioactivity. Some examples of this technique are:

(a) The measurement of the velocity of flow at the surface of a river by observation of the motion of a small float. The float should be only slightly less dense than the water and should have the least possible surface exposed to the wind.

(b) Observation of the flow of liquids by the local and momentary injection of dyes or salts which can be detected chemically.

(c) Observation of the flow of air by the local and momentary injection of smoke.

(d) The spark method invented by H. C. H. Townend. Here intense electric sparks pass periodically between electrodes placed in the current of gas whose motion is to be explored. Each spark heats the gas locally and the path of the heated element can be made visible by the shadow or Schlieren techniques (see § 5.12).

(e) When the flow to be explored is non-turbulent the following method can be used. Allow coloured fluid to exude into the stream from a fine bore tube and impose a small lateral movement on this tube periodically. This imposes a small wave or kink on the coloured filament and the motion of this can be observed.

Reference has already been made in § 5.7 to the use of reflective particles in the exploration of a field of flow.

A vane anemometer consists of a small vane-wheel mounted on bearings which are made as frictionless as possible (see Fig. 5.9,1). If the resistance to rotation of the wheel were strictly zero and the flow uniform over the region occupied by the wheel, each vane would move so that the fluid reaction on it was zero (apart from the inevitable skin friction). This would imply that the relative velocity of fluid and vane along the normal to the vane was zero. Let the plane of one of the vanes, assumed to be small, be inclined at the angle θ to the plane of rotation, let r be the radial distance from the axis of rotation to the centre of the vane, ω the angular velocity and V the velocity of the fluid, which we shall assume to be parallel to the axis of rotation. Then the component of V normal to the vane is $V \cos \theta$ while the component in this direction of the velocity of the centre of the vane is $r\omega \sin \theta$. When these are equal we have

$$\omega = \frac{V \cot \theta}{} \qquad (5.9,1)$$

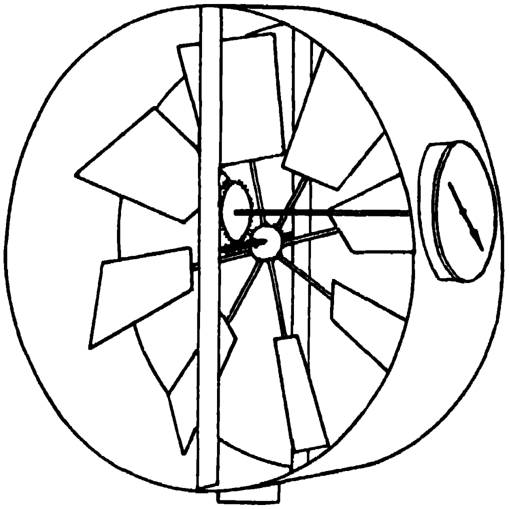

Fig. 5.9,1. Vane anemometer (Ower's type).

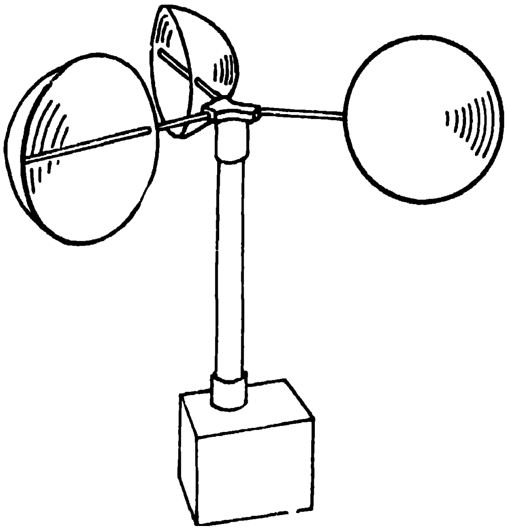

Fig. 5.9,2. Robinson cup anemometer.

and for a wheel of fixed geometry ω is proportional to V. This is the idealized characteristic equation of the instrument. Experiment shows that the graph of ω against V is a straight line, provided that V is not too small, but the line does not pass exactly through the origin. For accurate work a vane anemometer should be calibrated in an aerodynamic laboratory. For the greatest sensitivity the vane angle θ should be about 50°. The 'Patent log' sometimes used at sea may be regarded as a robust form of the vane anemometer and instruments of the same general type are sometimes used for the gauging of rivers.†

The Robinson cup anemometer, in an improved form, is shown in Fig. 5.9,2. There is no simple theory of this instrument and it requires calibration. Provided that the wind speed is not too low, the relation between angular velocity and wind speed is linear.

The really fundamental instrument for the measurement of the velocity of a fluid is the Pitot-static tube, although a simultaneous determination of density is also required. This instrument is shown diagrammatically in section in Fig. 5.9,3. The open end of the head, i.e. the Pitot tube, faces the stream and is connected by tubing to one side of a manometer. Since in steady conditions there is no flow into the manometer the fluid just inside the open end is at rest and the pressure there is the 'total pressure' i.e., the sum of the 'static pressure' and the 'dynamic pressure' (see § 3.4). The small holes in the side of the head allow the pressure in the outer tube to become equal to the pressure in the free stream, i.e. the 'static pressure'. This outer tube is connected to the other side of the manometer whose

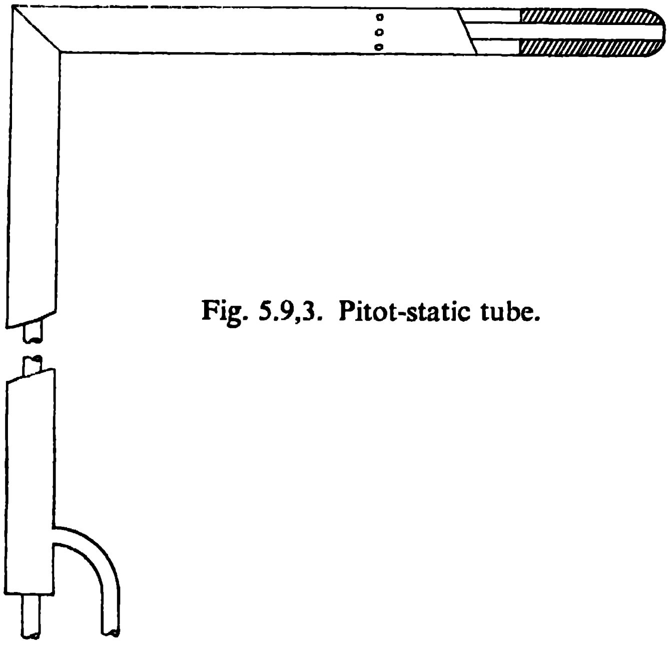

Fig. 5.9,3. Pitot-static tube.

differential reading is therefore equal to the 'dynamic pressure'. When the fluid is incompressible the pressure recorded by the manometer is accordingly

$$p_d = \tfrac{1}{2}\rho V^2 \qquad (5.9,2)$$

where V is the speed of the stream and ρ the constant density of the fluid. When the fluid is compressible and the Mach number not small, a modified equation must be used and a further modification is required when the velocity is supersonic and a shock wave lies ahead of the open end of the tube (see § 9.10). As already stated, it is necessary to know the value of ρ in order to find V from the reading of the manometer. However, for many purposes it is the product ρV^2 which is important rather than V itself. It is thus customary to derive a nominal speed from the measured pressure difference, calculated on the assumption that ρ has a standard value e.g. that of the standard atmosphere at sea level. In aeronautics the air speed derived in the manner described is called the equivalent airspeed (E.A.S.) and given the symbol V_e or V_i.

There is much experimental evidence to show that, except possibly for exceedingly small values of the velocity, the total pressure is accurately given by the Pitot tube. Such errors as arise in Pitot-static tubes are caused by incorrect registration of the static pressure. A small hole in a surface exposed to a current of fluid allows the pressure within the hole to become equal to that of the stream just outside the boundary layer (see § 6.3). Hence, by Bernoulli's theorem, the pressure in the hole will not be equal to that in the free stream (which is the so-called static pressure) unless the velocity just outside the boundary layer at the hole is equal to that in the free stream. There are two agencies tending to cause the velocity outside the boundary layer at the static holes to differ from that in the free stream:

(a) The proximity of the round nose of the instrument tends to raise the velocity and so to reduce the pressure.

(b) The presence of the stem behind the static holes tends to reduce the velocity and so to increase the pressure.

In the best Pitot-static tubes, such as those designed by the National Physical Laboratory and by Prandtl, the two effects cancel very nearly and the instrument reads correctly to within a fraction of 1 per cent.† A Pitot-static tube gives a true measurement of the dynamic pressure only when the axis of the head of the instrument is directed exactly along the wind direction but the error is inappreciable when the angle of yaw of the head does not exceed say 5°.

The determination of the *direction* of a fluid stream is effected by an instrument called a *yawmeter*. Yawmeters are of three main types:

(a) Pivoted vanes.

(b) Instruments depending on reducing a pressure difference to zero.

(c) Hot wire instruments. (These are described in § 5.13.)

A pivoted vane yawmeter consists of a small symmetrical aerofoil (or hydrofoil) which is pivoted about an axis in its plane and parallel to its span; when in stable equilibrium the angle of incidence on the aerofoil is zero, i.e. the fluid velocity vector lies in the plane containing the aerofoil and the pivot axis. A complete determination of direction would require two such instruments with (preferably) perpendicular pivot axes but a single cross-shaped vane mounted on a ball joint could be used. In any case, the friction at the pivot must be very small. In order that gravity shall not influence the reading of these instruments the centre of gravity of the complete vane assembly must lie on the pivot axis.

To explain differential pressure yawmeters we shall begin by considering the determination of the direction of a two-dimensional flow. Fig. 5.9,4 shows a circular cylinder with its axis normal to the plane of flow and provided with a pair of small holes (pressure tappings) P_1, P_2 which are connected to the two sides of a manometer. The stagnation point S is such that the line joining S to the centre O lies along the direction of the stream. When S does not lie exactly midway between P_1 and P_2 the pressures at the holes will differ. Hence the direction of the flow can be determined by rotating the cylinder until the differential pressure is zero. For a three-dimensional flow we may substitute a sphere for the cylinder; the sphere is provided with three pressure holes P_1, P_2, P_3 lying at the vertices of an equilateral triangle. P_1 and P_2 are connected to the two sides of a manometer and P_2 and P_3 to a second manometer. When *both* differential pressures are zero the pressures at P_1, P_2 and P_3 are equal and the velocity vector lies along the line joining the centre of the sphere to the centroid of the triangle $P_1P_2P_3$. There is an immense number of possible variations of the design of yawmeters of the differential pressure type but the principle is the same for all.

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† Experience shows that it is extremely difficult to measure the velocity of a fluid with an accuracy substantially better than 1 per cent.

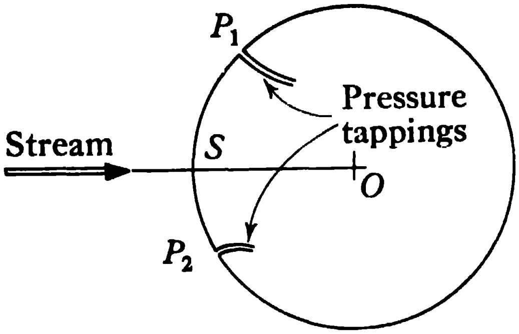

Fig. 5.9,4. Diagram to illustrate the use of a circular cylinder to determine the direction of a two-dimensional stream.

The measurement of the *mean* velocity of flow in a pipe or channel is discussed under "the measurement of discharge" (see § 5.10).

5.10 The Measurement of Discharge

When flow occurs in a pipe or channel we are usually interested in the total rate of discharge rather than in the velocity, which varies considerably across the section of the conductor; this amounts to saying that we are interested in the *mean* value of the velocity taken over a normal section rather than in the velocities at particular points. The rate of discharge is usually taken as a volume per unit time (cubic feet per second or *cusecs* in British consistent units, cubic metres per second in SI units) when the fluid is a liquid. For a gas the rate of discharge is quoted as a mass per unit time since the density varies as the gas moves along the pipe.

When the flow is steady the rate of discharge of a liquid is easily determined by collecting the liquid which passes in a measured interval of time. The volume may be obtained directly from the observed depth of liquid in the collecting tank by use of a calibration curve or indirectly by weighing the liquid and division by the specific weight.

The most important instrument for the measurement of discharge through pipes is the Venturi meter, of which the basic theory is given in Example 2 of § 3.4. A diagram of the apparatus is shown in Fig. 5.10,1 where the axis of the pipe is horizontal, but this may be inclined at any angle. The pipes connecting the pressure tappings to the manometer are full of the liquid and, when the rate of discharge is zero, the pressure difference across the manometer is zero whatever may be the levels of the tappings. In accordance with equation (3.4,21) the rate of discharge (volume per unit time) *for a perfect fluid* is given by

$$Q = \sqrt{\frac{2A_1^2A_2^2(p_1 - p_2)}{\rho(A_1^2 - A_2^2)}} \tag{5.10,1}$$

where A_1 is the cross-sectional area of the pipe upstream from the instrument, A_2 is the area at the throat, $(p_1 - p_2)$ is the pressure difference and ρ the density of the liquid. An alternative formula employing the difference of head $(h_1 - h_2)$ instead of the difference of pressure is given in equation (3.4,22); consistent units must be adopted whichever formula is used. For a real viscous fluid the discharge is less than that given by equation (5.10,1) and the ratio of the actual to the theoretical discharge is the *coefficient of discharge* C_d. For water the value of C_d varies from about 0·975 for a meter in a 2-inch (5 cm) pipe to about 0·985 for one in a 32-inch (80 cm) pipe. The coefficient is influenced by flow conditions upstream and depends on the proportions and surface finish of the tube; it also depends on Reynolds number. A decrease usually occurs with age owing to increase of roughness caused by corrosion or surface deposits.

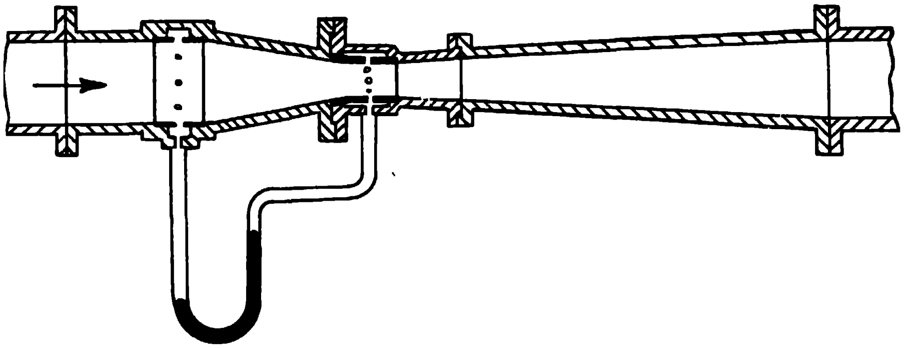

Fig. 5.10,1. The Venturi meter (with mercury manometer).

For simplicity and ease of construction, the *orifice gauge* is frequently used, the theory being similar to that for the Venturi meter. The device consists simply of a square edged circular orifice inserted concentrically in the pipe. It occupies less space than the Venturi tube, and the use of stainless steel has almost eliminated corrosion troubles. The coefficient of discharge is about 0·61, but depends on Reynolds number and area ratio. A nozzle with a faired contraction may be used in place of an orifice, but the coefficient for this (about 0·96 to 0·91) depends to a greater extent on the nozzle shape.

Small and steady discharges may be measured by passing the liquid into a tank from which it issues through a circular sharp-edged orifice, and the head (h) over the orifice is read from a glass gauge tube. The rate of flow is then

$$Q = C_d A\sqrt{2gh} \tag{5.10,2}$$

where, again, C_d is a coefficient of discharge and A is the area of the orifice (see § 8.6). Discharge may be measured to within 1 or 2 per cent by this method. For water, C_d is about 0·61 under most conditions, but in general C_d varies with the Reynolds number.[1]

Discharge in pipes or open channels may also be determined by measuring the velocity at a number of points on a cross-section and integrating. Traverses are made across suitable representative parts of the section, the velocity being measured by Pitot tube, Pitot-static tube, vane anemometer or propeller type current meter (see § 5.9). In an open channel, a fairly accurate value for the mean velocity over a vertical strip of the chosen cross-section may be obtained by measurement of the velocity at 0·4 of the depth above the stream bed.

Where knowledge is required of the total volume of flow in a pipe over a specified time, a *rotary meter* may be used. The apparatus consists of a current meter (§ 5.9) having a propeller of nearly the same diameter as the pipe, operating a counter which shows the total flow in gallons, cubic feet, etc. This type of meter is sufficiently accurate for commercial purposes over a wide range of flows and the loss of head is negligible. When accurate measurement of clean liquids is required, *positive meters* are used. These are positive displacement pumps (see § 12.1) in reverse or hydraulic motors running under no load. A greater loss of head occurs than in rotary meters.

For measurement of large discharges the most useful devices are the 90° V-notch and the rectangular weir (see § 8.6). In accordance with equation (8.6,15) the discharge over a V-notch is given by

$$Q = \frac{8}{15} C_d \tan\frac{\theta}{2} \sqrt{2g}\, h^{5/2} \qquad (5.10,3)$$

where θ is the total included angle of the notch and h is the head. For a 90° V-notch C_d has a value of about 0·59. The coefficient C_d is subject to slight variations as the head varies and BS 3680: Part 4A: 1965 contains tables giving values of C_d for $0{\cdot}05 \text{ m} < h < 0{\cdot}38 \text{ m}$. Head h is measured as the height of the free surface at a stagnation point above the apex of the notch. Stagnation points occur in the corners where the notch bulkhead meets the sides of the channel.

When the discharge exceeds 0·021 m^3/s a rectangular weir may be used. For rectangular weirs having complete contractions, B.S. 3680: Part 4A: 1965 gives the following relationship

$$Q = \tfrac{2}{3}\sqrt{2g}C_d b h^{3/2} \qquad (5.10,4)$$

in which

$$C_d = 0{\cdot}616(1 - 0{\cdot}1h/b) \qquad (5.10,5)$$

where b is the length of the weir and h is the observed head above the crest for negligible velocity of approach. This equation may be used for heads from 0·075 to 0·60 m provided that b/h is greater than 2. Care must be taken to have the approach channel sufficiently large, as detailed in the specification.

In broad shallow rivers and irrigation canals, where the loss of head at any metering device may have to be kept small, *broad-crested weirs* or *Venturi flumes* can be used. When no interference occurs from the backwater, the depth of flow over a broad-crested weir is $\frac{2}{3}h$ where h (fm.) is the still water level above the sill of the weir (see § 8.6). From equation (8.6,25) the discharge is then

$$Q = 1{\cdot}705C_d bh^{3/2} \tag{5.10,6}$$

where b is the weir length in metres and C_d is the coefficient of discharge, ranging from 0·9 up to 0·97. One measurement of h is sufficient. In practice a correction is required to allow for velocity of approach. This type of weir is easy to construct in broad shallow rivers. The Venturi flume, used frequently to measure irrigation or sewage water, consists of a constricted portion or throat in a channel. Accuracy is not great, but it is not liable to interference by deposition of silt. Only a small amount of head is lost, there are no mechanical moving parts and consequently there is no wear. The constriction may be formed either by raising a section of the bed or by reducing the width. Two wells may be used to measure the differential head, one at the throat, the other upstream, and the discharge may be found directly by using equation (5.10,1) with an appropriate coefficient (0·95 to 0·99) which should be found by calibration. When flow is produced in the throat at critical depth h_c (§ 8.5), and equation (5.10,6) is used, it is essential that no interference occurs from the backwater. The device is then properly styled a *standing wave flume*; upstream depth is unaffected by variation in downstream depth.

Measurements of rate of discharge can be made by the *salt dilution method.* A solution of common salt, or other soluble salt, of known high concentration is fed into the stream at a known steady rate at some convenient station. Samples of water are taken upstream of this and far enough downstream for the salt to have become uniformly diffused throughout the stream.

Let s_0 = mass of salt in unit volume of undosed water,
s_1 = mass of salt in unit volume of salt solution,
s_2 = mass of salt in unit volume of dosed water,
q = rate of discharge of salt solution,
Q = rate of discharge of stream.

Since the total weight of salt leaving the dosing station per second must be the same as that arriving at the sampling station,

$$Qs_0 + qs_1 = (Q + q)s_2 \text{ from which}$$

$$Q = q\left(\frac{s_1 - s_2}{s_2 - s_0}\right). \tag{5.10,7}$$

Sodium dichromate may be used instead of common salt and has the advantage that smaller quantities are effective. A *colorimetric method,* similar to the above in principle, may also be used to measure discharge. In the *salt velocity method* a quantity of suitable salt solution is suddenly injected into the pipe and measurements of the electrical conductivity of the water between immersed electrodes are made at stations downstream. It is found that the curve of electric current strength against time at any station exhibits a fairly definite peak and the mean velocity of flow can be calculated from the observed time for the peak to travel a known distance between stations. Radio-active tracers have also been used.

The *inertia-pressure method*, developed by N. R. Gibson,[1] is based on measurement of the increase of pressure generated when the valve at the downstream end of a pipe or conduit is closed. Equation (5.10,8) gives the time integral of the pressure rise caused by closure of a valve at the end of a pipe of length l in which liquid was initially moving with velocity v:—

$$\int p\, dt = l\rho v \tag{5.10,8}$$

It will be seen that if the integrated pressure rise due to inertia, the pipe length and the density are known, then the original velocity can be calculated, but there are important corrections to allow for changes in the pressure lost by fluid friction. Application may be extended to pipes having non-uniform cross-sectional areas.

5.11 The Measurement of Pressure and Head

Measurements of pressure are, in practice, always measurements of pressure difference. Very often the pressure datum is that of the ambient atmosphere and the quantity measured is the excess of pressure above atmospheric. Sometimes the instrument has two sides or tappings which can be connected through tubing to two sources of pressure and the reading of the instrument is then the pressure difference, subject possibly to certain corrections. When the instrument has only one connection or tapping (as for the ordinary Bourdon pressure gauge) the pressure which is effective and is indicated by the instrument is the excess pressure above atmospheric (or the defect below atmospheric in the case of a vacuum gauge). An absolute pressure is determined as the excess of pressure above that of a perfect vacuum.† Most pressure gauges are suitable only for the measurement of pressures which are steady or change slowly. Instruments to measure rapidly changing pressures are in a special class and must be adapted to the particular requirement. We shall discuss separately below the following types of pressure gauge:

(a) The U tube and its variants.
(b) The pressure balance.
(c) Instruments depending on elastic deformation under pressure.

A diagram of a U tube as arranged for the measurement of a pressure difference in a gas is shown in Fig. 5.11,1. When the system is in equilibrium the pressure in the liquid at a height z above a fixed horizontal datum is given by (see § 1.6)

$$p = p_0 - g\rho z. \qquad (5.11,1)$$

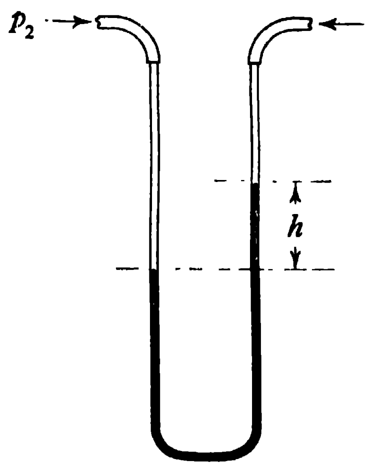

Fig. 5.11,1. Diagram of U tube manometer.

Let p_1 and z_1 be the pressure and height respectively at the free surface of the liquid in one branch of the U tube while p_2 and z_2 are the corresponding quantities for the other branch. Then by the last equation

$$p_1 = p_0 - g\rho z_1 \quad \text{and} \quad p_2 = p_0 - g\rho z_2$$

so

$$p_2 - p_1 = \Delta p = g\rho(z_1 - z_2) \qquad (5.11,2)$$

$$= g\rho h \qquad (5.11,3)$$

where $h = z_1 - z_2$ is the difference of level of the free surfaces. When using the instrument we read off h and multiply this by the specific weight $w = g\rho$ of the liquid in order to obtain the pressure difference. Since the density ρ depends somewhat on temperature, a reading of temperature must be taken in accurate work and the corresponding value of ρ used in calculating the pressure difference. Next, suppose that the instrument is used to measure a pressure difference in a liquid of density ρ_1 and let ρ_2 be the density of the liquid in the U tube.† We shall suppose that all the passages and spaces above the liquid in the U tube are filled with the liquid of density ρ_1 and for stability we must have $\rho_2 > \rho_1$. Equation (5.11,2) with ρ_2 substituted for ρ is still valid when the pressures p_1 and p_2 are measured *at the surfaces of separation of the liquids* but what is usually wanted *is the pressure difference across the instrument where the pressures are measured at the same level.* The pressure p_2' in the second leg of the instrument measured at the level of the free surface in the first leg is given by

$$p_2' = p_2 - g\rho_1 h. \qquad (5.11,4)$$

When we eliminate p_2 from this and the equation

$$p_2 - p_1 = g\rho_2 h$$

we obtain

$$\Delta p = p_2' - p_1 = g(\rho_2 - \rho_1)h \qquad (5.11,5)$$

and this is the pressure difference across the instrument at any common level. When ρ_1 is very small, as when the fluid concerned is a gas, it is usually neglected in applying (5.11,5).

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† The two liquids must not mix and there must be a clearly visible meniscus between them.

There is an immense variety of forms and modifications of the U tube made to meet special requirements and it is beyond our scope to describe these. We may mention, however, the device of widening one limb of the tube into a reservoir of large area at the free surface so as to make the changes of level there very small and inclining the tube so as to increase the linear displacement of the free surface corresponding to a given pressure difference. To facilitate accurate reading of differences of level a float bearing a hair line or similar device may be adopted.

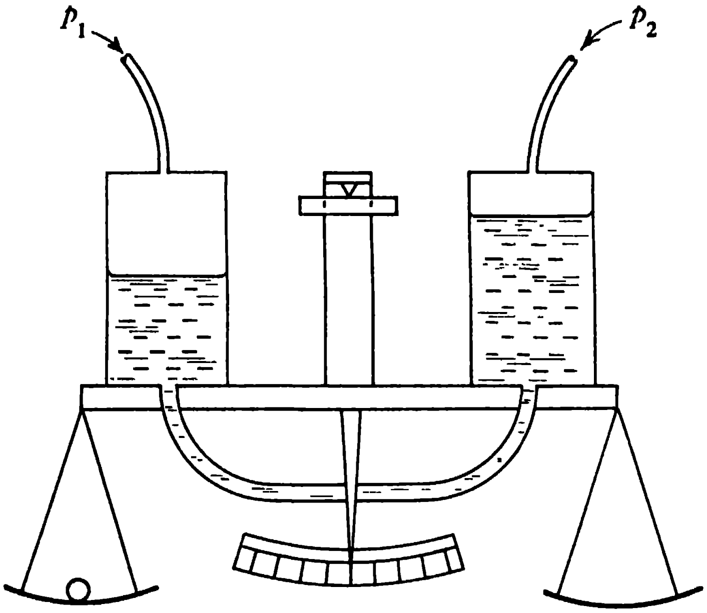

Fig. 5.11,2. Diagram to illustrate the principle of the pressure balance.

The principle of the pressure balance as applied to the measurement of pressure differences in a gas will be explained with the help of the diagram, Fig. 5.11,2. The cylindrical vessels have equal sectional areas and their axes are vertical; to facilitate the exposition we shall suppose that their plane bottoms are at the same level, though this is not necessary in practice. Each vessel has a pressure tapping and there is a connecting pipe which allows the liquid to flow between them until equilibrium is attained. We shall suppose that a steady state has been reached and that weights have been placed in the scale pans so as to put the balance in equilibrium with the axes of the vessels vertical. The pressures on the curved surface of either vessel are self-equilibrating and need not be further considered. Also the pressures on the bottoms of the vessels are the same in the state of equilibrium and give no unbalanced moment. Hence the unbalanced pressure moment is entirely due to the gas pressures on the upper surfaces of the vessels and its magnitude is

$$M = (p_2 - p_1)Al \tag{5.11,6}$$

where A is the internal cross-sectional area of each vessel and l is the perpendicular distance of their axes from the axis of the fulcrum. The pressures must be fed to the vessels through flexible tubes but a small amount of stiffness in these is tolerable since the balance is in its neutral position when a reading is taken. With the help of electric contacts and relays the pressure balance can easily be adapted for the automatic measurement of pressure or for the automatic maintenance of a given pressure. The greatest pressure difference which the instrument can deal with is fixed by the height of the vessels and the density of the liquid in them.

A pressure difference can be indicated or measured by the deformation of a suitably arranged elastic body. As a simple example we may take the steam engine indicator which consists essentially of a small piston sliding in a cylinder open to atmosphere at one end and connected to the source of pressure at the other; the displacement of the piston is resisted by a spring whose stiffness is chosen to suit the range of pressure to be dealt with. A direct reading pressure gauge may be constructed by gearing the movement of the piston suitably to the displacement of a pointer over a scale. The pressure sensitive element in a Bourdon pressure gauge consists of a tube of flattened section bent into a curve and fixed at one end. When the pressure inside the tube exceeds that outside the flattened section swells laterally so as to become more nearly circular and, since no bending moment is applied to the tube, its longitudinal curvature must become less, i.e., the tube tends to straighten itself. The free end of the tube therefore moves and by suitable gearing may be made to move a pointer over a scale. Another much used device is the pressure capsule, consisting of a shallow cylindrical vessel with a rigid wall and bottom but having a flexible metallic top, often provided with concentric corrugations. Alternatively, the metallic diaphragm may divide a cylindrical vessel into two parts which are connected to the sources of pressure.

Since the moduli of elasticity of most materials depend on temperature it would be necessary in accurate work to measure the temperature of the deformable element of a pressure gauge and correct the reading of the instrument accordingly. In any event, instruments of this class require calibration.

Estimation of pressure or head frequently entails measurement of the change in level of the free surface of a liquid. The simplest device for this is a *pointer* mounted above the surface with a vernier scale. Greater precision may be obtained by bending the pointer into a *hook gauge* so that the point is beneath the surface. The contact may be observed from above the surface, from below the surface through a glass window in the vessel, or by an optical device. In the last case contact occurs when the point and its image appear to meet. The contact of the pointer with the surface may be indicated by closure of an electrical circuit. Variable liquid levels may be measured by inserting an electrode into the liquid and observing the alteration in electrical resistance with level. Electrolytic troubles may be alleviated by using alternating current. Where it is impossible to use mechanical or electrical devices for the measurement of varying levels, reflection of a ray of light from the liquid surface on to a vertical scale may be used, provided that the surface is unruffled.

5.12 **Devices for Making Flow Visible**

We shall begin by considering arrangements for making streamlines visible and we shall suppose that the motion is steady, so the streamlines are identical with the paths of particles and filament lines (see § 2.2). The technique consists in injecting some substance which makes one or more filament lines visible. When the fluid is a liquid the injected substance may be a solution of a dye in the same liquid; in any case, the density of the injected liquid must not differ appreciably from that of the streaming liquid. When the fluid is a gas smoke is injected to make the filament lines visible. To be effective, the smoke must be very dense and it has been found that a satisfactory result is obtained by boiling a medium heavy mineral oil. An apparatus for generating the smoke has been developed at the National Physical Laboratory consisting of a vertical hard glass tube wound externally with an electric heating element and lagged with asbestos. In most cases it is advantageous to arrange for the visible substance to issue from a grid of evenly spaced orifices. Reference has already been made in § 5.7 to the use of reflective particles suspended in the streaming fluid in the determination of the rate and direction of flow.

Information about flow in gases can be obtained optically when there are local changes of density since these are accompanied by changes of the refractive index; the excess of the refractive index above unity is, for a given gas, proportional to the density. This device is chiefly of value for high speed two-dimensional flow. A collimated beam of light is passed through the gas normal to the plane of flow, and in the simplest application of the principle, a "shadow picture" is obtained on a plane white screen. The variations of density give rise to variations of brightness on the screen on account of what may be called a lens effect. This method is particularly successful in displaying the location of sharp changes of density, such as occur in shock waves (see § 9.8). A much refined and very sensitive version of this system is the Schlieren method.†

It is often necessary to know whether the flow over a surface is attached or separated (see § 6.2) and to establish the positions of the boundaries of the regions when both types of flow are present. This can sometimes be done by using smoke or the coloured filament technique to show the flow reversal near the surface when it exists as well as large eddies. Another method, which has been frequently applied in aerodynamic investigations, consists in using small streamers of silk, wool or the like. These may be attached at one end to some surface exposed to the stream (*surface tufts*) or be fixed to the outer ends of pins attached to the surface (*depth tufts*). When the motion is attached each streamer points more or less steadily downstream whereas in separated regions the streamers are agitated and may point upstream. Another important problem is that of locating the transition from laminar to turbulent flow in a boundary layer (see §§ 6.2 and 6.13). Many of the methods used to locate the transition depend on the fact that the rate of diffusion to or from the surface is greatly increased where the flow is turbulent. W. E. Gray was the first to employ this method and he applied a paint to the surface which changed colour when it came in contact with a reactive gas in the stream; where the flow in the boundary layer was laminar the colour change was absent or slight whereas it was intense in the turbulent regions.[1] One of the best variants of this method for laboratory use is the "china clay" method. Here the surface of the body is covered with a white paint whose pigment consists of very fine crystals of the china clay. When this paint is wetted with a liquid whose refractive index is the same as that of the crystals it becomes transparent so that the colour of the surface beneath is visible. The method consists in starting with the paint wetted with a liquid of the required refractive index and right degree of volatility. Since the evaporation is most rapid in the turbulent regions of the boundary layer, these are the first to become dry and therefore opaquely white. Suitable liquids are methyl salicylate (oil of wintergreen) and ethyl salicylate.†

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† See Pankhurst and Holder (*loc. cit.*) or Vol. II of *Modern Developments in Fluid Dynamics. High Speed Flow.* Interferometers have also been used.

5.13 The Hot Wire Anemometer

The hot wire anemometer consists of a fine wire, usually of platinum, which is heated by an electric current passing through it and exposed to the stream of gas whose velocity is to be measured. The rate of heat loss increases with the velocity of the stream and the temperature of the wire falls concurrently. Since the resistance of the wire depends on its temperature, a measurement of the resistance, when other factors are kept constant, can be used to determine the velocity. Obviously the hot wire anemometer is *not* an absolute instrument and requires calibration. For accurate work frequent calibration is, in fact, necessary. In the determination of a steady velocity the wire is mounted at right angles to the stream. The hot wire anemometer can be adapted as a yawmeter by using a pair (or more) of wires which are inclined to one another. For two-dimensional flow we take a pair of equal wires lying in the plane of flow and find the setting for which their resistances are equal; the velocity vector then bisects the angle between them. For three dimensional flow we may use three wires lying along the edges of a tetrahedron. Here again calibration is necessary for accurate work.

The application for which the hot-wire anemometer is pre-eminently suitable is the measurement of rapid fluctuations of velocity, such as occur in turbulent flow. This is a highly specialised technique which cannot be further discussed here.[1]

5.14 The Measurement of Temperature

The measurement of temperature in a streaming fluid presents no difficulty when the Mach number of the flow is very small but this is no longer true for flow at high subsonic, transonic and supersonic speeds. When the thermometric element (thermometer bulb, thermo-junction or resistance element) is at rest *relative to* the fluid it takes up the true or static temperature of the fluid in its vicinity. However, when the element moves relative to the fluid it will be in contact with fluid whose temperature has been modified by compression and fluid friction. The effect is negligible (except perhaps in work of exceptional accuracy) when the Mach number of the relative velocity is very small but, in other circumstances, the effect may be very important. A study of this question shows that the only temperature which can be measured at all accurately is the *stagnation temperature* sometimes called the *total temperature*. This is the temperature which would be reached in the stationary fluid inside a forward-facing cavity (or Pitot tube) if the boundary of the cavity were a non-conductor of heat and the influence of radiation negligible. Thermometers which are designed to measure the stagnation temperature have the thermometric element enclosed in a forward-facing cavity provided with a small leak which allows the fluid to flow slowly past the element. Since the material surrounding the cavity cannot, in practice, be nonconducting, it is necessary to allow a percolation of fluid in order to maintain the true stagnation temperature.† When the stagnation temperature has been measured, it will be possible to calculate the static temperature of the undisturbed fluid from a knowledge of the relative velocity and the properties of the fluid (see § 9.4).

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[1] L. V. King, "Precision Measurements of Air Velocity by Means of the Linear Hot Wire Anemometer", *Phil. Mag.*, **29**, 556 (1915). See also Pankhurst and Holder, *loc. cit.*, where other references are given.

† Further details may be obtained from Pankhurst and Holder, *loc. cit.*

CHAPTER 6

BOUNDARY LAYERS, WAKES AND TURBULENCE

6.1 Introduction

Until the turn of this century there appeared to be an unbridgeable gap between the highly theoretical and apparently unrealistic science of classical hydrodynamics of an inviscid fluid, and the very empirical and essentially practical subject of hydraulics. M. J. Lighthill has described the former as the study of phenomena which could be proved but not observed, and the latter as the study of phenomena which could be observed but not proved. For example, classical hydrodynamics showed that a body in steady motion through an unbounded fluid could experience neither drag nor lift, and the relative motion of fluid and surface could be other than zero at the surface, in manifest disagreement with reality. At the same time, theoretical hydrodynamicists were not content with the unexplained variability of the empirical 'constants' invented by the hydraulic engineers to describe their observations.

It is to the genius of Prandtl[1] that we owe the bridge that finally spanned the gap between these two subjects; the bridge was his theory of the *boundary layer*. With this theory the observations of the theoreticians and empirical engineers were reconciled, and the subject of aerodynamics, drawing heavily from both bodies of knowledge, was able to make the sure and rapid progress of the last half-century.

Most readers will, at some time or other, have observed the 'friction belt' adjacent to a ship on which they have been moving through water, i.e., a narrow belt of relatively slow moving fluid next to the surface of the ship. Such a belt or layer is in fact to be found whenever a body moves in a viscous medium, and the thickness of the layer increases with distance along the surface from the nose of the body and also decreases relative to the size of the body with increase of Reynolds number. At Reynolds numbers† in the range that is of normal aerodynamic interest, viz. 10^4 and above, the boundary layer, as it is called, is very thin and this is one of its most important characteristics. At the surface the fluid is at rest relative to the body, in conformity with the condition of *no-slip* that must hold with a viscous fluid that can be regarded as a continuum.‡ At the outer edge

[1] L. Prandtl, *Proc. III Intern. Math. Congr., Heidelberg* (1904).

† Based on the chord length of the body as typical length.

‡ In the case of gases at very low pressures and densities, such that the mean free path of the molecules is comparable to the dimensions of the body, this condition ceases to apply and slip is said to occur between the gas and the body (see also § 9.14).

of the layer the fluid has practically the full stream velocity relative to the the body; thus, there is a continuous change in velocity through the layer from zero at the surface to the full stream value at the outer edge of layer. It follows that for Reynolds numbers for which the layer is thin the rate of change of velocity with distance from the surface is large in the boundary layer, elsewhere it is small. We have learnt in § 1.4 that the effects of viscosity are made apparent in a fluid in the form of stresses which are directly proportional to rates of change of strain of the fluid, i.e. to the velocity gradients.† If we write x for distance along a surface and y for distance normal to it, and u and v for the corresponding velocity components, it follows that only in the boundary layer are large viscous stresses to be found and they will be shear stresses related to the dominant velocity gradient $\partial u/\partial y$; elsewhere the viscous stresses are small. Further, gradients of the stresses with respect to y are generally large compared with their gradients with respect to x. The essential approximations of boundary layer theory follow, namely, that the viscosity of the fluid need only be considered in the thin boundary layer and only the shear stresses arising from the gradient $\partial u/\partial y$ are of importance, while outside the boundary layer the fluid can be treated as inviscid. It is there that classical hydrodynamic theory applies. Since the boundary layer is thin these approximations lead to some welcome simplification of the general equations of motion of a viscous fluid (the Navier-Stokes equations)‡ and the resulting equations, the so-called boundary layer equations, have been found readily amenable to analysis and have helped us to determine what goes on inside the layer. The only problem that remains in principle is the matching of the solution of the boundary layer equations to the inviscid flow outside the boundary layer. This problem does not normally present any difficulty since the conditions at the outer edge of the boundary layer can generally be identified with those normally assumed as boundary conditions for inviscid flow past the body and the interaction of boundary layer and external flow is then, in effect, ignored. For some problems, however, allowance must be made for this interaction and then the problem of merging the boundary layer and external flows is more difficult.

The boundary layer can, in a sense, be regarded as a layer of 'tired' fluid, which has been retarded by the friction at the surface, having given up some of its momentum relative to the surface to balance the frictional drag of the body. Behind the body the boundary layers from its surfaces merge to form a stream of relatively slow moving fluid which is known as the *wake*. The part of the drag of the body which arises directly from the viscosity of the fluid, the so-called *boundary layer drag*, is thus made manifest by and can be readily related to the defect of momentum of the fluid in the wake relative to the fluid outside it.

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† See also § 11.1.
‡ See Ex. 6 Ch. 11.

In this chapter we shall confine ourselves to a discussion of boundary layers in steady two dimensional incompressible flow; for an extension of the analysis to unsteady or compressible flow reference can be made to Howarth[1] and Schlichting.[2] See also § 3.10.

6.2 General Outline of the Physics of Boundary Layer Flow

Over the forward part of a wing or body (see Fig. 6.2,1) we find that the flow in the boundary layer is *laminar*, that is, it is smooth and proceeds in streamlines roughly parallel to the surface, but at some stage, a transition takes place more or less rapidly to what is known as *turbulent flow*. In turbulent boundary layer flow there is a general mean motion roughly parallel to the surface but in addition there are rapid, random fluctuations in velocity direction and magnitude; the latter can be of the order of a tenth of the main stream velocity.

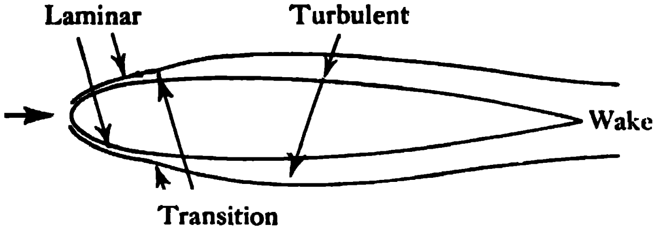

Fig. 6.2,1. Development of boundary layer on streamline shape (not to scale).

These fluctuations provide a powerful mechanism for mixing in the turbulent layer, usually much more powerful than that in the laminar layer where the sole mechanism for mixing is the very much smaller scale random molecular movement. Just as the molecular movements give rise to the viscous shear stresses the fluctuations in the turbulent layer give rise to so-called eddy shear stresses, which can also be related to the mean velocity gradient, $\partial u/\partial y$, but which are very much larger than the viscous stresses except very close to the surface.

There are in consequence important differences between the characteristics of laminar and turbulent boundary layers. Firstly, a typical turbulent boundary layer velocity profile on a flat plate at zero incidence is more nearly uniform over most of its thickness except very close to the surface (see Fig. 6.2,2). Consequently, the gradient $\partial u/\partial y$ at the surface tends to be much greater in the turbulent boundary layer than in the laminar boundary layer. Since τ_w, the frictional stress at the wall, is given by

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{6.2,1}$$

where μ is the coefficient of viscosity, it follows that the frictional stress or drag when the boundary layer is turbulent is usually much greater than when the boundary layer is laminar. Secondly, the more effective mixing that is characteristic of the turbulent layer results in a more ready mixing of the outside stream with the boundary layer and consequently the turbulent boundary layer tends to grow much more rapidly than does the laminar boundary layer. Thus, we find that the former grows approximately as $x^{4/5}$, whilst the latter grows as $x^{1/2}$. For example, at the trailing edge of a flat plate of chord of 2·5 m at a Reynolds number of 10^7 a fully laminar boundary layer would have reached a thickness of about 0·5 cm whilst a fully turbulent boundary layer would be about 4 cm thick.

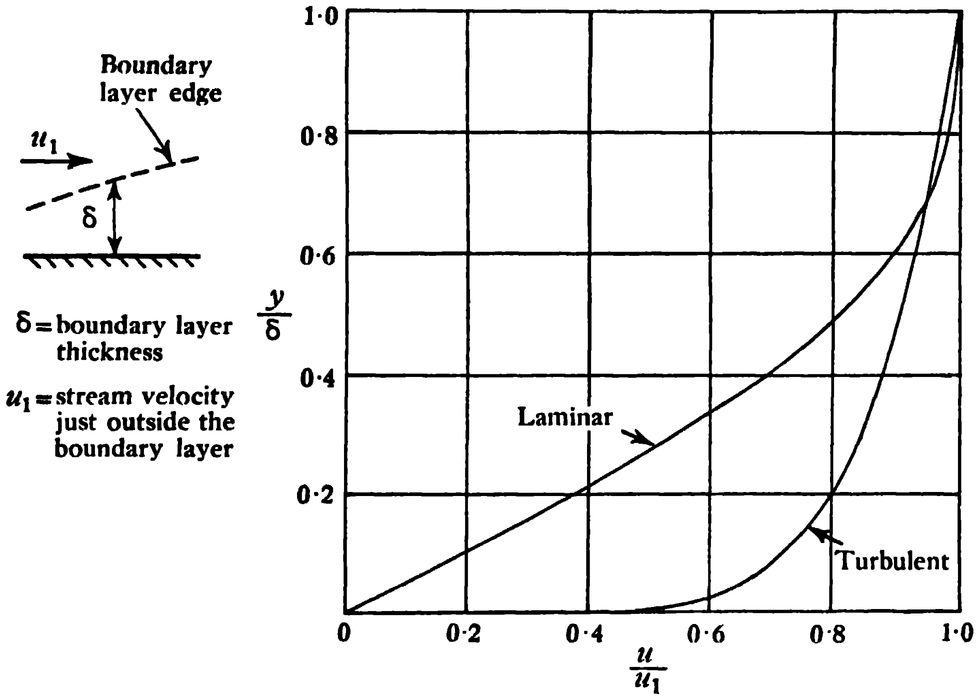

Fig. 6.2,2. Typical velocity distributions on a flat plate at zero incidence for laminar and turbulent boundary layers.

Thirdly, since the turbulent boundary layer has more capacity for mixing with and absorbing energy from the mainstream fluid than the laminar layer, it is much more robust and much less likely to separate from the surface under the influence of an external pressure which increases with distance along the surface (i.e., positive pressure gradient, dp/dx).† The effect of such a pressure gradient on the velocity profile of a boundary layer is illustrated in Fig. 6.2,3. We see that as the pressure rises with distance along the surface a compensating tendency to lose kinetic energy (and therefore velocity) occurs in the boundary layer and this tendency is only partially halted by mixing within the layer. In consequence the form of the velocity profile gets less full and the innermost layers show fractionally a greater reduction in velocity than the outer. Thus $(\partial u/\partial y)$ at the wall is reduced with increase of pressure gradient. With a sufficiently large pressure gradient, depending on the previous history of the boundary layer, a stage may be reached where $(\partial u/\partial y)$ at the wall is zero and fluid adjacent to the surface is then on the point of reversing its direction of flow; the boundary layer is then said to be on the point of *separation*. Subsequently, we may in general expect $(\partial u/\partial y)$ at the wall to become negative downstream of separation and an inner portion of the boundary layer then flows against the stream. The line of zero velocity which, upstream of the separation point, is coincident with the surface branches and one branch leaves the surface at the separation point and below it the fluid flows in the reversed direction. The boundary layer is then said to be separated.

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† As will be seen (§ 6.3) $\partial p/\partial y$ is generally negligible in the boundary layer.

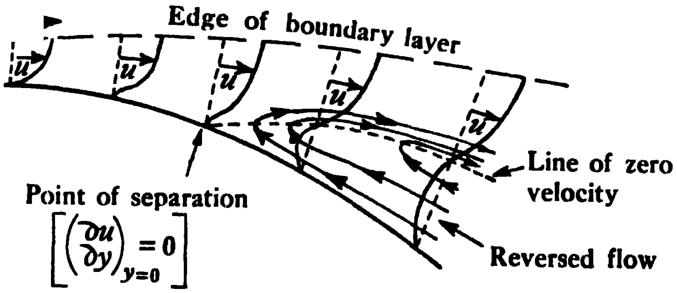

Fig. 6.2,3. Sketch illustrating boundary layer development in the presence of an adverse (positive) pressure gradient strong enough to cause separation.

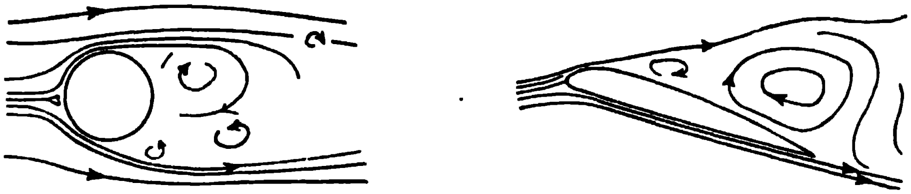

Fig. 6.2,4. Examples of eddying wakes behind bodies from which the flow has separated.

It will be seen from Fig. 6.2,3 that the reversed flow forms part of a large eddy under the outer part of the boundary layer. Such eddies are generally unstable in the sense that they move downstream from the body and new eddies then form. Thus, the wake of a body from which the flow has separated includes large scale eddies (see Fig. 6.2,4). The energy that is carried downstream and ultimately dissipated in such eddies is frequently very large and implies a corresponding large drag of the body which is manifest in the pressure distribution on the body. A body with extensive boundary layer separation and a wake with large scale eddies is referred to as a bluff body; in contrast, one on which the boundary layers remain attached over the whole surface to the rearmost point and then merge smoothly into the wake, without any large scale eddies being formed, is referred to as a streamline body.

It will be apparent that there is an important difference between the nature and magnitude of the drag of a bluff body and those of the drag of a streamline shape. The drag of a bluff body is usually very large, is almost entirely due to the energy wasted in the cast-off eddies, and can be determined with close accuracy from the pressure distribution acting normal to its surface. The drag of a streamline body is relatively small, and is mainly due to the viscous or frictional stresses which act tangential to its surface. Thus, if we hold a square plate normal to a stream, its drag expressed in coefficient form in terms of its area will be about 1·0, but if the same plate is held parallel to the stream its drag coefficient is only about 0·0015 with a fully laminar boundary layer and about 0·007 with a fully turbulent boundary layer at a Reynolds number of 5 × 10^6 (see Figs. 6.2,5, 6.2,6, 6.2,7 and 6.2,8). It is usual to refer to the part of the drag of a body that is given by the resultant in the drag direction of the normal pressures on the body as *pressure drag*, whilst the resultant of the frictional stresses is referred to as *skin-friction drag*. As we shall see later these two components of drag are not entirely independent. In two dimensional flow

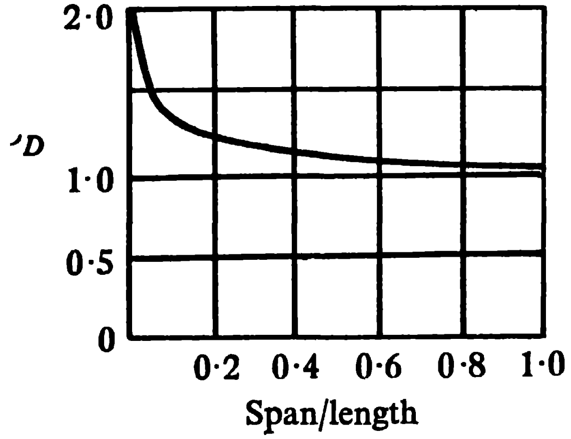

Fig. 6.2,5. Drag coefficient (based on frontal area) for a rectangular flat plate normal to the stream.

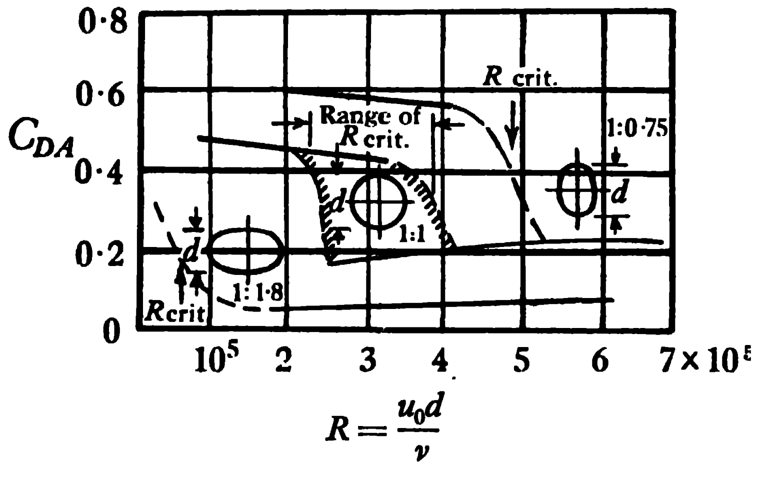

Fig. 6.2,6. Drag coefficient (based on frontal area) for sphere and spheroid.

and at speeds at which shock waves are absent (see Chapter 9) the former (sometimes called *form drag*) is entirely caused by the presence of the boundary layers and the sum of the two is then called the ***boundary layer drag*** or ***profile drag.***

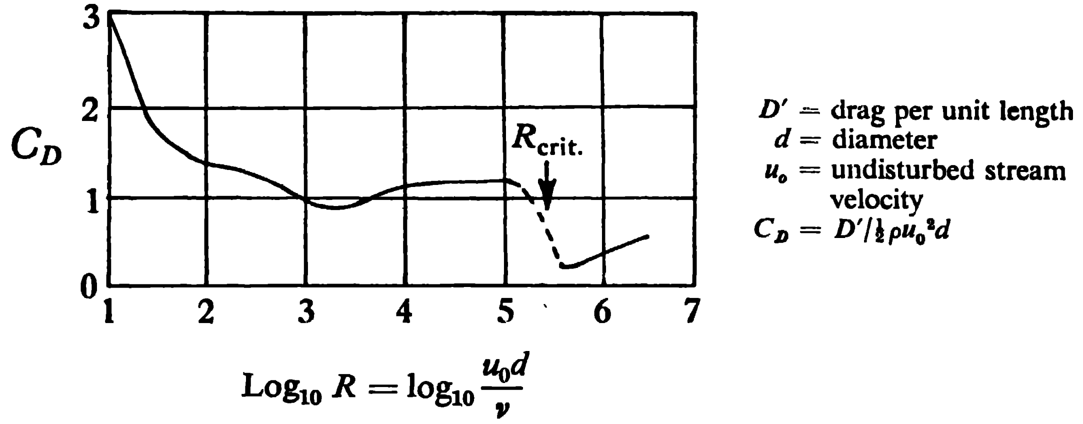

Fig. 6.2,7. Drag coefficient for an infinitely long, smooth, circular cylinder normal to the stream.

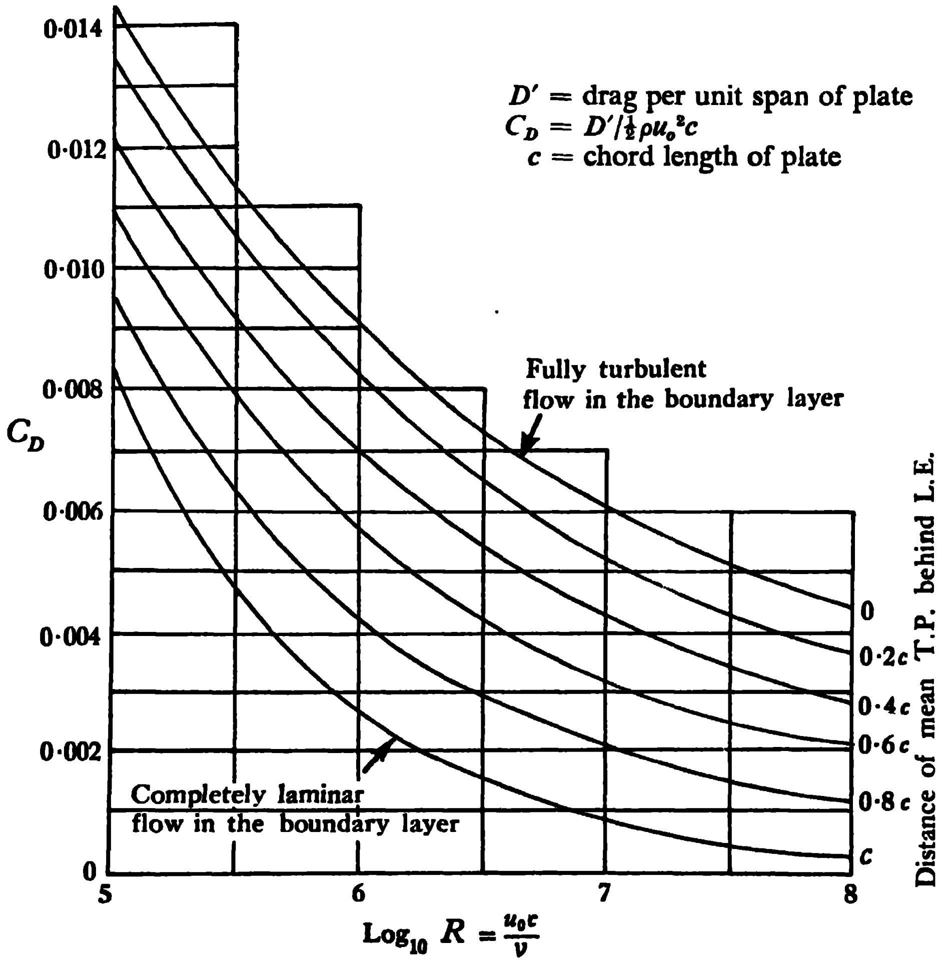

Fig. 6.2,8. Variation of calculated drag coefficient of a smooth flat plate, at zero incidence, with Reynolds Number (R) and position of mean transition point (T.P.).

For most applications we are concerned with the drag of good streamline shapes (e.g. aircraft) and with the problem of keeping their drag as low as possible. This leads us to consider the nature and extent of the transition process from laminar to turbulent flow and the factors that can influence it. At low Reynolds numbers (i.e. less than about 10^6 in terms of body length) transition occurs over a distance along the surface of significant extent. At the upstream end of the transition region the boundary layer is fully laminar whilst at the downstream end it is fully turbulent; in between the change in the characteristics from those of laminar to those of turbulent flow appears to be a gradual one. With increase of Reynolds number, however, the transition region becomes smaller, and for Reynolds numbers of the order of 2×10^6 the region is sufficiently small relative to body size to be referred to as a point—the *transition point.*

The precise physical nature of the process of transition and its causes are not at present fully understood, but something is known about the factors that can hasten or delay it. Thus, with an external pressure that falls with distance along the surface (i.e., an external negative pressure gradient) transition occurs much less readily than with an external pressure gradient that is positive. The reasons for this will be made clearer when we come to discuss the stability of the boundary layer. Roughness or imperfections of the surface tend to provoke early transition as does also a high degree of turbulence in the main stream outside the boundary layer. A most important factor, however, is Reynolds number. If we measure the Reynolds number in terms of the distance along the surface, then for values less than about 10^5, the laminar boundary layer is very stable and it is difficult to provoke transition. However, with increase of Reynolds number the stability of the laminar boundary layer decreases, transition is more and more easily provoked, and with a Reynolds number in terms of distance along the surface greater than about 2×10^6 considerable care must be taken in keeping the surface smooth and the external disturbances small if transition is not to be provoked. On an aeroplane wing at the Reynolds numbers characteristic of full scale flight it is generally regarded as unlikely even under the best conditions that transition will occur naturally much aft of the point on the surface where the minimum pressure is attained, since aft of that point the external pressure gradient is unfavourable (positive) for laminar flow. With some form of boundary layer control (see § 6.24), however, more extensive regions of laminar flow can be attained.

The effect of surface roughness in hastening transition has been noted in the preceding paragraph. The little roughness elements tend to act like bluff bodies and eddies are cast off them which disturb the laminar boundary layer and induce transition to turbulent flow and in consequence the drag is usually increased. Eddies are likewise cast off the roughness elements when the boundary layer is turbulent and then they directly increase the local drag. The larger the roughness elements the larger the drag increment that they cause. Since the drag of an irregular shaped bluff body tends to be much less dependent on Reynolds number than is the drag of a streamline shape, the drag of a rough surface tends to become less dependent on Reynolds number as the drag contribution of the roughnesses increases relative to the smooth surface drag. But with increase of Reynolds number the boundary layer gets thinner and the ratio of mean roughness height to boundary layer thickness increases, and thus proportionately more and more of the drag is caused by the roughnesses. Hence, we find that for a surface with a given roughness the drag coefficient tends to become constant at high enough Reynolds numbers. Fig. 6.2,9 illustrates these points.

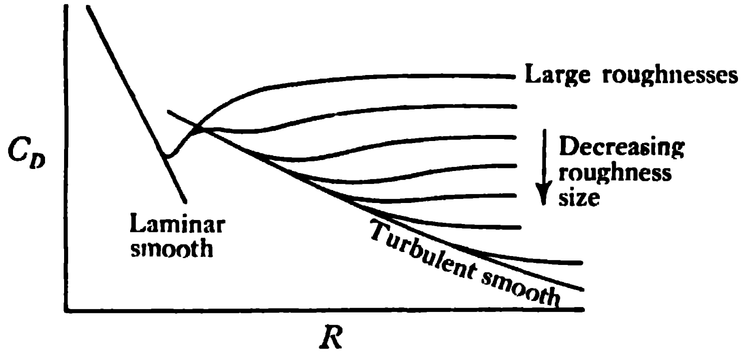

Fig. 6.2,9. Sketch illustrating variation of drag of a flat plate at zero incidence with roughness height.

It is of interest to note, however, that in certain cases it is possible to decrease the drag of a body by causing transition to occur earlier on it, e.g. by surface roughness or by the use of a trip wire. Such cases only occur with bluff bodies and at Reynolds numbers sufficiently low for the boundary layer to separate early in the laminar state; the wake is then wide and the pressure drag high. If transition is then provoked before separation, the boundary layer, when it does separate, is turbulent and hence separates much further aft from the body than does the less robust laminar boundary layer. The wake is in consequence narrower and the pressure drag is reduced, so that there is a marked net reduction in the total drag (see Fig. 6.2,10). Such a process can be observed in the case of a smooth bluff body like a sphere or cylinder as the Reynolds number is increased. At low Reynolds numbers the boundary layer is wholly laminar and separates early from the surface causing a large eddying wake; the drag coefficient is in consequence high. With increase of Reynolds number a critical value is eventually reached at which the boundary layer goes turbulent before separation and separation occurs later. The wake at that stage becomes reduced in size as does the drag coefficient. The Reynolds number at which this occurs is called the critical Reynolds number. These points are illustrated in Fig. 6.2,6, 6.2,7 and 6.2,10.

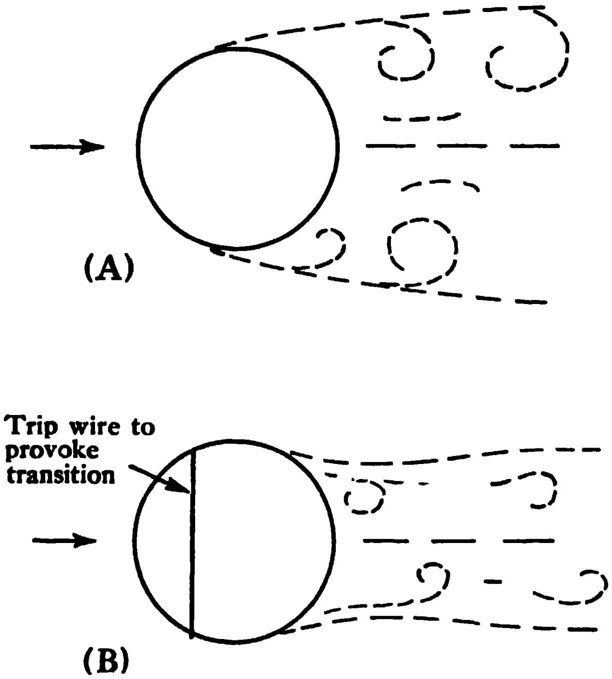

Fig. 6.2,10. (A) Flow past a sphere with laminar boundary layer separation. $C_{DA} \fallingdotseq 0.45$.
(B) Flow past a sphere with turbulent boundary layer separation. $C_{DA} \fallingdotseq 0.2$.
C_{DA} is drag coefficient based on frontal area.

6.3 Laminar Boundary Layer Equations for Steady Incompressible Two-Dimensional Flow

We have learnt (§§ 2.3 and 3.5) that the equations of continuity and motion for the steady flow of an incompressible, inviscid fluid in two-dimensions without body forces are:—

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{6.3,1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{6.3,2}$$

and

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{6.3,3}$$

where x and y are rectangular cartesian coordinates and u and v are the corresponding velocity components.

If we now consider a viscous fluid, then clearly the equation of continuity will be unchanged, but the viscous stresses will introduce additional terms into the equations of motion.

Let us confine our attention to the boundary layer on a flat plate with x measured along the plate and y normal to it. By the argument of the preceding section the only viscous stress that we need consider in (6.3,2) is the shear stress in the fluid acting parallel to the plate and denoted here by τ.

If we consider the small element of fluid of area $\delta x\, \delta y$ in the xy plane and unit length normal to the xy plane illustrated in Fig. 6.3,1 we see that acting on the lower surface there is a force, due to the shear stress, equal to $-\tau\,\delta x$ in the x direction. Likewise on the upper surface there is a force in the x direction

$$(\tau + \delta\tau)\,\delta x = [\tau + (\partial\tau/\partial y)\,\delta y]\,\delta x \text{ plus terms of order } \delta y^2\,\delta x$$

due to the shear stress there.

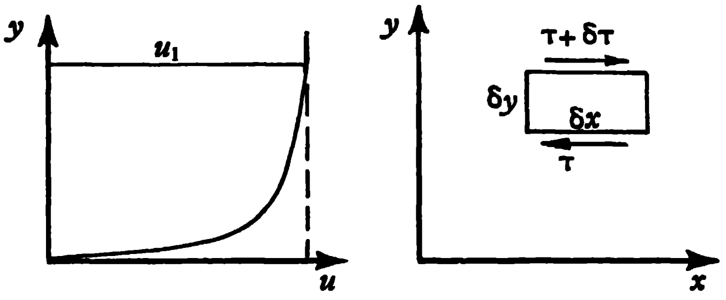

Fig. 6.3,1.

The net force in the x direction acting on the element is therefore

$$(\partial\tau/\partial y)\,\delta x\,\delta y \text{ plus terms of order } \delta y^2\,\delta x$$

and hence as δx and δy tend to zero the force per unit mass in the x direction arising from the viscous stresses remains

$$\frac{1}{\rho}\frac{\partial\tau}{\partial y}.$$

Since the left-hand side of equation 6.3,2 represents the acceleration in the x direction of a particle of fluid and the right-hand side represents the component of the force per unit mass in that direction due to the static pressure, it follows that we must now add the above term due to the viscous stresses to complete the equation of motion in the x direction in the boundary layer. This equation is then

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\frac{\partial\tau}{\partial y}. \qquad (6.3,4)$$

The second equation of motion (6.3,3) is, by the preceding argument, unchanged. More accurately, a complete analysis in which all the viscous stresses are taken into account demonstrates that a term is introduced which at the most is of the same order of magnitude as the inertia terms on the left-hand side of that equation.† Let us therefore examine the order of magnitude of these terms.

If U_1 is a representative velocity of the undisturbed flow, and c is a representative length for the plate (e.g. its chord) then we can say that

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† This viscous contribution derives from the terms (see § 11.1) $\frac{1}{\rho}\left[\frac{\partial}{\partial y}(p_{yy}) + \frac{\partial}{\partial x}(p_{xy})\right]$ and becomes $\nu\partial^2 v/\partial y^2$ for a laminar boundary layer.

$\partial u/\partial x$ is at most of the order of magnitude of U_1/c, by which we mean that the ratio of $\partial u/\partial x$ to U_1/c is unlikely to be large compared with unity. We represent this statement symbolically by writing

$$\partial u/\partial x = O(U_1/c),$$

where the symbol 'O' is shorthand for 'of the order of'.

From 6.3,1 it follows that

$$\partial v/\partial y = O(U_1/c),$$

and since $v = 0$ at $y = 0$, we see that

$$v = O[(U_1/c)\,\delta]$$

where δ represents the boundary layer thickness which we know to be small compared with c.

Thus
$$v/U_1 = O(\delta/c)$$

and hence v/U_1 is small compared with unity.

If we now examine the order of magnitude of the two terms on the left-hand side of equation 6.3,3 we see that they are both of order $U_1^2\,\delta/c^2$. As already remarked, the viscous term that a complete analysis demonstrates should strictly be included is at the most of the same order of magnitude. It follows that the pressure term in equation 6.3,3 must also be of this order of magnitude and hence

$$\frac{\partial p}{\partial y} = O\left[\rho U_1^2\frac{\delta}{c^2}\right].$$

Therefore the total variation of p throughout the boundary layer, Δp, must be such that

$$\frac{\Delta p}{\rho U_1^2} = O\left[\frac{\delta^2}{c^2}\right]$$

and hence this variation can be regarded as negligible. The second equation of motion therefore leads us to conclude that the pressure p can be taken as constant across the boundary layer. Hence we can replace the partial derivative, $\partial p/\partial x$, in equation (6.3,4) by the ordinary derivative dp/dx.

In the laminar boundary layer we have (see § 1.4 and 3.10)

$$\tau = \mu\frac{\partial u}{\partial y},$$

and hence the equation of motion (6.3,4) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (6.3,5)$$

where ν is the kinematic viscosity (μ/ρ).

The order of magnitude of the terms on the left-hand side as well as of the pressure term is readily seen to be U_1^2/c. This must therefore be the order of magnitude of the viscous term. But the order of magnitude of this term is also $\nu\ U_1/\delta^2$. It follows that

$$\nu U_1/\delta^2 = O(U_1^2/c)$$

or

$$\begin{aligned}\delta/c &= O(U_1 c/\nu)^{-\frac{1}{2}}\\ &= O(R^{-\frac{1}{2}})\end{aligned} \tag{6.3,6}$$

where R is the Reynolds number, U_1c/ν. This result provides an *a posteriori* justification for the basic assumption that the boundary layer can be regarded as thin at high Reynolds numbers. It also illustrates a general truth of the laminar boundary layer, namely, that at any station its thickness (or any length that is characteristic of its thickness) is proportional to $R^{-\frac{1}{2}}$. Further since

$$v/U_1 = O(\delta/c)$$

it follows that

$$v/U_1 = O(R^{-\frac{1}{2}}). \tag{6.3,7}$$

To summarise, then, the equations that govern the flow in the steady, two dimensional laminar boundary layer on a flat plate are:—

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6.3,1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{6.3,5}$$

and the pressure may be taken as constant across the layer.

In the case of flow past a curved surface we can use orthogonal curvilinear coordinates with x measured parallel to the surface and y normal to it.† We then find that in the boundary layer the above equations (6.3,1) and (6.3,5) apply unchanged provided the curvature κ of the surface is sufficiently small for $\kappa\delta$ to be small compared with unity, and further $\delta^2 d\kappa/dx$ must be small compared with unity. The second equation of motion now involves the centrifugal force term κu^2 and hence the pressure change across the boundary layer is such that

$$\Delta p/\rho U_1^2 = O(\kappa\delta).$$

Therefore if $\kappa\delta$ is small compared with unity we may still neglect the pressure variation across the boundary layer.

For axi-symmetric flow past a body of revolution we can similarly take x measured parallel to the surface in a meridian plane and y normal to the surface. We then find that again equation (6.3,5) applies, and the pressure can be taken as constant across the boundary layer provided $\kappa\delta$ and $\delta^2\, d\kappa/dx$ are small compared with unity. Here κ is the curvature of the meridian profile of the body. The equation of continuity, however, now becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{u}{r}\frac{\partial r}{\partial x}+\frac{v}{r}\frac{\partial r}{\partial y}=0,$$

where r is the normal distance to the axis from the point (x, y), but in the boundary layer the last term is of a smaller order of magnitude than the other terms and can be neglected. Hence the equation becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{u}{r}\frac{\partial r}{\partial x}=0. \qquad (6.3,8)$$

If δ/r is small, as is usually the case except over the extreme rear of a streamlined body, then r in the last term can be replaced by r_0, the radius of the body cross-section.

6.4 Boundary Conditions. Boundary Layer Thicknesses

The boundary condition for the boundary layer at the surface is simply the no-slip condition, viz.:—

$$u=v=0, \quad \text{when} \quad y=0. \qquad (6.4,1)$$

At the outer edge of the boundary layer we have the condition that the flow must merge smoothly with the external flow. However, we here meet the difficulty of defining the outer edge of the boundary layer. The thickness δ as used in the last paragraph was in no sense exactly defined but can be regarded as the limit of the region outside which the velocity gradient $\partial u/\partial y$ is sufficiently small compared with U_1/c for the viscous forces there to be negligibly small. However, what we are prepared to regard as negligibly small is to some extent a matter of choice. Any reasonable choice leads us to the conclusion that δ is small compared with the distance from the leading edge. In practice there is no difficulty in measuring the boundary layer thickness with a pitot probe traversing across the layer; the probe readily defines the point at which the total pressure begins to fall from the free stream value as the probe enters the layer. The accuracy of our measuring equipment then determines what order of smallness we are prepared to neglect, but we again find that even with exceptionally accurate equipment the measured boundary layer thickness is small compared with the distance from the leading edge. On the other hand, in strict mathematical terms, the boundary layer thickness is infinite, since the process of merging with the outside flow can never be said to be exactly complete at any finite distance from the wall, even though at all but small distances the difference from external flow conditions is for all practical purposes negligible. We must also note that what we assume to be the external flow is generally taken to be that of potential inviscid flow past the body; sometimes this is modified to allow for the usually small effects due to the boundary layer itself.

Thus the strict mathematical formulation of the external boundary condition is

$$u = u_1(x), \quad \text{when} \quad y \to \infty, \tag{6.4,2}$$

where $u_1(x)$ is taken to be known. Insofar as $u_1(x)$ is finite the conditions of smooth merging of boundary layer and external flow viz.:—

$$\partial^n u/\partial y^n = \partial^n u_1/\partial y^n = 0, \quad \text{for} \quad n = 1, 2, \text{etc.}, \tag{6.4,3}$$

are automatically satisfied.

If, however, it is convenient to define a finite boundary layer thickness δ, say, then both boundary conditions 6.4,2 and 6.4,3 must be considered to hold at $y = \delta$.

From the equation of motion (6.3,5) we see that at the surface $y = 0$ (denoted by suffix w)

$$\nu(\partial^2 u/\partial y^2)_w = \frac{1}{\rho}\frac{dp}{dx}, \tag{6.4,4}$$

whilst outside the boundary layer, by Bernoulli's theorem,

$$u_1 \frac{du_1}{dx} = -\frac{1}{\rho}\frac{dp}{dx}. \tag{6.4,5}$$

Hence, it follows that

$$\nu[\partial^2 u/\partial y^2]_w = -u_1(du_1/dx). \tag{6.4,6}$$

In contrast to the actual thickness of the boundary layer there are a number of other lengths which can be precisely defined and which can be regarded as measures of its thickness. They are the so-called displacement thickness, here denoted by δ^*, momentum thickness, θ, and energy thickness δ_E. They play important parts in boundary layer analysis and their definitions are as follows:—

Displacement thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_1}\right) dy. \tag{6.4,7}$$

It will be apparent from the above definition that $\rho u_1 \delta^*$ represents the total defect in rate of mass flow in the boundary layer as compared with the rate of mass flow in the absence of the boundary layer.

This defect in rate of mass flow must be manifest in an out flow into the external stream from the boundary layer. The fluid that is in fact pushed out into the main stream would, in the absence of the boundary layer, fill a layer of thickness δ^*; therefore the effect of the boundary layer on the external flow can be regarded as equivalent to displacing the surface outwards by the distance δ^*. Hence the name. A fuller discussion of this point is given in § 6.21,1.

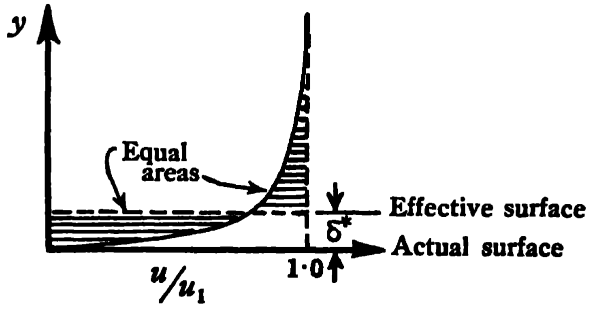

Fig. 6.4,1.

Momentum thickness

$$\theta = \int_0^\infty \frac{u}{u_1}\left(1 - \frac{u}{u_1}\right) dy. \tag{6.4,8}$$

Here we see that $\rho u_1^2 \theta$ represents the defect in rate of transport of momentum in the boundary layer as compared with the rate of transport of momentum in the absence of the boundary layer. We may expect that the momentum thickness distribution with distance x along a surface will be closely linked with the pressure and skin friction distributions.

Energy thickness

$$\delta_E = \int_0^\infty \frac{u}{u_1}\left[1 - \left(\frac{u}{u_1}\right)^2\right] dy. \tag{6.4,9}$$

Here, likewise, the quantity $\rho u_1^3 \delta_E/2$ represents the defect in rate of transport of kinetic energy in the boundary layer when compared with the rate of transport in the absence of the boundary layer. This quantity is closely related to the rate of dissipation of energy due to viscous effects in the boundary layer.

6.5 Flow in Wakes and Jets

Wakes and jets can likewise be regarded as regions where large gradients of the velocity component u with respect to transverse distance y can occur, and in consequence the shear stress component associated with $\partial u/\partial y$ is important and the other stress components (except of course the static pressure) can be neglected. Further, at Reynolds numbers of normal aerodynamic interest the transverse dimension of a wake at any section can be taken to be small compared with the distance from the leading edge of the body from which the wake trails. With a jet emerging from a nozzle mixing regions develop at the edge of the jet which grow progressively with distance downstream both inwards towards the jet axis and outwards into the external fluid. Across these mixing regions the flow velocity changes smoothly from that of the jet to that of the external fluid, and in consequence it is in these regions that large values of the velocity gradient $\partial u/\partial y$ can occur and the shear stress τ can be significant. Eventually the mixing regions from opposite edges of the jet meet at the jet axis; downstream of that point the whole of the jet becomes a mixing region and the velocity u on the axis falls steadily to the external stream value whilst the jet progressively widens with distance x. The developments of a wake and jet are illustrated in Fig. 6.5,1.

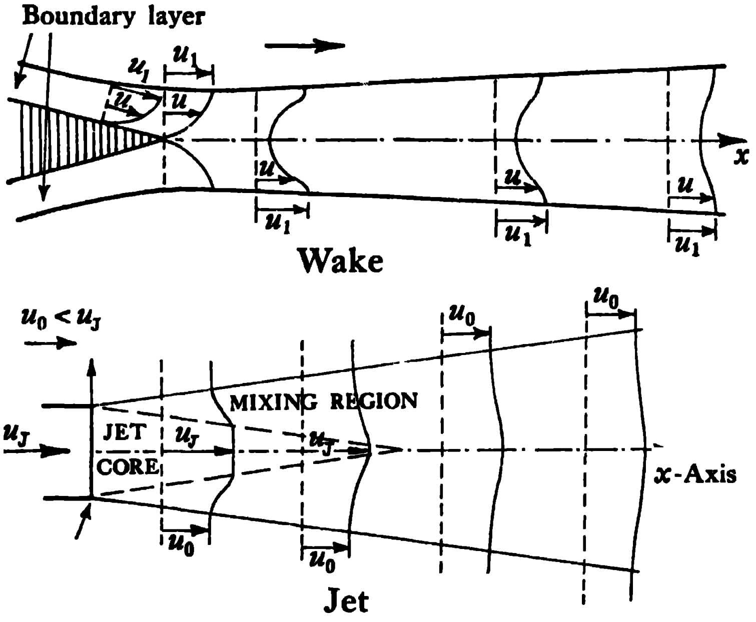

Fig. 6.5,1. Illustrations (not to scale) of flow in wake and jet.

Again, the widths of the mixing region or of the jet downstream of the point where the mixing regions merge (the tip of the so-called jet core) is small compared with the distance from the nozzle exit.

Thus, it is not surprising to find that the boundary layer equations of continuity and motion of § 6.3 apply unchanged to the flow in wakes and to the flow in the mixing regions of a jet including the flow downstream of the jet core.

The boundary conditions will differ, however, for these cases from those applicable to boundary layer flow. If in the wake we take the downstream streamline from the trailing edge as defining the x direction and the y direction is normal to it, then instead of the surface condition of no-slip of boundary layer flow we have the condition:

$$\partial u/\partial y = 0 \quad \text{when} \quad y = 0. \tag{6.5,1}$$

The conditions at the outer edge of the wake are the same as those for the outer edge of the boundary layer, and across any section of the wake away from the region close to the body the pressure can be taken as constant and equal to the pressure of the external flow there.

The same boundary conditions apply to the jet flow downstream of the jet core, where x is now taken along the jet axis or centre line. For the mixing regions upstream of the core tip the inner and outer boundary conditions are the same as the outer conditions for a boundary layer except that for the inner edge the flow must merge smoothly into a uniform flow of velocity equal to u_J, the jet velocity at the nozzle exit, and at the outer edge the flow merges smoothly into the external flow of velocity u_0. Again, across any section of the jet the pressure is constant and equal to the pressure of the external flow.

6.6 The Momentum Integral Equation of the Boundary Layer (Two-Dimensional Steady Flow)

The momentum integral equation of the boundary layer, first derived by v. Kármán, expresses the relation that must exist between the overall rate of flux of momentum across a section of the boundary layer, the frictional stress at the surface and the pressure gradient. It can be derived from an integration of the equation of motion of the boundary layer (equation 6.3,4) with respect to y from the surface to the outer edge of the boundary layer, but the following *ab initio* derivation is perhaps more instructive (for the momentum equations of a fluid see § 3.6).

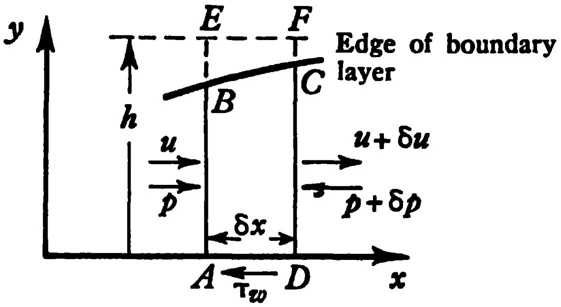

Fig. 6.6,1.

Consider an element of the boundary layer, $ABCD$, of unit length normal to the xy plane; the element is defined by two sections AB and CD normal to the surface AD, where AD is a small distance δx and BC is the edge of the boundary layer. We suppose AB and DC continued to E and F where $AE = DF = h$, say, and h is slightly greater than the local boundary layer thickness. As illustrated in Fig. 6.6,1, the surface is flat, but the following argument can be readily generalized to slightly curved surfaces and the same resulting formulae apply.

The rate of mass flow across AE into $AEFD$

$$= \int_0^h \rho u \, dy,$$

and the corresponding rate of mass flow across DF out of $AEFD$

$$= \int_0^h \rho u \, dy + \frac{d}{dx}\left[\int_0^h \rho u \, dy\right]\delta x + \text{smaller terms of order } \delta$$

Hence the net rate of mass flow across DF and AE, out of $AEFD$

$$= \frac{d}{dx}\left[\int_0^h \rho u \, dy\right]\delta x + \text{terms of order } \delta x^2.$$

If v_h denotes the mean value of v at height h above the surface over the length EF then the rate of mass flow out of $AEFD$ across EF

$$= \rho v_h\,\delta x.$$

But continuity of mass requires that there be no net rate of change of mass inside $AEFD$ and hence

$$\rho v_h = -\frac{d}{dx}\left[\int_0^h \rho u\,dy\right] + \text{terms of order } \delta x. \qquad (6.6,1)$$

Let us now consider, for the fluid in $AEFD$, the balance of rate of change of momentum in the x direction and components of applied forces in that direction.

The rate of transport of momentum in the x direction across DF minus the rate of transport of momentum in the x direction across AE is

$$\frac{d}{dx}\left[\int_0^h \rho u^2\,dy\right]\delta x + \text{terms of order } \delta x^2.$$

The rate of transport of momentum in the x direction across EF out of $AEFD$ is

$$\rho v_h u_1\,\delta x = -u_1\frac{d}{dx}\left[\int_0^h \rho u\,dy\right]\delta x + \text{terms of order } \delta x^2,$$

from equation 6.6,1.

The force in the x direction on $AEFD$ due to the pressure

$$= -h\,\delta p = -h\frac{dp}{dx}\,\delta x + \text{terms of order } \delta x^2,$$

while that due to the frictional stress at the wall

$$= -\tau_w\,\delta x,$$

where τ_w is the mean frictional stress over the length AD.

Equating the net increase in rate of flow of momentum to the sum of the forces acting in the x direction we have

$$\frac{d}{dx}\left[\int_0^h \rho u^2\,dy\right]\delta x - u_1\frac{d}{dx}\left[\int_0^h \rho u\,dy\right]\delta x = -h\frac{dp}{dx}\,\delta x - \tau_w\,\delta x$$
$$+ \text{residual terms of order } \delta x^2.$$

If we now divide this equation by δx, then the residual terms on the right-hand side become of order δx, and if we then let δx tend to zero these terms must likewise tend to zero leaving us with the equation

$$\frac{d}{dx}\left[\int_0^h \rho u^2\,dy\right] - u_1\frac{d}{dx}\left[\int_0^h \rho u\,dy\right] = -h\frac{dp}{dx} - \tau_w. \qquad (6.6,2)$$

This is the *momentum integral equation* but it is most usefully expressed in terms of the momentum and displacement thicknesses.

Thus, if we make use of equation 6.4,5 we can write 6.6,2 in the form

$$\frac{d}{dx}\left[\int_0^h \rho u(u-u_1)\,dy\right]+\frac{du_1}{dx}\left[\int_0^h \rho u\,dy\right]=h\rho u_1\frac{du_1}{dx}-\tau_w$$

or

$$\frac{d}{dx}\left[\int_0^h \rho u(u_1-u)\,dy\right]+\frac{du_1}{dx}\left[\int_0^h \rho(u_1-u)\,dy\right]=\tau_w.$$

If we now recall the definitions of the displacement thickness, δ^*, and the momentum thickness, θ (equations 6.4,7 and 6.4,8), we see that this equation can be written

$$\frac{d}{dx}(\rho u_1^2\theta)+\frac{du_1}{dx}\rho u_1\delta^*=\tau_w. \qquad (6.6,3)$$

The ratio δ^*/θ is commonly denoted by H and equation (6.6,3) can then be written

$$\rho u_1^2\frac{d\theta}{dx}+2\rho u_1\frac{du_1}{dx}\theta+\rho u_1\frac{du_1}{dx}H\theta=\tau_w$$

or

$$\frac{d\theta}{dx}+\frac{1}{u_1}\frac{du_1}{dx}\theta(H+2)=\frac{\tau_w}{\rho u_1^2}. \qquad (6.6,4)$$

This is the form in which *the momentum integral equation* is usually expressed. Its importance lies in the fact that it forms a ready basis for approximate methods of solving boundary layer problems. In such methods the aim is to satisfy this important integral relation in combination with more or less plausible assumptions approximating to the structure of the boundary layer. For problems in which we are interested in determining with adequate accuracy overall characteristics, e.g., skin friction or momentum thickness, rather than details of the flow, these methods can be extremely valuable and effective.

As derived above the equation is applicable to both laminar and turbulent flow.

As a simple example of its application consider the laminar boundary layer on a flat plate at zero incidence. In this case $du_1/dx=0$, and the momentum integral equation is simply

$$\frac{d\theta}{dx}=\tau_w/\rho u_1^2. \qquad (6.6,5)$$

The crudest and simplest assumption about the velocity distribution across a section of the boundary layer is that it is linear with y up to the edge of the boundary layer. If we accept this assumption then we can only satisfy the two boundary conditions $y = 0$, $u = 0$; and $y = \delta$, $u = u_1$. This assumption therefore leads to

$$\frac{u}{u_1} = \frac{y}{\delta}, \quad \text{for} \quad y \leqslant \delta. \qquad (6.6,6)$$

Hence
$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_w = \mu\frac{u_1}{\delta},$$

$$\delta^* = \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy = \frac{\delta}{2},$$

and

$$\theta = \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \frac{\delta}{6}.$$

Equation 6.6,5 then yields

$$\frac{1}{6}\frac{d\delta}{dx} = \frac{\nu}{u_1\delta}$$

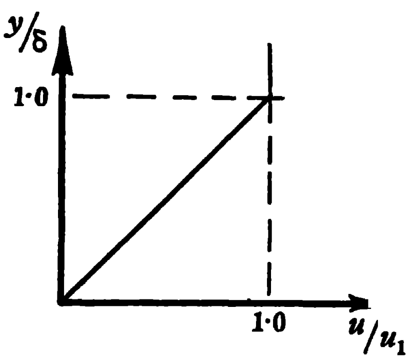

Fig. 6.6,2. Assumed linear velocity profile.

or

$$\frac{d}{dx}(\delta^2) = 12\frac{\nu}{u_1}.$$

Therefore $\delta^2 = 2\nu x/u_1$, since $\delta = 0$ when $x = 0$

or
$$\delta/x = 3{\cdot}464/\sqrt{R_x}, \qquad (6.6,7)$$

where
$$R_x = u_1 x/\nu.$$

Therefore, the local skin friction coefficient is

$$c_f = \frac{2\tau_w}{\rho u_1^2} = \frac{2\nu}{u_1\delta}$$
$$= 0{\cdot}577/\sqrt{R_x}, \qquad (6.6,8)$$

and the overall skin friction coefficient for one surface of a plate of chord c is

$$C_F = \frac{1}{c}\int_0^c c_f\,dx = 1{\cdot}155/\sqrt{R} \qquad (6.6,9)$$

where R is the Reynolds number of the plate $= u_1 c/\nu$. The momentum thickness is given by

$$\theta/x = \delta/6x = 0{\cdot}577/\sqrt{R_x} \qquad (6.6,10)$$

and the displacement thickness is given by

$$\delta^*/x = \delta/2x = 1{\cdot}732/\sqrt{R_x}. \qquad (6.6,11)$$

In spite of the crudity of the assumption that the velocity distribution is linear, these results are fairly close to the results given by an exact analysis of the boundary layer equations (see § 6.7). Thus the exact solution gives

$$c_f = \theta/x = 0{\cdot}664/\sqrt{R_x},$$

$$C_F = 1{\cdot}328/\sqrt{R},$$

and

$$\delta^*/x = 1{\cdot}729/\sqrt{R_x}.$$

Better agreement with the exact solution can of course be obtained by more plausible assumptions for the velocity distribution satisfying more of the boundary conditions, e.g. higher order polynomials. The following table lists a few assumed velocity distributions with the resulting values for the main boundary layer characteristics obtained by solving the momentum equation as above.

Assumed form for u/u_1	$c_f\sqrt{R_x}$	$C_F\sqrt{R}$	$\frac{\theta}{x}\sqrt{R_x}$	$\frac{\delta^*}{x}\sqrt{R_x}$	$\frac{\delta}{x}\sqrt{R_x}$	H
y/δ	0·577	1·155	0·577	1·732	3·464	3·00
$\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$	0·646	1·292	0·646	1·740	4·640	2·70
$2\frac{y}{\delta} - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$	0·686	1·372	0·686	1·752	5·840	2·55
$\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$	0·654	1·310	0·654	1·741	4·789	2·66
Exact solution	0·664	1·328	0·664	1·721	—	2·59

The reader is advised as an exercise to investigate the number and nature of the boundary conditions satisfied by the assumed velocity distributions listed above and to derive the values of the quantities listed.

It will be seen that the cubic form of velocity distribution gives much better agreement with the exact solution than does the linear form, but the answers given by the quartic form can hardly be described as better than those given by the cubic form. They all agree in providing correctly for the laminar boundary layer the form of the relations between c_f and $\sqrt{R_x}$, c_F and $\sqrt{R}$ and between the boundary layer thicknesses and $\sqrt{R_x}$. We note that the thicknesses all increase with x as $x^{\frac{1}{2}}$.†

We may also note that although the exact solution does not yield any finite value for the boundary layer thickness, since it satisfies the outer boundary condition at $y = \infty$, nevertheless for $u/u_1 = 0{\cdot}999$ the corresponding value of $(y/x)\sqrt{R_x}$ is about 6·0, see § 6.7. These values show that for all practical purposes the resulting boundary layer thicknesses given by the assumed velocity distributions other than the linear one are of the right order.

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† It should be noted that the form of these relations is a consequence of the assumption that u/u_1 is a function of y/δ only. This assumption is valid for the laminar boundary layer but for the turbulent boundary layer it is not applicable since the eddy shear stress is a dominant parameter of the flow except very close to the surface. In consequence the corresponding relations are different in form (see § 6.14–6.17).

6.7 The Exact Solution for the Laminar Boundary Layer on a Flat Plate at Zero Incidence (Blasius Solution)

With the external pressure constant the boundary layer equations become

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \end{aligned}\right\} \tag{6.7,1}$$

with the boundary conditions $y = 0$, $u = v = 0$; and $y = \infty$, $u = u_1$.

Since the flow has no characteristic absolute length, we may expect that the velocity distributions across the boundary layer for different values of x will be similar. By this we mean that the distribution for one value of x can be obtained from that for another value of x by a simple change of scale of the y ordinates, the scaling factor being determined by the corresponding x coordinates.

This condition of similarity can be expressed in the form

$$\frac{u}{u_1} = f(\zeta), \quad \text{when} \quad \zeta = y/\phi(x), \tag{6.7,2}$$

where f and ϕ are functions of ζ and x, respectively, and ζ is non-dimensional.

This implies that equations (6.7,1) which are partial differential equations in the two independent variables x and y, can be transformed into an ordinary differential equation in terms of ζ, the solution of which should give the function $f(\zeta)$.

To guide us in the correct choice of ζ we have the results of other investigations, such as those of § 6.6, to show that the linear dimension of the boundary layer in the y direction varies as $x/\sqrt{R_x}$ or as $(u_1/\nu x)^{-\frac{1}{2}}$ and hence we consider

$$\zeta = \frac{y}{2}\left(\frac{u_1}{\nu x}\right)^{\frac{1}{2}}. \tag{6.7,3}$$

The factor 1/2 is introduced for convenience in the subsequent analysis. Further, the equation of continuity implies the existence of a stream function ψ (see § 2.4), such that

$$u = -\partial\psi/\partial y, \quad v = \partial\psi/\partial x \tag{6.7,4}$$

we can therefore reduce our two equations (6.7,1) to one equation by expressing the first in terms of ψ.

We seek then to express ψ as a function of ζ but to do so we require first to divide ψ by a combination of u_1, x or ν, on which it must depend, to make it non-dimensional. Such a combination is $(u_1 x\nu)^{\frac{1}{2}}$; thus we write

$$\psi = -(u_1 x\nu)^{\frac{1}{2}} F(\zeta) \qquad (6.7,4)$$

and we endeavour to determine the function $F(\zeta)$. From (6.7,4) we have

$$\begin{aligned} u = -\partial\psi/\partial y = -\frac{\partial\psi}{\partial\zeta}\frac{\partial\zeta}{\partial y} &= (u_1 x\nu)^{\frac{1}{2}} \frac{F'}{2}\left(\frac{u_1}{\nu x}\right)^{\frac{1}{2}} \\ &= u_1 F'/2, \qquad (6.7,5) \end{aligned}$$

where the dash denotes differentiation with respect to ζ. From the form of (6.7,5) we see that the condition of similarity, equation (6.7,2), is satisfied.

Further,
$$v = \frac{\partial\psi}{\partial x} = \frac{1}{2}\left(\frac{u_1\nu}{x}\right)^{\frac{1}{2}}(F'\zeta - F). \qquad (6.7,6)$$

Hence,

$$\frac{\partial u}{\partial x} = -\frac{u_1}{2}F''\frac{y}{4x}\left(\frac{u_1}{\nu x}\right)^{\frac{1}{2}}, \quad \frac{\partial u}{\partial y} = \frac{u_1 F''}{4}\left(\frac{u_1}{\nu x}\right)^{\frac{1}{2}}, \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = \frac{u_1{}^2}{8x\nu}F'''$$

and on substitution of these expressions in the first equation 6.7,1 we obtain finally

$$FF'' + F''' = 0. \qquad (6.7,7)$$

The boundary conditions are

$$\zeta = 0, \quad F = 0, \quad F' = 0; \qquad \zeta = \infty, \quad F' = 2.$$

Thus, we see that we have obtained an ordinary differential equation in ζ and the boundary layer flow can be completely described by its solution.

Equation (6.7,7) is a non-linear equation of the third order and cannot be solved analytically. It must therefore be solved numerically, and the first numerical solution as well as the derivation of the equation are due to Blasius[1]. Details of his method of solution cannot be given here, but it involves an expansion in power series in ζ for small ζ, which can be made to satisfy the boundary conditions at $\zeta = 0$ in terms of the unknown $F''(0)$. The value of the latter is then determined by the requirements of continuity with an asymptotic solution for $\zeta = \infty$ satisfying the boundary conditions there.

Subsequently, other methods of numerical solution were developed† that were either speedier and simpler to apply than Blasius's method or could be adapted to give a more accurate solution. Today, the solution can be readily obtained using a digital computer.

The following table gives some details of the solution to four figures:—

$\zeta = \frac{y}{2}\left(\frac{u_1}{\nu x}\right)^{\frac{1}{2}}$	F	$\frac{F'}{2} = \frac{u}{u_1}$
0	0	0
0·1	0·0066	0·0664
0·2	0·0266	0·1328
0·3	0·0597	0·1989
0·4	0·1061	0·2647
0·5	0·1656	0·3298
0·6	0·2380	0·3938
0·8	0·4203	0·5168
1·0	0·6500	0·6298
1·2	0·9223	0·7290
1·4	1·2310	0·8115
1·6	1·3691	0·8761
1·8	1·9295	0·9233
2·0	2·3058	0·9555
2·5	3·2833	0·9916
3·0	4·2796	0·9990
3·5	5·2793	0·9999
4·0	6·2792	1·0000

The velocity ratio u/u_1 is shown as a function of ζ in Fig. 6.7,1.

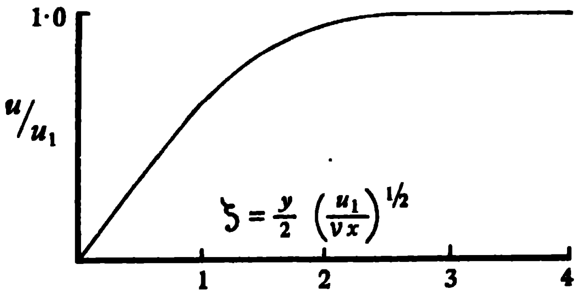

Fig. 6.7,1. Velocity distribution in laminar boundary layer of flat plate at zero incidence, according to solution of Blasius equation.

It should be noted that the lateral velocity component, v, increases continuously with ζ to an asymptotic value for $\zeta = \infty$ (see Fig. 6.7,2), i.e. there is a finite value of v at the outer edge of the boundary layer. This value is given by v_∞ where

$$v_\infty/u_1 = 0{\cdot}865/\sqrt{R_x}. \qquad (6.7,8)$$

The value of $F''(0)$ is 1.328 and from this we can determine the skin friction and drag coefficients of the plate. Thus, the frictional stress at the surface is, from equations 6.7,5 and 6.7,3

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_w = \mu\frac{u_1}{4}\left(\frac{u_1}{\nu x}\right)^{\frac{1}{2}}F''(0)$$

and the local skin friction coefficient is

$$c_f = \frac{2\tau_w}{\rho u_1^2} = \left(\frac{\nu}{u_1 x}\right)^{\frac{1}{2}} \frac{F''(0)}{2}$$

$$= 0{\cdot}664/R_x^{\frac{1}{2}}. \qquad (6.7,9)$$

The total skin friction coefficient for one side of a flat plate of chord c is

$$C_F = \frac{1}{c}\int_0^c c_f\,dx = 1{\cdot}328/R^{\frac{1}{2}}, \qquad (6.7,10)$$

where $R = u_1 c/\nu$,

and the drag coefficient of the plate, taking both sides into account, is

$$C_D = 2{\cdot}656/R^{\frac{1}{2}}. \qquad (6.7,11)$$

Since, from the momentum equation for the flat plate (equation 6.6,5),

$$\frac{d\theta}{dx} = \frac{c_f}{2}$$

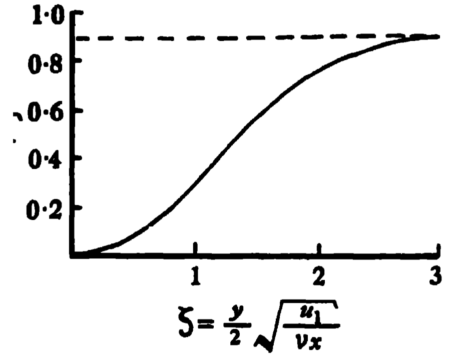

Fig. 6.7,2. Distribution of lateral velocity component v in laminar boundary layer on flat plate.

it follows that

$$\frac{\theta}{x} = \frac{1}{2x}\int_0^x c_f\,dx = c_f = 0{\cdot}664/R_x^{\frac{1}{2}}, \qquad (6.7,12)$$

and the displacement thickness is found to be given by

$$\frac{\delta^*}{x} = 1{\cdot}721/R_x^{\frac{1}{2}}. \qquad (6.7,13)$$

6.8 Flow in a Pipe or Channel of Constant Cross-section

If we consider steady and initially uniform flow in a circular pipe of constant cross-section we find developing from the entry a boundary layer which eventually grows to fill the pipe some diameters downstream of the entry. In this initial section of entry flow, as it is called, the velocity profile across the pipe continuously changes with distance downstream; the central core of non-boundary layer flow diminishes in section but its velocity increases with distance downstream, since the overall rate of mass flow must be constant for all sections of the pipe (see Fig. 6.8,1).

As always, the boundary layer is initially in a laminar state but it eventually becomes turbulent and transition may occur either in the entry flow region or subsequently. The flow aft of the entry region is characterized by the fact that the cross-sectional velocity profile is the same for all sections where the flow is laminar and is similarly constant for all sections where the flow is turbulent, but it changes at transition from laminar to turbulent flow. It is referred to as *fully developed pipe flow*, and since the velocity profile is constant it follows that the force due to friction at the pipe wall exactly balances that due to the pressure gradient down the pipe required to maintain the flow. The flow in a channel of constant section can be described in closely similar terms.

The development of the entry flow region boundary layer is not essentially different from that of the boundary layer along a plane wall with an external velocity u_1 which increases with x, with u_1 being determined by the requirement of constancy of rate of mass flow for all cross-sections. It can be dealt with by methods that are discussed in later sections of this Chapter, but for further details readers are referred to Goldstein.[1]

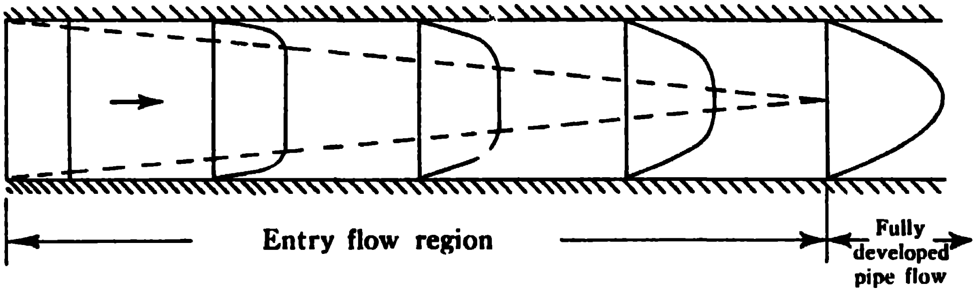

Fig. 6.8,1. Rough sketch illustrating the development of laminar boundary layer in a pipe (not to scale).

The theory of steady laminar flow in pipes of circular and other sections is given in § 7.6. References to fully developed turbulent flow in pipes and channels are made in various sections of the present chapter, as the subject cannot be treated separately from that of the turbulent boundary layer on a plate. A further discussion of turbulent flow in pipes is, however, given in § 7.7 and likewise turbulent flow in open channels is treated in § 8.2.

6.9 Similar Solutions of the Laminar Boundary Layer Equations

Paragraph 6.7 gave one example where the velocity profiles at different stations were similar in the sense described (see equation 6.7,2), and the question naturally arises whether other such solutions exist. Mathematically they offer obvious and welcome simplifications, since in their case the problem reduces to the solution of an ordinary differential equation with one independent variable, whereas in the general case we have partial differential equations in two variables to solve.

This question has been considered in general terms and the answer[2,3] is that such solutions exist when either:

(a) u_1 is of the form

$$\text{const. } x^m, \text{ where } m \text{ is a constant,}$$

or (b) u_1 is of the form

$$\text{const. } \exp(\alpha x), \text{ where } \alpha \text{ is a positive constant.}$$

Consider case (a) with

$$u_1 = u_0\left(\frac{x}{L}\right)^m \quad \text{say,} \tag{6.9,1}$$

where u_0 has the dimensions of a velocity, and L is some standard length. We adopt the variable

$$\zeta = \frac{y}{2}\left(\frac{u_1}{\nu x}\right)^{\frac{1}{2}} = \frac{y}{2}\left(\frac{u_0}{\nu}\right)^{\frac{1}{2}} \cdot \frac{x^{\frac{1}{2}(m-1)}}{L^{m/2}}, \tag{6.9,2}$$

and we write the stream function ψ in the form

$$\begin{aligned}\psi &= -(u_1\nu x)^{\frac{1}{2}}F(\zeta)\\ &= -\frac{(u_0\nu)^{\frac{1}{2}}x^{\frac{1}{2}(m+1)}}{L^{m/2}}F(\zeta).\end{aligned} \tag{6.9,3}$$

Then $$u = -\partial\psi/\partial y = u_1F'/2,$$

and hence the condition of similarity is satisfied. If as before we determine from (6.9,2) and (6.9,3) expressions for v, $\partial u/\partial y$, $\partial u/\partial x$, and $\partial^2 u/\partial y^2$ in terms of F and its derivatives and substitute these expressions into the equation of motion:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_1\frac{du_1}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

we obtain eventually the equation

$$F''' + (m+1)FF'' - 2mF'^2 + 8m = 0. \tag{6.9,4}$$

This is, as required, an ordinary differential equation for F, and the boundary conditions are

$$\zeta = 0, \quad F = F' = 0; \qquad \zeta = \infty, \quad F' = 2.$$

It is sometimes convenient to use $\eta = \zeta[2(m+1)]^{\frac{1}{2}}$, and $f(\eta) = \sqrt{\frac{m+1}{2}}F(\zeta)$ and then (6.9,4) becomes

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{6.9,5}$$

where $\beta = 2m/(m+1)$ and the boundary conditions are

$$\eta = 0, \quad f = f' = 0; \qquad \eta = \infty, \quad f' = 1{\cdot}0.$$

This equation was first considered and solved numerically for various values of m by Falkner and Skan[1] and subsequently Hartree[2] gave solutions of greater accuracy. Their results are illustrated in Fig. 6.9,1.

It is of interest to note that for positive m equation (6.9,1) describes the velocity in potential flow near the stagnation point on the surface of a symmetrical wedge of semi-angle $\pi\beta/2$ where β is related to m as above. For the values of m illustrated in Fig. 6.9,1 the corresponding values of β are:

m	β
−0·091	−0·199
−0·0654	−0·14
0	0
1/9	0·2
1/3	0·5
1	1
4	1·6

The solution $m = 1$ corresponds to the flow in the neighbourhood of a stagnation point of a round-nosed aerofoil ($\beta = 1{\cdot}0$). We note that when

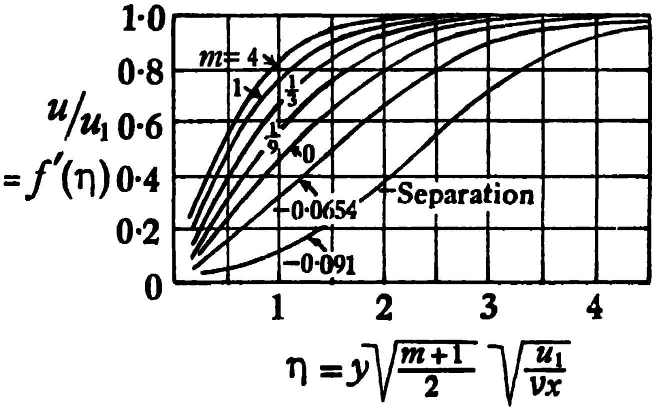

Fig. 6.9,1. Velocity profiles for various similar solutions of the laminar boundary layer equations with u_1 = const. x^m.

m becomes negative, and the external pressure gradient becomes positive, the velocity profiles show a point of inflection. This result can be quite simply deduced from the boundary condition (6.4,6). Thus, it follows from that condition that if $du_1/dx < 0$, then $(\partial^2u/\partial y^2)_w > 0$. Hence $(\partial u/\partial y)$ must increase with y near the wall from its positive value at the wall. But at the outer edge of the layer $\partial u/\partial y = 0$, and hence at some point in the layer $(\partial u/\partial y)$ must be stationary, i.e., $\partial^2u/\partial y^2 = 0$. Hence the velocity profile must show a point of inflection.

Another interesting feature of the results illustrated in Fig. 6.9,1 is the relatively large effect quite a small adverse external pressure gradient has on the velocity profile. Thus, the boundary layer is on the verge of separation with $m = -0{\cdot}091$.

Another special case arises with $m = -1$ and with $u_0 < 0$; this corresponds to the flow in the boundary layer on one wall of a convergent channel. In this case the resulting equation can be solved in analytic closed form. For further details see Schlichting.[1]

6.10 The General Problem and the Series Solution

The general problem of the flow in the boundary layers on a cylinder with prescribed pressure distribution is one of formidable complexity, involving as it does partial differential equations with two independent variables.† The most effective analytic attack on it has been by the so-called series solution method to which Blasius,[1] Hiemenz[2], Howarth[3], Frössling[4], and Ulrich[5] have made the most important contributions.

Details of this method cannot be given here, but the essential features are as follows. The prescribed external velocity distribution $u_1(x)$ is expressed as a power series in x, and the stream function ψ is then expressed as a related power series in which the coefficients are functions f_1, f_2, etc., of the non-dimensional $\eta = y\sqrt{\frac{u_{11}}{\nu}}$, where u_{11} is the coefficient of x in the power series for $u_1(x)$. After substituting the resulting expressions for u, $\partial u/\partial x$, $\partial u/\partial y$, etc. in the equation of motion and equating coefficients of powers of x we obtain a series of ordinary differential equations for the functions f_1, f_2, etc. The first equation involves only f_1, is non-linear and is precisely the same equation as that for flow in the neighbourhood of a stagnation point ($m = 1$) referred to in para 6.9. The subsequent equations are linear and each can be solved when the previous equations have been solved. In principle, therefore, once a sufficient number of these equations are solved and the functions f_1, etc. tabulated the boundary layer flow for any prescribed $u_1(x)$ can be readily determined. However, the determination of these successive functions, f_1, etc., rapidly becomes very complex and tedious, particularly as the accuracy with which the earlier functions are required in evaluating the later functions becomes more stringent as the order of the latter increases. A further difficulty arises from the fact that many external velocity distributions of practical interest are not readily represented by a conveniently small number of terms of a power series in x.

For the case of a cylinder that is symmetrical about the axis only the odd terms in the power series are relevant, and for this case tables have been evaluated which enable the functions f_1 to f_9 to be determined. The method has been applied to the flow past a circular cylinder, and predicts a point of separation about 110° round from the nose: for fuller details see Schlichting.[6] These results are appropriate to the external velocity distribution given by potential flow; however, the extensive flow separation that occurs behind a circular cylinder considerably alters the external velocity distribution from that of potential flow theory. If an experimental velocity distribution is taken as basis for the calculations, the predicted separation point is found to be in good agreement with experiment† and lies forward of the potential flow position in the region of 80°.

The series method has also been applied by Howarth,‡ with small modifications, to the flow along a plane wall with a constant external velocity gradient i.e.

$$u_1(x) = u_0 - \alpha x$$

where α can be positive or negative. He determined the auxiliary functions (f_1, etc.) up to the sixth order, and calculated a series of velocity profiles for different values of the non-dimensional $x^* = \alpha x/u_0$. He concluded that with an adverse external pressure gradient ($\alpha > 0$), separation would occur when $x^* = 0{\cdot}12$.

6.11 The Laminar Wake

The wake immediately downstream of a body is subject to the pressure field of the body and the determination of the wake flow in that region presents a complicated problem which will not be considered here. Far downstream, however, where the pressure has returned to the undisturbed stream value the influence of the body can be ignored apart from the fact that its drag is manifest in the reduced momentum of the fluid in the wake as compared with that outside (see Fig. 6.11,1). Thus, we can argue from first principles that if D' is the drag per unit span of a wing section then it must be equal to the integrated flux per unit span of momentum loss in the x direction across the wake, provided the integration is taken where the pressure is that of the undisturbed stream. A formal proof of this will be given later (see § 6.21,2). Thus we can write

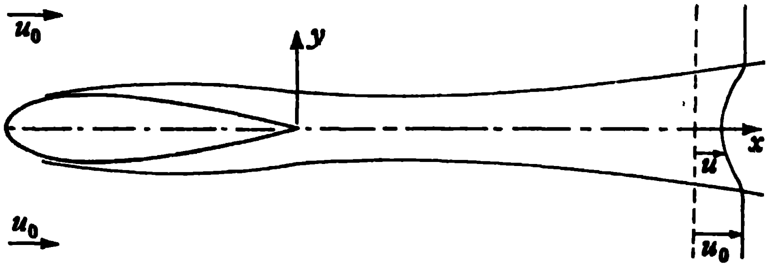

Fig. 6:11,1.

$$D' = \int_{-\infty}^{\infty} \rho u(u_0 - u)\, dy, \qquad (6.11,1)$$

where the integral is taken across the wake section and u_0 is the undisturbed stream velocity. Hence in the development of the wake this integral must remain constant.

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† As long as transition does not occur before laminar separation can take place, i.e., as long as the Reynolds number is less than the critical.

‡ L. Howarth, *loc. cit.*

Write

$$u_2 = u_0 - u \tag{6.11,2}$$

and we shall assume that at the stage in the development of the wake that we are considering u_2 is small compared with u_0 so that terms of order $(u_2/u_0)^2$ can be neglected.

The equation of motion, with zero external pressure gradient, is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}.$$

From the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

it can be argued that $v = 0[(\delta/x)u_2]$ where x is measured downstream from the body, and δ is some practical measure of the wake width, hence

$$v\,\partial u/\partial y = 0(u_2{}^2/x).$$

But

$$u\,\partial u/\partial x = -(u_0 - u_2)\,\partial u_2/\partial x = -u_0\,\partial u_2/\partial x + \text{term of order } (u_2{}^2/x)$$

and we have agreed that u_2/u_0 is small, hence the above equation of motion reduces to

$$u_0\frac{\partial u_2}{\partial x} = \nu\frac{\partial^2 u_2}{\partial y^2}. \tag{6.11,3}$$

The boundary conditions are

$$y = 0,\quad \frac{\partial u_2}{\partial y} = 0; \qquad y = \infty,\quad u_2 = 0.$$

As before we take the non-dimensional independent variable

$$\zeta = \frac{y}{2}\sqrt{\left(\frac{u_0}{\nu x}\right)}.$$

Then
$$D' = \int_{-\infty}^{\infty} \rho u_0 u_2\,dy = \int_{-\infty}^{\infty} \rho u_0 u_2 2\sqrt{\left(\frac{\nu x}{u_0}\right)}d\zeta$$

and if this integral is to be independent of x, we must choose the following form for the dependence of u_2 on x and ζ

$$u_2 = Ku_0\left(\frac{x}{L}\right)^{-1/2} g(\zeta) \tag{6.11,4}$$

where K is a constant, L is some standard length, e.g. wing chord, and $g(\zeta)$ is a function of ζ to be determined.

If we now substitute for u_2 from (6.11,4) into equation (6.11,3) we obtain finally the equation

$$g'' + 2\zeta g' + 2g = 0.$$

This equation is readily integrated to give, when account is taken of the boundary conditions,

$$g = \exp(-\zeta^2) \tag{6.11,5}$$

any multiplying constant being absorbed in the constant K of equation (6.11,4).

It follows that
$$D' = \frac{2K\rho u_0^2}{\sqrt{R}} L \int_{-\infty}^{\infty} g\, d\zeta,$$

where
$$R = u_0 L/\nu,$$

and since
$$\int_{-\infty}^{\infty} \exp(-\zeta^2)\, d\zeta = \sqrt{\pi}$$

or
$$\left.\begin{aligned} D' &= 2K\rho u_0^2 L\sqrt{(\pi/R)} \\ C_D &= \frac{2D'}{\rho u_0^2 L} = 4K\sqrt{(\pi/R)}. \end{aligned}\right\} \tag{6.11,6}$$

Hence, the constant K is given by

$$K = \frac{C_D}{4}\sqrt{\frac{R}{\pi}}. \tag{6.11,7}$$

For example, if the wing is a flat plate, we have seen that (equation 6.7,11)

$$C_D\sqrt{R} = 2{\cdot}656$$

and then
$$K = 0{\cdot}375.$$

In general, with a laminar boundary layer, $C_D\sqrt{R}$ would be constant for a given wing section.

We have finally

$$\frac{u_2}{u_0} = \frac{C_D\sqrt{R}}{4\sqrt{\pi}}\left(\frac{x}{L}\right)^{-1/2} \exp\left(\frac{-y^2 u_0}{4\nu x}\right). \tag{6.11,8}$$

For further details of the wake flow, particularly near the body, see Goldstein[1].

It is important to note, however, that except at very low Reynolds numbers the flow in the wake of a wing is in practice likely to be turbulent. Even in cases where the boundary layer on the body is completely laminar, the laminar flow in the wake tends to be very unstable and transition occurs close to the trailing edge.

The flow in a plane jet far downstream of the nozzle from which it emerges is amenable to a similar if more complicated analysis. In this case the net rate of momentum flux across the jet is constant and equal to its thrust. For further details of the analysis reference should be made to Schlichting[1] and Bickley[2]. It is found that the jet spreads as $x^{2/3}$ whilst the maximum velocity in the jet falls as $x^{1/3}$. In contrast we may note that a round jet spreads as x, and the maximum velocity falls as x.

6.12 Approximate Methods

6.12.1 Introductory Remarks

The considerable numerical difficulties involved in obtaining exact solutions of the boundary layer equations for the general case has led to much attention being paid to the development of approximate methods. Usually such methods have been developed with the limited objective of predicting reliably overall characteristics of the boundary layer, e.g. momentum thickness, displacement thickness and skin friction, rather than details of the boundary layer flow. The momentum integral equation generally provides the basis for such methods, and the approximations are manifest in the assumptions adopted to solve that equation. There are many such methods and they cannot all be considered here. Only three will be discussed in some detail, namely, Pohlhausen's method[3], Thwaites' method[4] and Young's method[5]. Pohlhausen's method is described because it was one of the first and is a classic of boundary layer literature; from it have stemmed many other methods, and its basic premises and approach have proved fruitful in tackling a variety of boundary layer problems. Thwaites' method is based on an analysis of the results of existing exact solutions and offers a fair degree of accuracy combined with simplicity. Young's solution results in practically the same formula but from a different approach and has lent itself to ready extension to boundary layer problems in compressible flow. All three methods make use in differing ways of the simplifying assumption that laminar boundary layer velocity profiles can be regarded as a uni-parametric family. This assumption is reasonably close to the truth when the external pressure gradient is favourable (negative) or weakly adverse but when the gradient is strongly adverse, as when separation is approached, the assumption loses validity and requires modification.†

6.12.2 Pohlhausen's Method

In essence, this method extends and generalises the application of the momentum integral equation (equation 6.6,4) described in § 6.6.

We begin with that equation

$$\theta' + \frac{u_1'}{u_1}\theta(H + 2) = \frac{\tau_w}{\rho u_1^2} \tag{6.12,1}$$

where the accent now denotes differentiation with respect to x.

A finite thickness δ for the boundary layer is specified so that the boundary conditions (§ 6.4) are

$$\left.\begin{aligned} &y = 0, \quad u = 0, \quad \nu\left(\frac{\partial^2 u}{\partial y^2}\right)_w = -u_1 u_1'; \\ &y = \delta, \quad u = u_1, \quad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 u}{\partial y^3} = \ldots = 0. \end{aligned}\right\} \tag{6.12,2}$$

It is now assumed that u/u_1 can be expressed as some function of $\eta = y/\delta$ of chosen form, the form being such that a suitable number of the boundary conditions can be satisfied to specify the function completely. The form that is normally associated with this method is a quartic, viz:

$$\frac{u}{u_1} = a\eta + b\eta^2 + c\eta^3 + d\eta^4, \tag{6.12,3}$$

but it is as well to note that other forms, including transcendental forms, can be used. We shall, however, develop the analysis for the above quartic.

To determine the coefficients a, b, c and d we can satisfy four of the above boundary conditions in addition to the condition $u = 0$ when $\eta = 0$ which is already satisfied. We choose therefore to satisfy

$$\eta = 0, \quad \nu\, \partial^2 u/\partial y^2 = -u_1 u_1'; \qquad \eta = 1{\cdot}0, \quad \frac{u}{u_1} = 1{\cdot}0, \quad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 0.$$

The first of these conditions is of prime importance as it is the only one in which the external velocity gradient enters directly to influence the flow in the boundary layer.

With these conditions satisfied we readily find that

$$a = 2 + \frac{\lambda}{6}, \quad b = -\frac{\lambda}{2}, \quad c = -2 + \frac{\lambda}{2}, \quad d = 1 - \frac{\lambda}{6}, \tag{6.12,4}$$

where λ is a dimensionless parameter given by

$$\lambda = \frac{\delta^2}{\nu} u_1'. \tag{6.12,5}$$

Hence the velocity distribution becomes

$$\left.\begin{aligned} \frac{u}{u_1} &= F(\eta) + \lambda G(\eta) \\ \text{where} \qquad F(\eta) &= 2\eta - 2\eta^3 + \eta^4 \\ \text{and} \qquad G(\eta) &= \tfrac{1}{6}\eta(1-\eta)^3. \end{aligned}\right\} \qquad (6.12,6)$$

We see, therefore, that the method effectively assumes that the velocity distributions of the boundary layer are uni-parametric in form, with form parameter λ, which is directly related to the local external velocity gradient.

Our problem now reduces to determining the unknown δ, and it is at this stage that we make use of the momentum integral equation. We have that

$$\frac{\tau_w}{\rho u_1^2} = \frac{\mu\left(\dfrac{\partial u}{\partial y}\right)_w}{\rho u_1^2}$$

$$= \frac{\nu}{u_1\delta}[F'(0) + \lambda G'(0)] = \frac{\nu}{u_1\delta}\left(2 + \frac{\lambda}{6}\right), \qquad (6.12,7)$$

$$\frac{\delta^*}{\delta} = \int_0^1 [1 - F(\eta) - \lambda G(\eta)]\, d\eta$$

$$= \frac{3}{10} - \frac{\lambda}{120}, \qquad (6.12,8)$$

and

$$\frac{\theta}{\delta} = \int_0^1 [F(\eta) + \lambda G(\eta)][1 - F(\eta) - \lambda G(\eta)]\, d\eta$$

$$= \frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072}. \qquad (6.12,9)$$

If we now substitute these expressions in equation (6.12,1) we obtain a differential equation in δ. We find it convenient to use the variable

$$Z = \delta^2/\nu = \lambda/u_1', \qquad (6.12,10)$$

and the resulting equation for Z is then of the form

$$\left.\begin{aligned} \frac{dZ}{dx} &= h(\lambda)\frac{d^2u_1}{dx^2}Z^2 + g(\lambda).\frac{1}{u_1}, \\ \text{where} \qquad h(\lambda) &= \frac{4(24 + 5\lambda)}{3(12-\lambda)(444+25\lambda)} \\ \text{and} \qquad g(\lambda) &= \frac{4(45360 - 8352\lambda + 237\lambda^2 + 5\lambda^3)}{3(12-\lambda)(444+25\lambda)}. \end{aligned}\right\} \qquad (6.12,11)$$

This equation can easily be solved numerically by a step by step process with the aid of tables of values of $h(\lambda)$ and $g(\lambda)$. However a starting value of Z at the forward stagnation point is required. At that point $u_1 = 0$ and so to ensure a finite value of dZ/dx there the function $g(\lambda)$ must vanish there. The zeros of $g(\lambda)$ occur when $\lambda = -70$, 17·75 and 7·052. However, λ cannot be negative at the stagnation point since u_1' is positive there, so that the value -70 can be excluded. If the starting value of λ were 17·75 then, since λ is negative where the external velocity gradient is negative, at some stage λ would pass through the value 12. But when $\lambda = 12$ we see that both $g(\lambda)$ and $h(\lambda)$ become infinite and the solution becomes physically meaningless. Thus, the only acceptable starting value of λ must be 7·052, and this corresponds to a starting value of Z' given by

$$Z'_{(x=0)} = -5{\cdot}391 u_1''/u_1'^2. \tag{6.12,12}$$

From comparisons with the results of more exact calculations this method is found to give accurate results in regions of negative pressure gradient, but in regions of strong positive pressure gradient the method is somewhat less reliable. In particular the condition for separation, which is $\lambda = -12$, appears to over-estimate by a significant amount the distance to separation; more accurate calculations give values of λ at separation in the region of -7 to -8. For the flat plate at zero incidence the method gives the results quoted in the table of § 6.6, where it will be seen that the error in skin friction as compared with the Blasius solution is about $3\frac{1}{2}$%.

The method has been extended to bodies of revolution by Tomotika[1].

A modified and somewhat more convenient form of Pohlhausen's method has been developed by Holstein and Bohlen[2] using variables involving θ in place of δ.

6.12.3 Thwaites' Method

Two non-dimensional parameters l and m are defined by the equations

$$l = \frac{\theta}{u_1}\left(\frac{\partial u}{\partial y}\right)_w, \quad m = \frac{\theta^2}{u_1}\left(\frac{\partial^2 u}{\partial y^2}\right)_w. \tag{6.12,13}$$

The parameter l is directly related to the skin friction, whilst m is related to the external pressure (or velocity) gradient through the only boundary condition in which the external pressure gradient appears, viz:—

$$\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_w = \frac{1}{\rho}\frac{dp}{dx} = -u_1 u_1'$$

so that

$$m = -u_1'\theta^2/\nu. \tag{6.12,14}$$

Thus, m is a very important parameter of the velocity profile and if we assume that the laminar boundary layer velocity profiles form a uni-parametric family, then we can regard m as the form parameter. In that case,

l must be a unique function of m for all velocity profiles, as must be $H = \delta^*/\theta$. The momentum integral equation can be written

$$\theta' + \theta(H+2)\frac{u_1'}{u_1} = \frac{\tau_w}{\rho u_1^2} = \frac{\nu}{u_1\theta}\,l$$

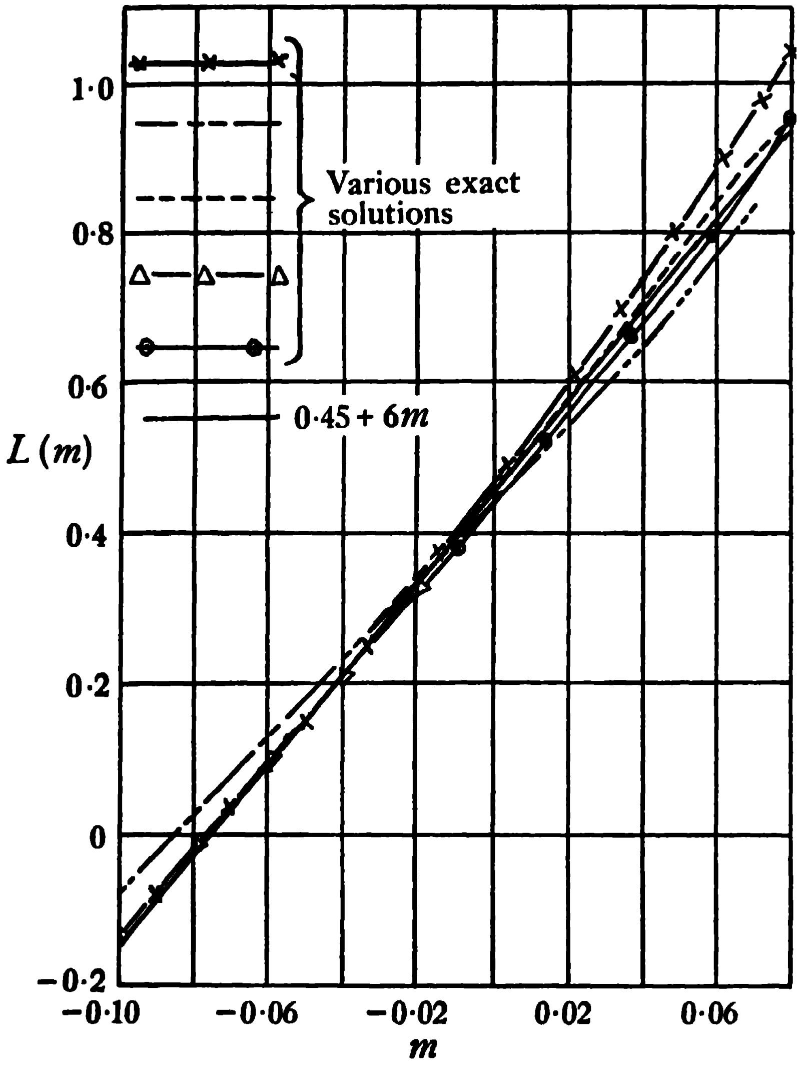


Fig. 6.12,1.

and if we make use of equation (6.12,14) this becomes

$$u_1\frac{d}{dx}(\theta^2) = 2\nu[m(H+2)+l]$$

$$= \nu L(m), \qquad (6.12,15)$$

where
$$L(m) = 2[m(H+2)+l]. \qquad (6.12,16)$$

Thwaites evaluated the expression represented by the right hand side of (6.12,16) for the differing velocity profiles given by known exact solutions covering a wide range of pressure gradients and found that the resulting curves lay close to a straight line for which the equation was (see Fig. 6.12,1)

$$L(m) = 0{\cdot}45 + 6m. \qquad (6.12,17)$$

Equation (6.12,15) therefore can be written as

$$u_1 \frac{d}{dx}(\theta^2) - 0{\cdot}45\nu + 6u_1'\theta^2 = 0$$

or

$$\frac{d}{dx}(u_1^6\theta^2) = 0{\cdot}45\nu u_1^5$$

and hence

$$[\theta^2]_{x_1} = \frac{0{\cdot}45\nu}{(u_1^6)_{x_1}} \int_0^{x_1} u_1^5\, dx, \qquad (6.12,18)$$

since either θ or $u_1 = 0$ when $x = 0$. The suffix x_1 indicates that quantities are evaluated at $x = x_1$.

Thus, from equation (6.12,18) the value of θ^2 can be determined at any point by a simple quadrature. Once θ^2 is determined the parameter m can be obtained from (6.12,14). However, the plots of l and H as functions of m determined by Thwaites from the analysis of some known solutions showed significant scatter for $m > 0$ (see Figs. 6.12,2 and 6.12,3).

This scatter reflects the inadequacy of the assumption that the boundary layer profiles form a uni-parametric family. Thwaites, therefore, selected certain of the exact solutions most relevant to the flow past aerofoils as a guide, particularly the case of a linear adverse velocity gradient, and on the basis of these solutions he devised a table of l and H as unique functions of m. This table is recommended as offering a reasonable overall accuracy for many problems of practical interest.

Table of Thwaites' Suggested Values of l and H as Functions of m.

m	$l(m)$	$H(m)$
0·082	0	3·7
0·0816	0·016	3·66
0·0808	0·030	3·61
0·080	0·039	3·58
0·078	0·055	3·47
0·074	0·076	3·30
0·070	0·089	3·17
0·064	0·104	3·05
0·056	0·122	2·94
0·048	0·138	2·87
0·032	0·168	2·75
0·016	0·195	2·67
0	0·220	2·61
−0·032	0·268	2·49
−0·064	0·313	2·39
−0·10	0·359	2·28
−0·14	0·404	2·18
−0·20	0·463	2·07
−0·25	0·5	2·0

It may be noted that these tabulated values are not completely consistent with equation (6.12,17), implying that some accuracy in satisfying the momentum integral equation has been sacrificed to mitigate the error involved in the assumption of a uni-parametric family of velocity profiles.

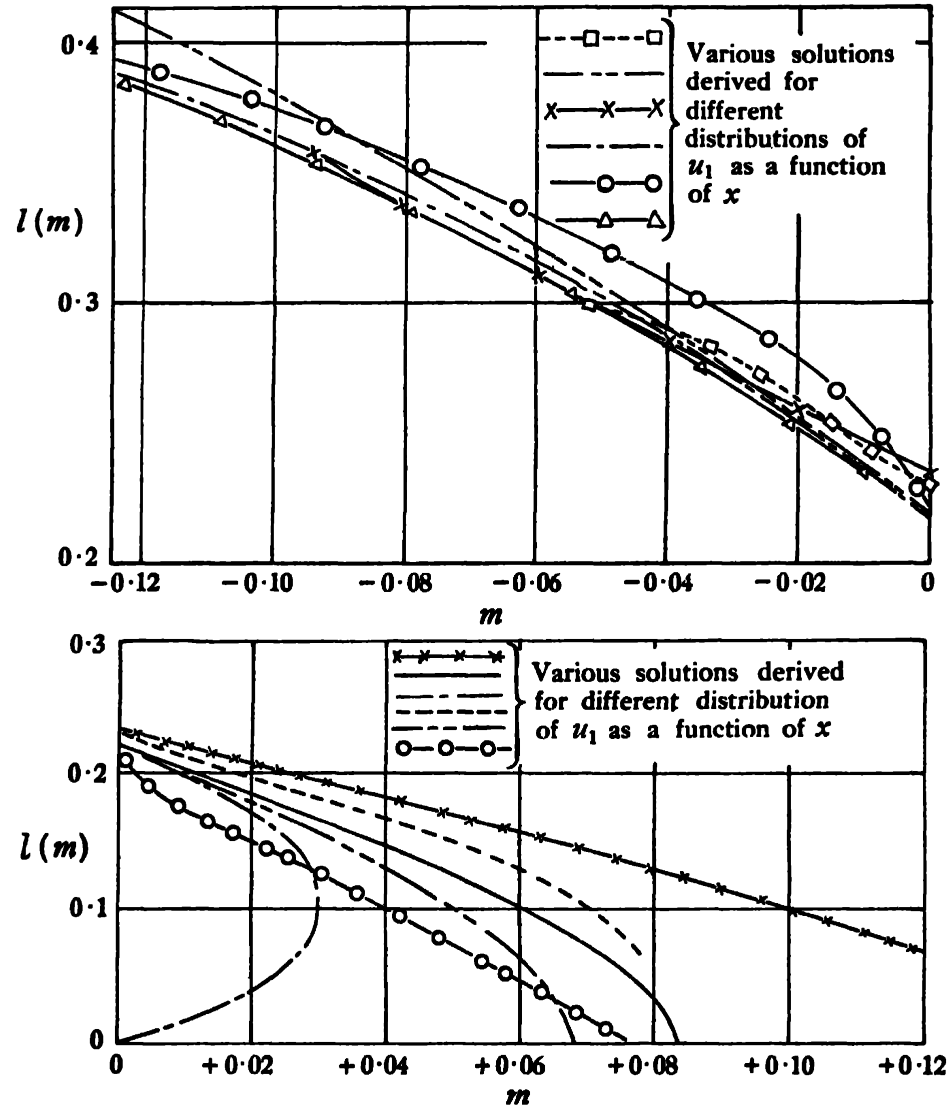


Fig. 6.12,2.

From the values of l and θ at any point the local skin friction coefficient can be determined since

$$\tau_w = \mu\left(\frac{\partial u}{\partial y}\right)_w = \frac{l u_1}{\theta}\mu$$

and therefore

$$c_f = \frac{2\tau_w}{\rho u_0^2} = \frac{2\nu l u_1}{\theta u_0^2} \qquad (6.12,19)$$

where u_0 is the undisturbed stream velocity.

The velocity profile can also be determined with some approximation in the form of a cubic, given l, m and H. Thus, with

$$y/\theta = a_1\xi + a_2\xi^2 + a_3\xi^3 \qquad (6.12,20)$$

where $$\xi = u/u_1,$$

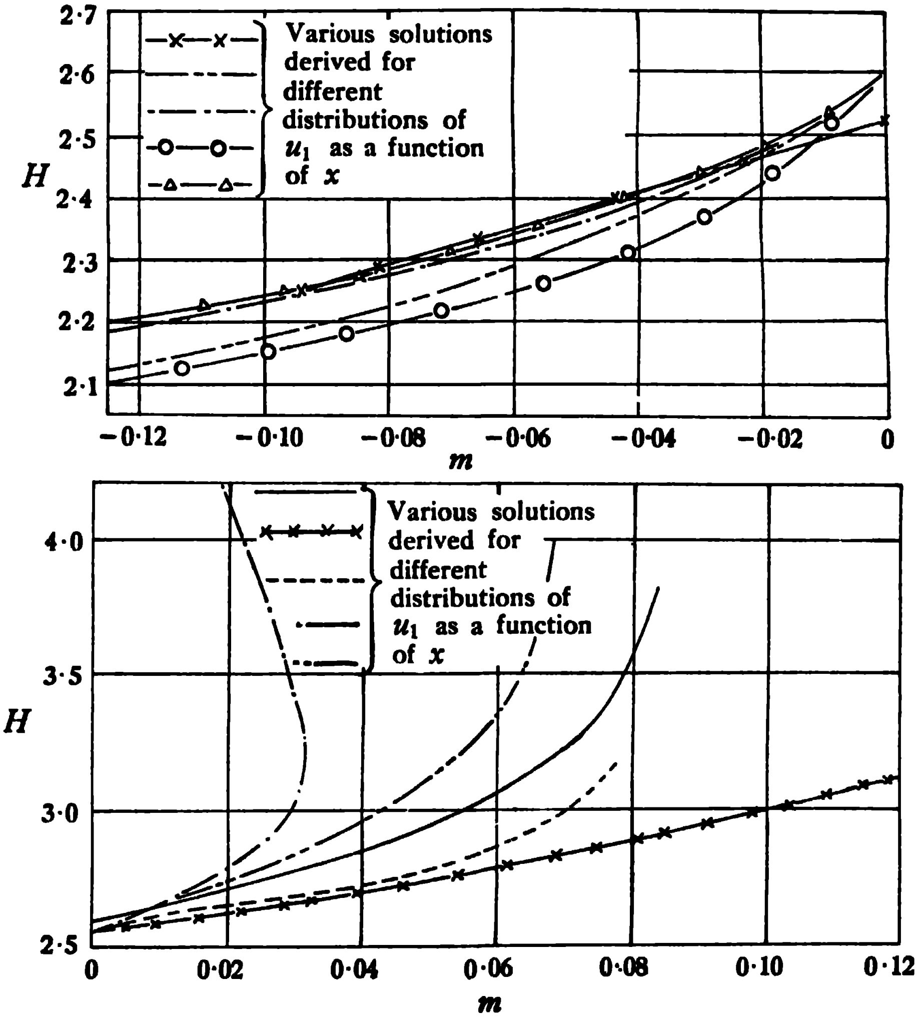


Fig. 6.12,3.

then $$\frac{u_1}{\theta}\left(\frac{\partial y}{\partial u}\right)_{u=0} = \frac{1}{\theta}\left(\frac{\partial y}{\partial \xi}\right)_{\xi=0} = a_1$$

and hence $$\frac{1}{l} = a_1.$$

Further, if we express (6.12,20) in the form $y/\theta = f(\xi)$,

then $$\theta\,\partial\xi/\partial y = 1/f'(\xi)$$

and $$\theta\,\partial^2\xi/\partial y^2 = -\left[\frac{f''(\xi)}{f'(\xi)^2}\right]\Big/\frac{\partial y}{\partial \xi} = \frac{-f''(\xi)}{\theta f'(\xi)^3}.$$

Hence
$$[\partial^2\xi/\partial y^2]_{y=0} = -f''(0)/[\theta^2 f'(0)^3]$$
$$= -2a_2/(\theta^2 a_1^3).$$

Therefore
$$a_2 = \frac{-\theta^2}{2u_1 l^3}\left[\frac{\partial u}{\partial y^2}\right]_{y=0} = -\frac{m}{2l^3}.$$

Also
$$H = \delta^*/\theta = \int_0^1 (1-\xi)\,dy/\theta$$
$$= \frac{1}{\theta}\int_0^1 (1-\xi)\frac{\partial y}{\partial \xi}\,d\xi = \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4}.$$

Hence
$$a_3 = 4H - \frac{2}{l} + \frac{2m}{3l^2}.$$

We have finally
$$y/\theta = \frac{\xi}{l} - \frac{m\xi^2}{2l^3} + \left(4H - \frac{2}{l} + \frac{2m}{3l^2}\right)\xi^3, \tag{6.12,21}$$

and therefore given l, m, H and θ we can determine the value of y for a series of values of $\xi = u/u_1$ from 0 to 1·0 and so determine the profile.

The method is found to give very good results for the evaluation of the skin friction and is in general more reliable than the Pohlhausen method for regions of positive (adverse) external pressure gradient. This follows from the use of exact solutions to formulate the mean functional dependence of l and H on m. In particular the method predicts the position of separation with much better accuracy than does the Pohlhausen method. The velocity profiles, however, are not predicted in general with satisfactory accuracy because of the inadequacy of a cubic representation.

We may note that if $u_1 = \beta x$, say, when x is small, as near a stagnation point, then from equation (6.12,18) it readily follows that
$$(\theta^2)_{x=0} = 0{\cdot}45\nu/6\beta,$$
and hence θ is not zero at the origin.

6.12.4 Young's Method

We make use of the relation (equation 6.12,7) between τ_w and the parameter $\lambda = \delta^2 u_1'/\nu$ given by the assumption of a quartic velocity distribution. Thus
$$\frac{\tau_w}{\rho u_1^2} = \frac{\nu}{u_1\delta}\left(2 + \frac{\lambda}{6}\right). \tag{6.12,7}$$

Write
$$\delta/\theta = f, \text{ say.} \tag{6.12,22}$$

The momentum integral equation
$$\theta' + \theta(H+2)\frac{u_1'}{u_1} = \frac{\tau_w}{\rho u_1^2}$$

then becomes, with the aid of equation (6.12,7),

$$\theta' + \theta(H+2)\frac{u_1'}{u_1} = \frac{u_1' f\theta}{6u_1} + \frac{2\nu}{f\theta u_1}$$

or

$$\theta' = \frac{u_1'\theta^2}{u_1}\left[\frac{f}{6} - (H+2)\right] + \frac{2\nu}{fu_1}$$

or

$$\left.\begin{aligned} \frac{d}{dx}(\theta^2) + \frac{u_1'\theta^2}{u_1}g &= \frac{4\nu}{fu_1} \\ \text{where} \qquad g &= 2\left[(H+2) - \frac{f}{6}\right] \end{aligned}\right\} \qquad (6.12,23)$$

The Pohlhausen quartic formulae show that for the extreme cases of the stagnation point ($\lambda \approx 7$) and of separation ($\lambda = -12$) the corresponding values of g are about 5·5 and 8·0, respectively, whilst for moderate pressure gradients the variation of g is small. Therefore, for the purpose of integrating equation (6.12,23) we shall assume g to be constant. Then, on integration, we have

$$[\theta^2]_{x_1} = \frac{4\nu}{[u_1^{g}]_{x_1}}\int_0^{x_1}\frac{u_1^{g-1}}{f}\,dx. \qquad (6.12,24)$$

We see that, since g enters as an exponent of u_1 in both the numerator and denominator on the right hand side, the resulting value of θ^2 is not very sensitive to the value of g, and this provides an *a posteriori* justification for the assumption that g is constant.

For a flat plate at zero incidence equation (6.12,24) simplifies to

$$[\theta^2]_{x_1} = 4\nu x_1/fu_1,$$

and, dropping the suffix 1 for x, we can write this as

$$\frac{\theta\sqrt{R_x}}{x} = 2\left(\frac{1}{f}\right)^{\frac{1}{2}}.$$

But the Blasius solution gave (equation 6.7,12) $\frac{\theta\sqrt{R_x}}{x} = 0{\cdot}664$, and hence to obtain the correct value of θ for the zero incidence case we must then have

$$f = \frac{4}{(0{\cdot}664)^2} = 9{\cdot}072.$$

Also in the case of the flat plate at zero incidence the Blasius solution gave $H = 2{\cdot}59$, and hence we have

$$g = 6{\cdot}16.$$

We now make the further assumption that f is constant and equal to 9·072, and then equation (6.12,24) becomes

$$[\theta^2]_{x_1} = \frac{0{\cdot}441\nu}{[u_1^{6\cdot16}]_{x_1}} \int_0^{x_1} u_1^{5\cdot16}\,dx. \tag{6.12,25}$$

It will be seen that this equation is identical in form with that given by Thwaites' method (equation 6.12,18), but the numerical constants involved differ slightly. However, in any particular problem the answers for θ given by the two methods differ very little.

Again, we note that if $u_1 = 0$ at $x = 0$ and is of the form $u_1 = \beta x$ near $x = 0$, then it follows from (6.12,25) that

$$[\theta^2]_{x=0} = \frac{0{\cdot}441}{6{\cdot}16\beta}\nu = 0{\cdot}0716\frac{\nu}{\beta}.$$

Having obtained θ, we have

$$\lambda = \frac{\delta^2 u_1'}{\nu} = \frac{f^2\theta^2 u_1'}{\nu}$$

and from (6.12,7) the local skin friction coefficient is

$$c_f = \frac{2\tau_w}{\rho u_1^2} = \frac{2\nu}{u_1 f\theta}\left(2 + \frac{\lambda}{6}\right). \tag{6.12,26}$$

The velocity profile of the boundary layer can be obtained from the quartic form of the Pohlhausen method (equations 6.12,3 and 6.12,4) once λ is determined.

In general this method gives good results in regions of favourable or small adverse pressure gradients, but like the Pohlhausen method it becomes less reliable in regions of large adverse pressure gradients, particularly those approaching the condition of separation. Thus, the method is especially suitable to problems of boundary layer flow on aerofoil sections at small angles of incidence. It can, however, be modified to deal with large adverse gradients by treating f and H as slowly varying functions of λ, determined from exact solutions. It then readily lends itself to extension to compressible flow problems, including problems involving heat transfer to or from the fluid.†

6.12.5 Some Remarks on Other Approximate Methods. The Energy Integral Equation

The modified form of Pohlhausen's method due to Holstein and Bohlen leads to an equation of the form

$$\frac{dZ_\theta}{dx} = \frac{F(K)}{u_1},$$

where $$Z_\theta = \theta^2/\nu, \quad K = u_1' Z_\theta,$$

---

† See R. E. Luxton and A. D. Young, *R & M* No. 3233 (1962).

and $F(K)$ is a function of K that is independent of the body shape. Walz[1] noted that $F(K)$ could be approximated fairly closely as a linear function of K, viz.

$$F(K) = a - bK,$$

and with this approximation the equation for Z_θ can be integrated to yield

$$u_1\theta^2/\nu = (a/u_1{}^{b-1})\int_0^x u_1{}^{b-1}\,dx,$$

a form which we note is identical with that derived by Thwaites (equation 6.12,18) and Young (equation 6.12,25). The particular values of a and b favoured by Walz are $a = 0{\cdot}47$, and $b = 6$.

Various authors have sought to develop approximate methods based on the satisfaction of another overall relation of the boundary layer in addition to the momentum integral equation. Such a relation is afforded by considering the balance of energy flux and dissipation in the boundary layer as expressed in the so-called *energy integral equation.* This is perhaps most simply obtained by multiplying the equation of motion by u and integrating with respect to y between the limits 0 and h, where h is some length slightly greater than δ the boundary layer thickness. We have then

$$\int_0^h u^2\frac{\partial u}{\partial x}\,dy + \int_0^h uv\frac{\partial u}{\partial y}\,dy = \int_0^h uu_1\frac{du_1}{dx}\,dy + \int_0^h \nu u\frac{\partial^2 u}{\partial y^2}\,dy.$$

But
$$\int_0^h uv\frac{\partial u}{\partial y}\,dy = \left[v\frac{u^2}{2}\right]_0^h - \int_0^h\frac{u^2}{2}\frac{\partial v}{\partial y}\,dy$$

and from the equation of continuity

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}, \quad\text{and}\quad v_h = -\int_0^h\frac{\partial u}{\partial x}\,dy.$$

Hence we have

$$\int_0^h\frac{3}{2}u^2\frac{\partial u}{\partial x}\,dy - \frac{u_1{}^2}{2}\int_0^h\frac{\partial u}{\partial x}\,dy - \int_0^h uu_1\frac{du_1}{dx}\,dy = \int_0^h \nu u\frac{\partial^2 u}{\partial y^2}\,dy$$

or
$$\frac{d}{dx}\left[\int_0^h u\left(\frac{u^2}{2} - \frac{u_1{}^2}{2}\right)dy\right] = \int_0^h \nu u\frac{\partial^2 u}{\partial y^2}\,dy$$
$$= \left[\nu u\frac{\partial u}{\partial y}\right]_0^h - \int_0^h \nu\left(\frac{\partial u}{\partial y}\right)^2 dy.$$

But at $y = 0$, $u = 0$ and at $y = h$, $\partial u/\partial y = 0$, and therefore

$$\frac{d}{dx}\left[\int_0^h \rho u\left(\frac{u_1{}^2}{2} - \frac{u^2}{2}\right)dy\right] = \int_0^h \mu\left(\frac{\partial u}{\partial y}\right)^2 dy. \qquad (6.12,27)$$

---

[1] A. Walz, *Lilienthal-Bericht*, **141**, 8 (1941).

This is the energy integral equation. We can identify the left-hand side as the rate of change with x of the flux of defect of kinetic energy across a section of the boundary layer whilst the right-hand side represents the rate at which the viscous stresses are dissipating kinetic energy into heat.

The *energy thickness* of the boundary layer was defined as (equation 6.4,9)

$$\delta_E = \int_0^h \frac{u}{u_1}\left[1 - \left(\frac{u}{u_1}\right)^2\right] dy$$

and we see that the energy integral equation can be written

$$\frac{\rho}{2}\frac{d}{dx}[u_1^3\delta_E] = \int_0^h \mu\left(\frac{\partial u}{\partial y}\right)^2 dy. \tag{6.12,28}$$

Consideration of the energy integral equation introduces the possibility of developing approximate solutions involving two parameters, or alternatively if only one parameter is retained a solution can be sought in which the energy integral equation is satisfied instead of one of the boundary conditions at the wall.

Two-parameter methods are necessarily more complex and tedious to apply than one-parameter methods, as they involve the simultaneous numerical solution of two equations derived from the momentum integral and energy integral equations. The extra work and complication can be justified, however, where a high degree of accuracy is required for the velocity profile of the boundary layer or where the external pressure gradient is changing rapidly with distance. In such cases single parameter methods cannot properly take account of the upstream history of the boundary layer. Amongst the first to develop a two-parameter method was Wieghardt[1] who assumed a polynomial of the eleventh degree for the velocity distribution. This polynomial was designed so that the first seven derivatives of u with respect to y vanished at the outer edge of the boundary layer; therefore, once the additional conditions $u = 0$, $y = 0$ and $u = u_1$, $y = \delta$ were satisfied, two disposable constants were left which could be regarded as form parameters related to the slope and curvature of the velocity profile at the surface. This is equivalent to regarding the parameters l and m of Thwaites' method as independent.

Later Head[2] generalised Wieghardt's method by recognising that it is equivalent to writing the velocity profile in the form (compare equation 6.12,6)

$$\frac{u}{u_1} = F(\eta) + \lambda_1 G_1(\eta) + \lambda_2 G_2(\eta)$$

where $F(\eta)$ is the Blasius profile for the flat plate at zero incidence, λ_1 and λ_2 are in effect form parameters determined by conditions at the surface. The functions $G_1(\eta)$ and $G_2(\eta)$ were determined numerically from known exact solutions, and the important boundary layer characteristics H, $H_e = \delta_E/\theta$, $D^* = \int_0^{\delta/\theta} (\theta/u_1)^2(\partial u/\partial y)^2\, d(y/\theta)$, and y/θ for various values of u/u_1 from 0 to 0·995 are presented graphically by Head as functions of the parameters l and m. With these data the simultaneous step-by-step solution of the suitably transformed momentum and energy integral equations is straightforward. The method is designed to deal with the velocity profiles obtained with suction at the surface as well as pressure gradient.

For methods in which only one parameter is used and the energy integral equation is satisfied whilst dropping the first compatibility condition at the wall (equation 6.4,4) reference should be made to Walz[1] and Tani[2].

6.13 Stability of Laminar Flow. Reynolds Stresses. Transition

6.13.1 Introductory Remarks

We have already remarked that the ease with which the laminar flow in a boundary layer changes to turbulent is very dependent on Reynolds number. If we define the Reynolds number in terms of momentum thickness (i.e. $R_\theta = u_1\theta/\nu$), then we find that on a flat plate it is impossible to provoke transition when $R_\theta < 200$, approx.; on the other hand, values of R_θ greater than about 5×10^3 with a laminar boundary layer are rare, although they can be attained if extreme care is taken.

We have still to achieve a proper understanding of the physical mechanism of transition to turbulent flow in a boundary layer. Much effort has been and is being devoted to this problem and there is no reason to suppose that it will not be solved. An important contribution to our knowledge has been the study of the stability of the laminar boundary layer to small disturbances, and this aspect of the problem is now well understood. It is by no means the complete answer, however, as we know nothing of what happens between the onset of instability of this classical kind and the development of turbulence. Further it provides no clue to the profound effect of finite disturbances, e.g. surface imperfections or external turbulence, which are known to provoke turbulence under conditions when the boundary layer is stable to small disturbances.

Before we can review current knowledge and ideas on this topic we must discuss in broad terms the factors that can be expected to augment or decrease the energy of disturbances. To do this we must first consider the stresses set up in a boundary layer when fluctuating disturbances are present.

6.13.2 Reynolds Stresses

We will denote the mean velocity components relative to cartesian axes (x, y, z) by (u, v, w), the disturbance velocity components by (u', v', w'), and the total velocity components by (U, V, W). Thus

$$U = u + u', \quad V = v + v', \quad W = w + w'. \tag{6.13,1}$$

The mean of any quantity f, say, at time t_1 will be denoted by $\bar{f}$ where

$$\bar{f} = \frac{1}{2T}\int_{t_1-T}^{t_1+T} f\,dt$$

and where it is assumed that T can be taken to be a sufficiently long time compared with a typical period of the disturbances for $\bar{f}$ to be sensibly independent of T, but T is short compared with some period representative of long term variations in $\bar{f}$. It follows that

$$u = \bar{U}, \text{ and } \overline{u'} = \frac{1}{2T}\int_{t_1-T}^{t_1+T} u'\,dt = 0, \tag{6.13,2}$$

and similar relations hold for the other velocity components.

Now consider a small area δA in the plane defined by the y and z axes. Through such an area there passes fluid in time δt of mass

$$(\delta A \rho U)\,\delta t.$$

This will be associated with the transport of momentum of components

$$(\delta A \rho U^2)\,\delta t, \quad (\delta A \rho U V)\,\delta t, \quad (\delta A \rho U W)\,\delta t.$$

The mean values of the components of the rates of transport of momentum per unit area across δA are therefore

$$\rho\overline{(u + u')^2}, \quad \rho\overline{(u + u')(v + v')}, \quad \rho\overline{(u + u')(w + w')}$$

or

$$\rho u^2 + \overline{\rho u'^2}, \quad \rho u v + \overline{\rho u' v'}, \quad \rho u w + \overline{\rho u' w'} \tag{6.13,3}$$

from equation (6.13,2).

The term $\overline{\rho u'^2}$ which derives from the disturbances represents a normal stress acting on the plane yz in the negative x direction, similarly $\overline{\rho u'v'}$ and $\overline{\rho u'w'}$ represent tangential stresses in that plane in the negative y and z directions. Thus, the disturbances give rise to the following apparent stress components acting on the plane yz:

$$\tau_{xx} = -\overline{\rho u'^2}, \quad \tau_{xy} = -\overline{\rho u'v'}, \quad \tau_{xz} = -\overline{\rho u'w'}. \tag{6.13,4}$$

Exactly similar expressions are obtained for the stress components acting on the planes zx and xy, viz.

$$\begin{aligned} \tau_{xy} &= -\rho u'v', \quad \tau_{yy} = -\rho v'^2, \quad \tau_{yz} = -\rho v'w' \\ \tau_{xz} &= -\overline{\rho u'w'}, \quad \tau_{yz} = -\rho v'w' \quad \tau_{zz} = -\overline{\rho w'^2} \end{aligned}$$

These stress components are referred to as the *Reynolds stresses*, and sometimes as *eddy stresses*. They are, of course, additional to the viscous stresses which arise from the much smaller scale random molecular movements. Reynolds stresses are normally considered only in connection with turbulence, but they exist when any form of fluctuating disturbance is present in the laminar layer.

In the two-dimensional turbulent boundary layer the eddy stress $\tau_{xy} = -\rho\overline{u'v'}$ is of first importance and, except in a very thin layer adjacent to the surface where the velocity fluctuations become small, the so-called laminar sublayer, the magnitude of this stress is far greater than the viscous stress $\mu\partial u/\partial y$ or the other eddy stresses. In general, fluid elements arriving at a point from strata nearer the surface (v' positive) arrive with roughly the mean velocities of those strata and hence with negative values of u'; conversely fluid elements from higher strata (v' negative) generally arrive with positive values of u'. Thus the product $u'v'$ is generally strongly negative, so that the stress $-\rho\overline{u'v'}$ is positive and usually large. We then speak of u' and v' as being strongly correlated in the sense that the sign and magnitude of the one has an important influence on the sign and magnitude of the other and their product is not completely random with time.

6.13.3 Energy Balance in Disturbed Laminar Flow†

Associated with any disturbance pattern will be disturbances in pressure as well as the Reynolds or eddy stresses, and both pressure disturbances and eddy stresses will do work and may thereby enable energy to be transferred from the mean motion to the disturbances or vice versa. If we define $(u'^2 + v'^2)/2$ as the kinetic energy of the disturbances per unit mass, then consideration of the energy balance of a fluid particle leads to the result that the rate of change of this disturbance kinetic energy is equal to the rate of work done by the apparent stresses and pressure fluctuations minus the rate at which energy of the disturbing motion is dissipated by viscosity and converted into heat. When the resulting equation is integrated over a volume of fluid of large extent compared with the eddies, we find that on the average the contribution to the energy balance due to the work done by the pressure fluctuations can be neglected, and for two-dimensional flow with u as a function of y only we are left with an equation of the form

$$\frac{DE'}{Dt} = -\iint \rho u'v' \frac{du}{dy}\,dx\,dy - \mu\iint \zeta'^2\,dx\,dy \qquad (6.13,5)$$

where $\zeta' = \partial v'/\partial x - \partial u'/\partial y$, i.e. the vorticity of the disturbances, and E' is the integrated kinetic energy of the disturbances in the volume considered. The first term on the right-hand side can be identified with the rate of work done by the eddy stress $-\rho u'v'$, whilst the second can be shown to be the rate of dissipation of the disturbance energy due to viscosity. Thus the right-hand side can be written in the form

$$\rho(M - \nu N)$$

where M can be of either sign, depending on the nature of the disturbance and the sign of du/dy, whilst N is always positive. We note that the ratio $M/\nu N$ is unaffected by the amplitude of the disturbance. If h is a characteristic length and u_0 a characteristic velocity, then we can write

$$\rho(M - \nu N) = \rho u_0^2[hu_0M' - \nu N'] = \rho u_0^2\nu[RM' - N'] \qquad (6.13,6)$$

where M' and N' are non-dimensional forms of M and N and R is the Reynolds number u_0h/ν.

We see therefore that if $R < N'/M'$ the energy balance is such that the disturbances are damped out by viscous action and the flow must be stable to the particular type of disturbance considered. This argument by itself does not prove that there exists a Reynolds number for any given boundary layer below which there is complete stability to all possible forms of disturbance, although in certain cases such a Reynolds number has been shown to exist by this argument, but there is ample experimental and other evidence for its existence. We can however infer that as R increases the possibility of instability increases, since it is always possible to conceive of a disturbance pattern for which M' is positive.

To take this argument further, we require to consider specific types of disturbances that are hydrodynamically possible, and this has led to the detailed consideration of the stability of a laminar boundary layer to infinitesimal disturbances of harmonic form. Fourier analysis readily enables us to treat any wavelike disturbance as a sum of such disturbances.

6.13.4 Classical Stability Analysis†

We consider a two dimensional motion in which the mean velocity u is a function of y only and $v = 0$. From the equation of continuity

$$\frac{\partial}{\partial x}(u + u') + \frac{\partial v'}{\partial y} = 0.$$

It follows that

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

and hence we can introduce a stream function ψ, say, such that

$$u' = -\partial\psi/\partial y, \quad v' = \partial\psi/\partial x. \qquad (6.13,7)$$

The function ψ is then expressed in the form

$$\left.\begin{aligned}\psi &= F(y)\exp\left[i\alpha(x-ct)\right]\\ &= F(y)\exp\left[i(\alpha x-\beta t)\right]\end{aligned}\right\} \qquad (6.13,8)$$

Here $\alpha = 2\pi/l$, and l is the wave length of the disturbance, c is in general complex

$$= c_r + ic_i$$

where c_r = wave (or phase) velocity,
and, if $c_i > 0$, the disturbance is amplified with time, i.e. the boundary layer is unstable,
if $c_i < 0$, the disturbance damps out with time.

We see that

$$\beta_r/\alpha = c_r, \quad \text{and} \quad \beta_i/\alpha = c_i.$$

The quantity β_i can be regarded as a coefficient of amplification. If we now express lengths non-dimensionally in terms of δ^*, say, and velocities in terms of some standard velocity u_1, and we write

$$F(y) = u_1\,\delta^*\phi(\eta) \qquad \text{where} \qquad \eta = y/\delta^*,$$

then an equation for ϕ of the following form is obtained from the equations of motion when linearised in the disturbance components:

$$(u-c)(\phi''-\alpha^2\phi) - u''\phi = \frac{1}{i\alpha R_{\delta^*}}(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi) \qquad (6.13,9)$$

where $R_{\delta^*} = u_1\delta^*/\nu$, with the boundary conditions

$$\eta = 0,\ \phi = 0,\ \phi' = 0; \quad \eta = \infty,\ \phi = 0,\ \phi' = 0.$$

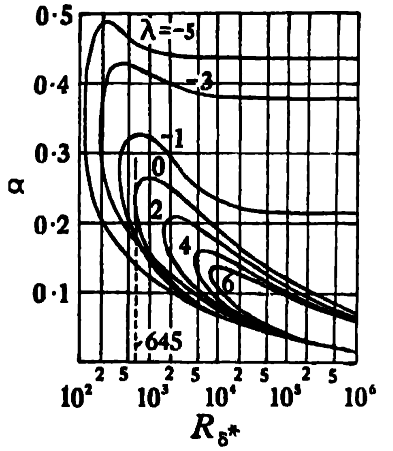
The mathematical details of the solution of this equation are too complex to be presented here, and we shall confine ourselves to a brief discussion of certain salient features of the results obtained.

The equation and its boundary conditions can be shown to constitute what is known as a boundary value problem, and this implies that for any given velocity profile, $u(y)$, there exists a functional relation between the parameters α, R_{δ^*} and c which the solution must satisfy. Thus, if R_{δ^*} and wavelength $l = 2\pi/\alpha$ are specified, c_r and c_i are determined by this parametric relation in the form

$$c_r = c_r(\alpha, R_{\delta^*}), \quad c_i = c_i(\alpha, R_{\delta^*}).$$

Since neutral stability is obtained when $c_i = 0$, it follows that α can be determined as a function of R_{δ^*} for neutral stability from the second of these relations, and this function can be presented graphically on an α, R_{δ^*} diagram separating areas of boundary layer stability and instability. From the first of the above parametric relations the phase velocity of the disturbance, c_r, can be determined. An alternative presentation of the neutral stability curve can therefore be made on a c_r, R_{δ^*} diagram. The amplification factors can also be determined in the regions of instability ($c_i > 0$).

Calculated neutral stability curves determined by Schlichting[1] are shown in Fig. 6.13,1 for the boundary layer on a flat plate with zero pressure gradient and with various positive and negative pressure gradients corresponding to various values of the parameter $\lambda = (\delta^2/\nu)u_1'$. Here $\alpha = 2\pi\delta^*/l$.



The interior of each curve represents a region of instability, the exterior is a region of stability. (Reproduced by kind permission of Prof. Schlichting from *Boundary Layer Theory*, Pergamon Press, 1955.)

Fig. 6.13,1. Curves, calculated by Schlichting, of neutral stability for laminar boundary layer profiles determined by the single parameter Pohlhausen method. The parameter $\lambda = u_1'\delta^2/\nu$.

For any one velocity profile the boundary layer is unstable for values of α and R_{δ^*} lying within the appropriate neutral curve and stable for values outside. We note that in each case there is a minimum Reynolds number, R_{δ^*}, below which the boundary layer is completely stable to all small disturbances, but for Reynolds numbers greater than this minimum there is a band of wavelengths of disturbances to which the boundary layer is unstable. Hence, the tendency to instability of a given velocity profile can be measured by two factors—firstly, this minimum or critical Reynolds number, the higher this is the longer is instability delayed on a given surface; secondly, the relative extent of the band of wavelengths for which the boundary layer is unstable at a given Reynolds number. We see therefore that there is a marked deterioration of the stability of the boundary layer as the external pressure gradient goes from zero to positive ($\lambda < 0$), whilst a negative pressure gradient ($\lambda > 0$) is associated with some improvement of stability. The degree of instability of the boundary layer can be assessed by the degree of amplification that occurs in the region of instability and this can be gauged by the maximum value of β_i in that region. Pretsch[2] has determined the neutral curves and amplification factors for the similar velocity profiles associated with external velocity distributions $u_1 = \text{const.}\,x^m$, discussed in § 6.8. The following table lists the critical Reynolds numbers and maximum amplification factors obtained by Pretsch for various values of *m*.

m	$-0{\cdot}048$	0	$0{\cdot}111$	$0{\cdot}25$	$0{\cdot}43$	$1{\cdot}0$
R_{δ^*}	126	660	3200	5000	8300	12600
$(\beta_i)_{\max}10^4$	155	34·5	14·6	—	9·1	7·3

In so far as instability may be regarded as a prelude to transition, these results are in close accord with experience, since it is well known that the effect of a positive (adverse) pressure gradient is to provoke transition very readily at all but very low Reynolds numbers. Likewise transition can be delayed by a negative (favourable) pressure gradient provided that there are no extraneous finite disturbances in the form of surface imperfections or large scale turbulence in the external stream.

It will be noted that the theory involves the not inconsiderable assumption that the stability of the boundary layer at any station on a surface is a function only of its local velocity profile, and the previous history of the layer and its development with distance along the surface play no explicit part.† Further, it tells us nothing about the known effect of finite disturbances in provoking transition nor does it tell us anything about the mechanism of transition. In some cases the calculated position of the beginning of instability is found to be close to the measured position of transition, but in other cases transition is found to occur considerably after instability is calculated to begin. The former cases usually involve strong adverse external pressure gradients where in any case separation would ensue soon after instability develops if transition to turbulent flow did not occur. For these reasons the theory was treated with reserve in some quarters until a classic series of experiments by Schubauer and Skramstad in America[1] provided conclusive evidence that within the context of instability to small disturbances the theory was substantially right, at least for the boundary layer on a flat plate in a stream of low turbulence. Schubauer and Skramstad demonstrated the existence of a neutral curve such as is illustrated in Fig. 6.13,1 and obtained substantial quantitative agreement with the theoretical predictions of the detailed behaviour of the oscillations in the boundary layer. They demonstrated that if the turbulence of the main stream was low (i.e. its intensity (see § 6.14) was less than 0·03% of the main stream velocity) sinusoidal oscillations of the kind considered by the theory appeared naturally in the boundary layer well ahead of transition, and they were able to investigate the behaviour of artificially induced oscillations over a wide range of frequency. These were produced by vibrating a metal ribbon held close to the surface and subjected to an oscillating electro-magnetic field.

The foregoing refers to the stability of the boundary layer on a plane wall and it will be noted that only two-dimensional disturbances are considered in the theory. However, Squire[1] has shown that instability due to such disturbances must occur earlier than that due to three-dimensional disturbances if the wall is plane, so that the analysis covers the most critical case. However, if the surface is concave then Görtler[2] has shown that instability can develop to a form of three-dimensional disturbance consisting of vortex-like eddies with their axes in the streamwise direction, the rotation in neighbouring eddies being in opposite directions (see Fig. 6.13,2).

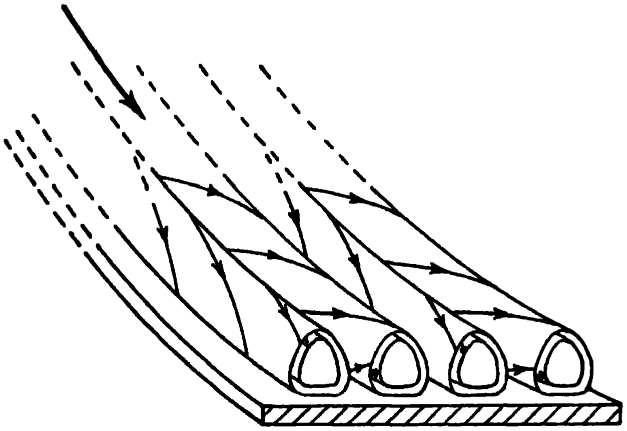


Fig. 6.13,2. Pattern of three-dimensional disturbances that develop in the flow on a concave wall, according to Görtler.

Very similar eddy formations were previously shown by G. I. Taylor[3] to develop in the analogous flow between two concentric cylinders, with the outer cylinder at rest and the inner one rotating (see also § 3.10). Görtler's theory shows that instability can develop if

$$\frac{u_1\theta}{\nu}\sqrt{\frac{\theta}{r}} > 0{\cdot}58$$

where r is the radius of curvature of the wall. In practice this implies a very early tendency to instability even for quite large values of r.† Experimental investigations by Liepmann[4] have confirmed this tendency with consequent transition; the latter was found to occur for a value of θ given by

$$\frac{u_1\theta}{\nu}\sqrt{\frac{\theta}{r}} = 7{\cdot}3.$$

6.13.5 The Transition Process and Turbulence

It would seem from the experimental evidence that the onset of turbulence requires the presence of disturbances of some definite size, and the Reynolds number must be sufficiently high. In the absence of extraneous sources of such disturbances, e.g. external turbulence or surface imperfections which shed eddies into the flow, they may be the result of amplification of small disturbances in conditions of instability as predicted by the stability theory. Outstanding questions that arise are what is the physical process whereby turbulence develops as a result of the introduction of such disturbances, and how large must the disturbances be to provoke transition?

These questions are far from answered at present, but a few pointers towards their answers are worth summarising. G. I. Taylor[1] suggested that if the pressure gradients associated with the disturbances were large enough when combined with the main stream gradient to cause local transient separation, then transition would follow almost immediately. On the basis of this hypothesis he obtained good agreement with the results of experiments on a sphere in a wind tunnel with a high degree of turbulence in the main stream. However, there is as yet no direct experimental confirmation of this hypothesis. If it were correct it would provide an answer to the second question postulated above, but it throws no obvious light on the first.

Perhaps the most promising development as far as the first question is concerned is the work of Emmons[2] and subsequently of Schubauer and Klebanoff[3]. Emmons suggested, as a result of experiments with a water table, that turbulence develops initially not along a continuous front but at isolated spots, and each spot as it passes downstream then grows into an expanding region of turbulent flow. The spots occur with a random distribution in space and time. As the fluid flows downstream from the beginning of the transition region the number of spots increases and eventually the resulting expanding regions of turbulent flow coalesce to form continuous and fully developed turbulent flow in the boundary layer. Where conditions are such as to favour instability, e.g. adverse pressure gradient, surface imperfections, external turbulence, high enough Reynolds number, the frequency of occurrence of the spots is high and the transition region is short, but in favourable conditions this frequency is low and either transition will not occur at all or else will take place very gradually. From a continuous source of disturbance, e.g. a small protuberance, a continuous stream of rapidly growing turbulence spots pass downstream and almost immediately coalesce to form a wedge of turbulent flow (see Fig. 6.13,3). Such wedges have been long observed experimentally; the apex angle is generally of the order of 20°, although it is somewhat dependent on external pressure gradient, surface curvature, etc.

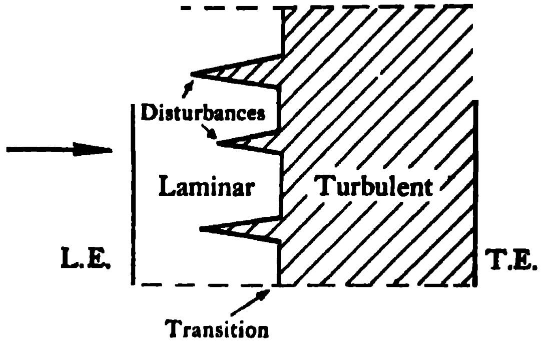


Fig. 6.13,3. Typical turbulence 'wedges' from isolated protuberances on a wing surface.

Schubauer and Klebanoff investigated disturbance patterns in a laminar boundary layer on a flat plate in a wind tunnel, using the 'hot wire' technique for measuring the disturbance velocity components.† They confirmed Emmons' hypothesis and provided further details of the shape and speed of the individual turbulent regions as they grew from the initial spots. Such disturbance spots were introduced by a spark discharged to the surface through the boundary layer. Typically the resulting turbulence region was roughly triangular in form with apex downstream, as illustrated in Fig. 6.13,4. The leading edges moved downstream with a velocity very nearly the main stream velocity whilst the base moved at about half the main stream velocity, and it expanded outwards at such a rate as to define the sides of a typical 'turbulence wedge' with apex at the point of initiation.

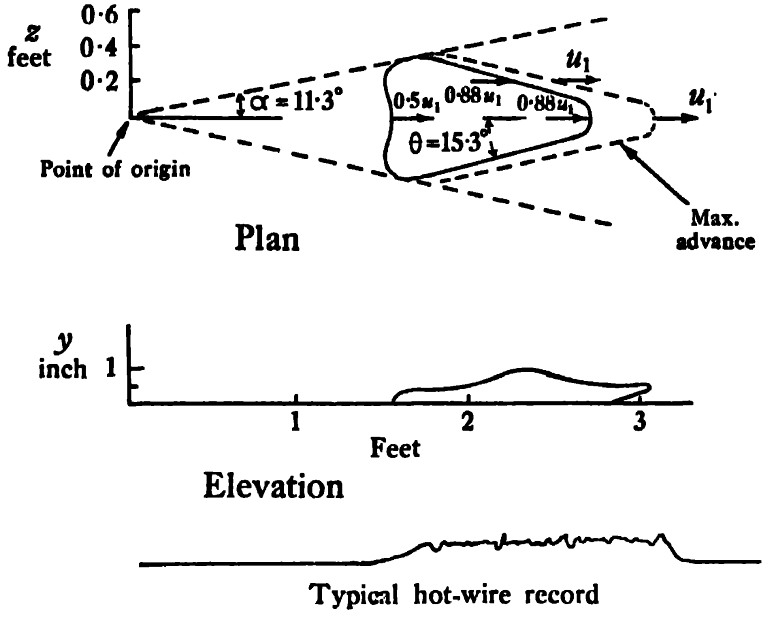


Fig. 6.13,4. Turbulent region moving downstream after initiation at disturbance spot by spark discharge.

An interesting feature of these turbulent regions was that each was followed, according to its hot wire oscillograph records, by a region of unusual steadiness to small disturbances. This feature may be explained by the argument that the velocity profile of the boundary layer after passage by the turbulent region is close to that of turbulent flow and is therefore particularly stable to small disturbances, but after a short period it reverts to the laminar velocity profile and its immunity to small disturbances disappears.

Another result of some importance demonstrated by Schubauer and Klebanoff is that the mean velocity profiles as measured by a pitot-static tube change continuously from typically laminar to typically turbulent profiles with distance downstream from the front of the transition region, where spots in significant numbers first begin to appear, to the rear of the transition region, where the resulting triangular areas of turbulence have coalesced. This is illustrated in Fig. 6.13,5.

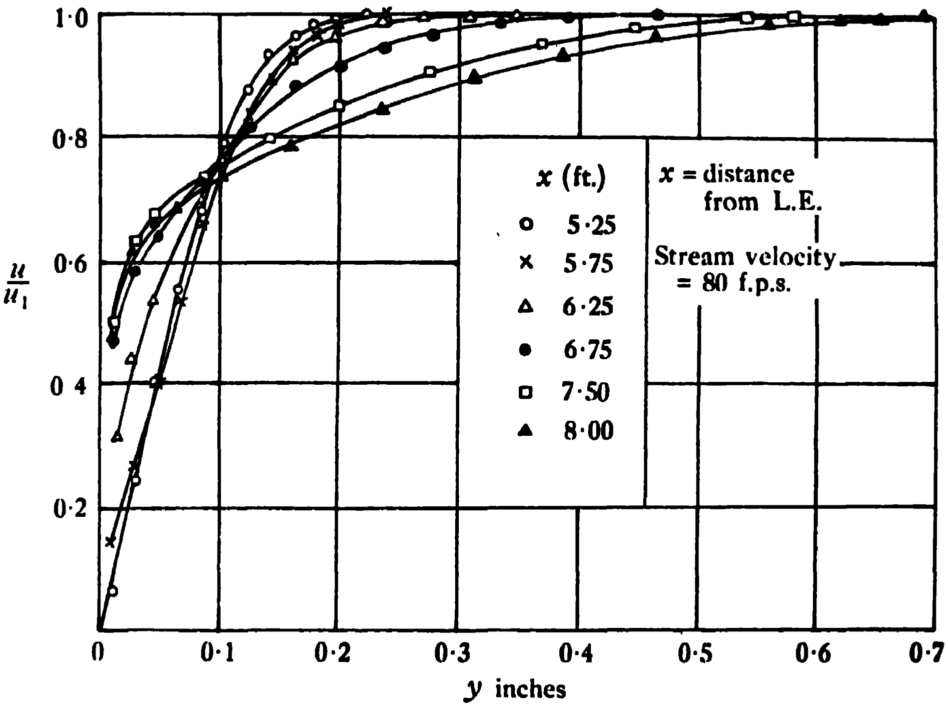


Fig. 6.13,5. Typical mean velocity profiles through transition region, as measured by Schubauer and Klebanoff on a flat plate at zero incidence.

As a result of further investigations Schubauer[1] suggested that the two-dimensional waves of classical stability theory become turbulence spots because of pre-existing irregularities which cause a spanwise warping of the waves. According to this theory, this warping increases with distance downstream, forming peaks and valleys of high and low disturbance intensity with different amplification rates. At a certain stage the rate of growth of the peaks exceeds the amplification rate predicted by linearised perturbation theory, whilst that of the valleys becomes less than that predicted by the theory, and the development of the spanwise velocity fluctuation component, w', causes a significant transfer of energy from the valleys to the peaks. The peaks thus form streets of high intensity disturbances which at some stage develop high frequency fluctuations and each peak can then be identified with a turbulence spot. These then grow with convection downstream as already described.

It remains for theory to examine the stability and response of the boundary layer to finite disturbances, and in some way to explain the development of turbulence spots at points of high disturbance intensity. A step in this direction has been achieved by Stuart[1] who has considered the energy balance in the interaction of finite disturbance waves and the mean boundary layer flow, taking account of the effect on the latter of the Reynolds stresses associated with the disturbances. He has shown that quite marked changes in the velocity profile can occur as a result of these Reynolds stresses, and this raises the possibility of the changed profile being unstable to some new pattern of disturbance and so on. Thus it is possible to conceive of a kind of chain reaction in which we end finally with a velocity profile in energy equilibrium with a wide spectrum of disturbances, including disturbances of high frequency, and the boundary layer can then be identified as turbulent.

6.14 Turbulence and the Structure of the Turbulent Boundary Layer

Although the physical nature of turbulence and its development is far from clear, it can be studied both experimentally and theoretically as a statistical phenomenon, and in this way many important aspects of the structure of turbulent flows have been revealed. This statistical approach to turbulence is not one that can be dealt with here, and only a very few salient points will be touched upon. For further details of this and other aspects of turbulent flow reference should be made to monographs on the subject by Batchelor[2], Townsend[3] and Rotta[4].

The mean intensities of the turbulent velocity components are conveniently represented by their root mean squares, viz.:—

$$\sqrt{(\overline{u'^2})},\ \sqrt{(\overline{v'^2})},\ \sqrt{(\overline{w'^2})}$$

and the overall mean turbulence intensity is represented by

$$\tfrac{1}{3}[\sqrt{(\overline{u'^2})} + \sqrt{(\overline{v'^2})} + \sqrt{(\overline{w'^2})}].$$

The correlation between turbulent velocity components at two points, u_1' and u_2', say, is represented by the correlation coefficient:

$$R = \frac{\overline{u_1'u_2'}}{\sqrt{(\overline{u_1'^2})}\sqrt{(\overline{u_2'^2})}}. \qquad (6.14,1)$$

Complete correlation, in the sense that u_2' can be represented as a unique function of u_1', implies $|R| = 1{\cdot}0$, whilst if $R = 0$ the two flow turbulence patterns are not in any sense linked. A close link between the two patterns associated with a value of R near unity suggests that both points are readily included in an average eddy; conversely, if $R = 0$ we can infer that the distance between the points is larger than an average eddy. It follows that a useful measure of the size of average eddy in the region of point 1 is

$$L = \int_0^\infty R(s)\, ds \qquad (6.14,2)$$

where s is the distance between points 1 and 2 and the position of 2 is varied along a line through 1 in some preferred direction.

It will be evident that a number of such double and higher order correlation coefficients can be studied; these correlation coefficients play a big part in the statistical interpretation of turbulent flow.

Another important concept in the statistical theory of turbulence is that of the energy spectrum. Each turbulence component can be regarded as compounded of oscillations of differing frequency and amplitude, so that if the proportion of the total disturbance energy associated with that component lying within a narrow frequency band n and $n + \delta n$ is $f(n)\, \delta n$, say, then the function $f(n)$ is referred to as the energy spectrum function of the turbulence. A typical distribution of $f(n)$ for channel flow is shown in Fig. 6.14,1.

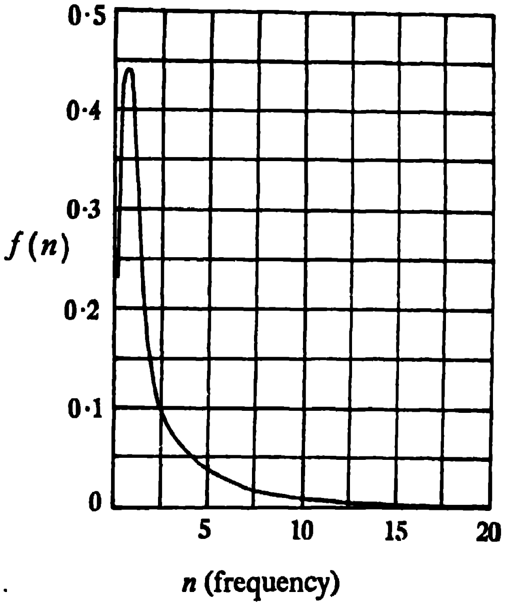


Fig. 6.14,1. Typical spectral distribution of energy for $\overline{u'^2}$ for channel flow (Motzfeld[1]).

The general pattern of eddy distribution in the turbulent boundary layer is broadly speaking as follows:[2,3] Over the outer part of the layer we find a region of predominantly large scale eddies. In this region the turbulence is not continuous but is intermittent. If an intermittency factor, γ, is defined by the fraction of time over a reasonably long period that turbulence is recorded at a given point, then γ increases from 0 at a distance from the surface equal to about $1{\cdot}2\delta$ to unity at about $0{\cdot}4\delta$, where δ is the boundary layer thickness as indicated by a total head tube. The intermittent nature of the eddy formations in this region and the large size of the eddies, which are of the order of δ, result in the instantaneous edge of the boundary layer presenting an irregular wavy appearance (see Fig. 6.14,2). This region is one of relatively uniform mean velocity due to the considerable and continuous entrainment of fluid from outside the layer that results from the large eddies. It is also a region of relatively low shear stress. It is frequently referred to as the *outer region.*

Below this region of large scale intermittent eddies we find a fully turbulent region (the *inner region*) extending from about $0{\cdot}4\delta$ down to the *laminar sub-layer.* This sub-layer is a very narrow region of flow adjacent to the wall, in which the turbulent fluctuations become small in absolute magnitude,† and the dominant shear stress is the purely viscous one, $\mu\, \partial u/\partial y$, which is constant in the sub-layer. In the fully turbulent inner region above it, on the other hand, the shear stress is dominated by the turbulent contribution, $-\rho\overline{u'v'}$ which is very much greater than the viscous stress. In this region a wide spectrum of turbulence frequencies is found, with the larger, low frequency eddies further away from the surface, and it appears that the larger eddies extract energy from the mean flow and pass it downwards to the smaller eddies. The smallest eddies dissipate the energy as heat, due to the action of viscosity. The eddy shear stress reaches a maximum in this region, near the surface it is roughly constant and equal to the viscous shear stress in the laminar sub-layer.

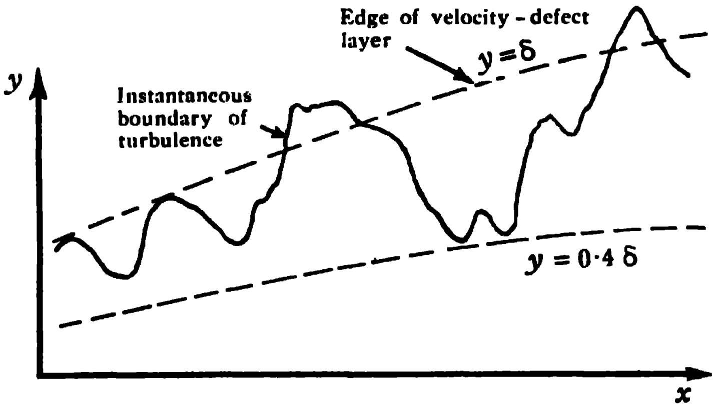

Edge of velocity - defect layer
$y = \delta$
Instantaneous boundary of turbulence
$y = 0{\cdot}4\,\delta$
y
x

Fig. 6.14,2. Sketch illustrating intermittent character of turbulent flow in outer part of boundary layer.

The boundary layer in a pipe or channel presents a similar picture, except that the 'inner region' extends over a much greater proportion of the layer. Further, whereas in the boundary layer on a plate the turbulence intensity falls off to near zero as the edge of the boundary layer is approached from the wall, in a pipe or channel the turbulence intensity decreases less rapidly to a value at the centre of the pipe channel about a quarter of the maximum intensity found close to the wall (see Fig. 6.14,3).

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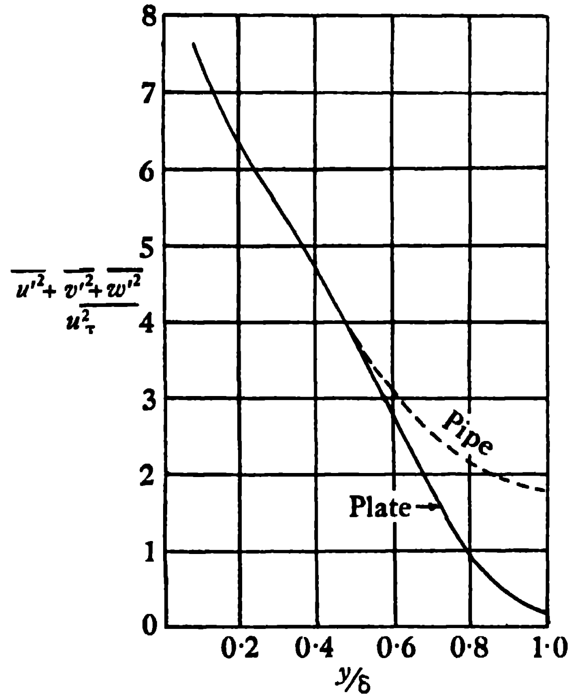
† The ratio of these fluctuations to the local mean velocity remains large, however.

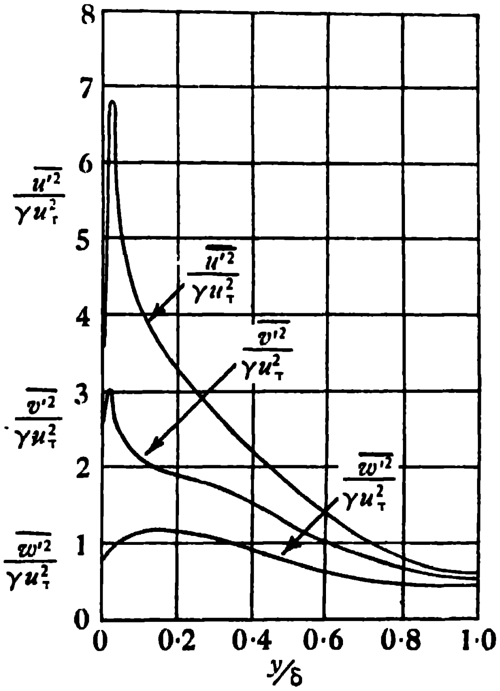
A convenient reference velocity for the turbulent boundary layer is the so-called friction velocity, u_τ, where

$$u_\tau = (\tau_w/\rho)^{\frac{1}{2}} \tag{6.14,3}$$

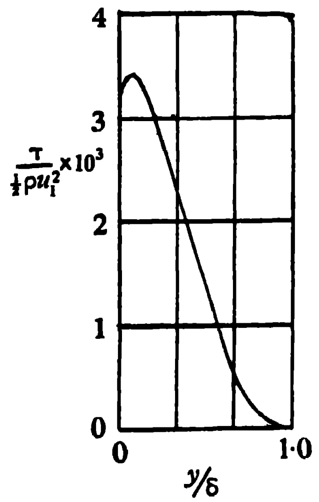
and τ_w is the frictional stress at the wall.



Typical distribution of total turbulent energy in boundary layer of flat plate and pipe. (Schubauer)



Typical distribution of turbulent components in boundary layer on flat plate (Schubauer). γ denotes intermittency factor



Typical shear stress distribution in boundary layer on flat plate (Schubauer and Klebanoff)

Fig. 6.14,3.

In the laminar sub-layer the mean velocity is practically linear with distance from the wall, i.e.

$$u = y\left(\frac{\partial u}{\partial y}\right)_w = \frac{y}{\mu}\tau_w$$

or

$$\frac{u}{u_\tau} = \frac{yu_\tau}{\nu} \tag{6.14,4}$$

The thickness of the laminar sub-layer, y_l, is given by

$$y_l u_\tau/\nu = 10, \text{ approx.} \tag{6.14,5}$$

Typical distributions of turbulent velocity components in the boundary layer on a flat plate at zero incidence and in a pipe and of shear stress in the boundary layer on the plate are also shown in Fig. 6.14,3. These are due to Schubauer and Klebanoff.[1,2]

The total shear stress in the two dimensional turbulent boundary layer can be written

$$\tau = \mu \frac{\partial u}{\partial y} - \rho\overline{u'v'} \tag{6.14,6}$$

although, as already remarked, in all but the narrow laminar sub-layer the viscous term $\{\mu(\partial u/\partial y)\}$ is negligible compared with the eddy stress term.

The boundary layer momentum equation (§ 6.3) is

$$(u + u')\frac{\partial}{\partial x}(u + u') + (v + v')\frac{\partial}{\partial y}(u + u')$$
$$= -\frac{1}{\rho}\frac{d}{dx}(p + p') + \nu\frac{\partial^2}{\partial y^2}(u + u')$$

and taking the mean of this equation we get

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} - \overline{u'\frac{\partial u'}{\partial x}} - \overline{v'\frac{\partial u'}{\partial y}}. \tag{6.14,7}$$

But the equation of continuity is

$$\frac{\partial}{\partial x}(u + u') + \frac{\partial}{\partial y}(v + v') = 0$$

of which the mean yields

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{and hence} \qquad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0.$$

Therefore

$$-u'\frac{\partial u'}{\partial x} - v'\frac{\partial u'}{\partial y} = -u'\frac{\partial v'}{\partial y} - v'\frac{\partial u'}{\partial y} - \frac{\partial}{\partial x}(u'^2) = -\frac{\partial}{\partial y}(u'v') - \frac{\partial}{\partial x}(u'^2).$$

On the basis of the usual boundary layer assumptions we can neglect the last term on the right-hand side and thus (6.14,7) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial y}(\overline{u'v'}). \tag{6.14,8}$$

We can confirm that, with the shear stress as given by equation (6.14,6) above, this equation is the same as equation (6.3,4).

The argument which led to the conclusion in § 6.3 that the pressure could be taken as constant across the laminar boundary layer applies essentially unchanged to the turbulent boundary layer. The boundary layer thickness, however, is no longer of the order indicated by equation (6.3,6), since the eddy stress is a dominant parameter of the flow and the viscous stress is no longer significant except in the laminar sub-layer. However, as we shall see, the boundary layer thickness still remains small, and it satisfies an equation of the form of equation (6.3,6) but with the exponent of R about $-1/5$ instead of $-1/2$.

6.15 Mixing Length Theories. Velocity Distributions

6.15.1 General Remarks

The rough analogy that exists between the relatively large scale or molar random movements of fluid elements in turbulent motion and the small scale random movements of molecules in a gas stimulated workers at an early stage in the history of the subject to explore the consequences of postulating a mixing length for turbulent motions analogous to the mean free path of molecular movements. The mixing length was thus thought of as a mean distance travelled by a fluid particle over which it retained its initial values of velocity or vorticity, and at the end of which it mixed with its surroundings.

Three main theories were developed, the Momentum Transport Theory, due to Prandtl[1], in which it is assumed that momentum was conserved over the mixing length, the Vorticity Transport Theory, due to Taylor[2], in which vorticity was conserved, and the Similarity Theory[3], due to v. Kármán, which made use of the assumption of similarity of turbulence patterns at all points. The first has the virtues of simplicity and ease of application; the assumptions of the second are easier to accept, since the conservation of vorticity is more readily justified than is the conservation of momentum, but the analysis and application are much more complicated; the third method is somewhat more elegant but is no more easily justified than the other two.

With the aid of suitable adjustments of empirical constants all three methods can be made to provide answers to particular problems in reasonable agreement with experimental results, and on that evidence no one method is consistently superior to the other two, although for any specific application one method may be found to be somewhat better than the other methods. However, the physical bases of these methods do not bear close examination. The concept of a mixing length over which a fluid particle retains its identity and certain properties are conserved is not readily consistent with motion in a continuous medium in which the motion of a fluid element is dependent on and influences the motion of the surrounding fluid. The energy to maintain turbulent motion is provided by the work done by the Reynolds stresses on the mean flow, whilst the mean flow is strongly affected by the diffusive action of the turbulence. These aspects of the intimate interaction between turbulence and mean flow hardly figure in a mixing length theory. Further, it is an assumption of the mixing length theories that the mixing length is small compared with the boundary layer thickness, so that changes in mean flow quantities over a distance normal to the wall equal to the mixing length can be readily related to the local gradient of the mean flow quantities with respect to distance from the wall. Experimental data, however, yield on analysis estimates for the mixing length that are frequently too large for this assumption to apply.

Nevertheless, it can be argued that mixing length theories serve a useful purpose in yielding semi-empirical formulae of considerable practical utility, and in providing a starting point for more detailed and comprehensive investigations. It must, however, be noted that such success as has attended the applications of the theories can be shown to derive more from their common sub-stratum of general dimensional reasoning than from the particular physical assumptions which appear to characterise them. Indeed, dimensional arguments can by themselves be used to provide most of the more important results without reference to mixing lengths.[1,2]

In what follows a brief description will be given of the application of the Momentum Transport Theory and the Similarity Theory to the flow in the inner region of the boundary layer over a plane wall with zero external pressure gradient. These are presented for their didactic as well as historic interest, as much may be learnt from them if the above points are borne in mind. The Vorticity Transport Theory is not included because of its greater complication; a full description of it will be found, however, in *Modern Developments in Fluid Dynamics*, Vol. I.† An argument based on dimensional analysis due to Squire is presented for comparison.

6.15.2 **Momentum Transport Theory**

Consider a mean flow such that u is a function of y only (see Fig. 6.15,1). The turbulence is assumed to be two dimensional with components u' and v'. The momentum transport theory then says, in effect, that fluid particles at height y_1, with mean velocity $u(y_1)$, say, will on the average have come from levels $y_1 - l_1$, and $y_1 + l_1$, where l_1 is the mixing length, retaining their initial velocities $u(y_1 - l_1)$ and $u(y_1 + l_1)$, respectively, before mixing at height y_1. Thus, these velocities represent on the average the overall spread of instantaneous velocities at that height, i.e. $u - \sqrt{(\overline{u'^2})}$, and $u + \sqrt{(\overline{u'^2})}$. Since, with the assumption that l_1 is small, we can write

$$u(y_1 - l_1) = u(y_1) - l_1\frac{du}{dy}, \qquad u(y_1 + l_1) = u(y_1) + l_1\frac{du}{dy},$$

we have therefore

$$\sqrt{(\overline{u'^2})} = l_1\frac{du}{dy}. \qquad (6.15,1)$$

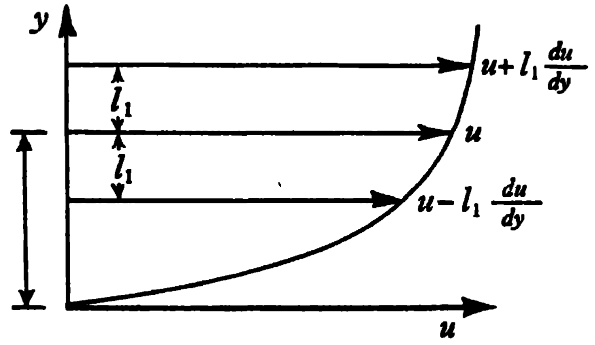


Fig. 6.15,1.

We see that a strong negative correlation between u' and v' is implicit in this argument, since a positive u' is associated with a negative v' and a negative u' is associated with a positive v'.

It is now argued that u' and v' are in general of the same order of magnitude but of opposite sign, so that the Reynolds shear stress can be written

$$\tau = -\rho\overline{u'v'} = \text{const. } \rho l_1{}^2\left(\frac{du}{dy}\right)\left|\frac{du}{dy}\right|,$$

since τ is positive when du/dy is positive. The unknown constant can be compounded with the equally unknown $l_1{}^2$ so that we write

$$\tau = \rho l^2\left|\frac{du}{dy}\right|\left(\frac{du}{dy}\right) \qquad (6.15,2)$$

where l is proportional to the mixing length.

We see from equation (6.15,2) that $\rho l^2|du/dy|$ can be regarded as an effective viscosity coefficient, and it is usually denoted by ε. Thus the total stress (eddy stress plus viscous stress) is

$$\tau = (\varepsilon + \mu)\, du/dy, \qquad (6.15,3)$$

but we note that, except in the laminar sub-layer, $\varepsilon \gg \mu$.

If we confine our attention to the lower part of the inner region of the turbulent layer where the eddy stress is practically constant, and equal to τ_w, the shear stress at the wall, and we exclude the laminar sub-layer, then it follows that

$$\tau_w = \varepsilon\, du/dy = \rho l^2\left|\frac{du}{dy}\right|\frac{du}{dy} \qquad (6.15,4)$$

We must now make some further assumption about the length l. The simplest assumption is that it is proportional to the distance y from the wall, i.e. we write

$$l = ky \tag{6.15,5}$$

where k is some constant.

Then, with du/dy positive, equation (6.15,4) becomes

$$\tau_w = k^2 \rho y^2 \left(\frac{du}{dy}\right)^2 \quad \text{or} \quad \frac{du}{dy} = u_\tau/ky,$$

where u_τ is the friction velocity, $(\tau_w/\rho)^{\frac{1}{2}}$.

Integrating this last equation with respect to y we obtain

$$u = \frac{u_\tau}{k} \cdot \ln y + \text{const.} \tag{6.15,6}$$

This formula obviously breaks down for $y = 0$, i.e. in the close neighbourhood of the wall. It is clear from the analysis that the method must become invalid there, since it takes no account of the laminar sub-layer where the eddy stress ceases to be dominant and the viscous stress becomes important. Thus, if we assume that equation (6.15,6) is valid down to $y = y_l$ at the edge of the laminar sub-layer then it follows that

$$u - u(y_l) = \frac{u_\tau}{k} \ln (y/y_l).$$

But on dimensional grounds we may expect y_l to depend only on the frictional velocity u_τ and the kinematic viscosity ν, it follows that

$$y_l = \text{const.}\ \nu/u_\tau = \beta\nu/u_\tau, \text{ say,}$$

and we find from experimental results that β is of the order of 10.

From equation (6.14,4)

$$u(y_l) = y_l u_\tau^2/\nu = \beta u_\tau \qquad \text{and therefore} \qquad u - \beta u_\tau = \left[\frac{u_\tau}{k} \ln \left(\frac{y u_\tau}{\beta \nu}\right)\right]$$

or

$$\frac{u}{u_\tau} = A \ln \left(\frac{y u_\tau}{\nu}\right) + B \tag{6.15,7}$$

where $A = 1/k$, $B = \beta - (1/k) \ln \beta$.

The theory is not such that we could hope to rely on more than the form of equation (6.15,7), but since in the inner region u must be a function of u_τ, y, and ν, if we neglect the influence of the previous history of the boundary layer, we would expect on dimensional grounds that

$$\frac{u}{u_\tau} = f(y u_\tau/\nu), \tag{6.15,8}$$

and the theory therefore indicates the nature of the function f. In this form it is sometimes referred to as the *inner velocity law* or *law of the wall*.

6.15.3 Von Kármán's Similarity Theory

The basic assumption of this theory is that the turbulence patterns at all points in the turbulent boundary layer are similar except for changes of linear scale and time. It is then argued that the length l can depend only on the local values of u, du/dy, d^2u/dy^2, d^3u/dy^3, etc. However, since u can be arbitrarily changed by altering the velocity of the frame of reference it cannot influence the length l, and, if we neglect the possible influence of d^3u/dy^3 and higher derivatives, we are left with l depending only on du/dy and d^2u/dy^2. By dimensional reasoning, since the only non-dimensional quantity that can be formed from l, du/dy, and d^2u/dy^2 is

$$l\frac{d^2u}{dy^2}\bigg/\frac{du}{dy}$$

it follows that

$$l = \text{const.}\left|\frac{du}{dy}\bigg/\frac{d^2u}{dy^2}\right|. \tag{6.15,9}$$

Thus the theory provides a formula for l.

Further, the similarity hypothesis leads one to conclude that $\sqrt{(\overline{u'^2})}$ and $\sqrt{(\overline{v'^2})}$ are dependent only on l, du/dy and d^2u/dy^2, and again dimensional reasoning shows that these mean turbulence components are of the form

$$\text{const.}\, l\frac{du}{dy}\bigg/f\left[l\bigg/\left(\frac{du}{dy}\bigg/\frac{d^2u}{dy^2}\right)\right]$$

and, in view of (6.15,9), we deduce that

$$\sqrt{(\overline{u'^2})} = \text{const.}\, l\left|\frac{du}{dy}\right|, \qquad \sqrt{(\overline{v'^2})} = \text{const.}\, l\left|\frac{du}{dy}\right|.$$

Hence, it follows that the eddy stress

$$\tau = \text{const.}\ \rho l^2\left|\frac{du}{dy}\right|\left(\frac{du}{dy}\right)$$

and as before we can absorb the constant in l^2 and we arrive at the relation

$$\tau = \rho l^2\left|\frac{du}{dy}\right|\left(\frac{du}{dy}\right)$$

which is the same as the momentum transport theory relation (equation 6.15,2).

Considering, as before, the region of constant shear stress with du/dy positive, we see that

$$l = u_\tau\bigg/\frac{du}{dy}.$$

which with equation (6.15,9) gives

$$\left(\frac{du}{dy}\right)^2 = \text{const. } u_\tau \frac{d^2u}{dy^2} = -\frac{u_\tau}{k}\frac{d^2u}{dy^2}, \text{ say.}$$

The solution of this equation is of the form

$$u = \frac{u_\tau}{k} \ln (y - y_\varepsilon) + \text{const.} \qquad (6.15,10)$$

in which y_ε can be interpreted as a shift of the origin of y from the wall to a suitable point nearby where similarity of turbulence patterns can be assumed to hold. Apart from this possible shift of origin it will be seen that equation (6.15,10) and the corresponding equation of the momentum transport theory, equation (6.15,6), are identical. This agreement can be seen to derive from the fact that for this particular problem of a simple shear flow with the eddy stress constant the relation between the mixing length and y as given by equation (6.15,9) reduces to a linear one, as assumed in the momentum transport theory (equation 6.15,5). However, in general, for problems involving a variation of the eddy stress with y the two methods result in different formulae, although the numerical results are frequently close.

It may be noted that experimental measurements of turbulence in a boundary layer do not lend any general support to the assumption of similarity of turbulence patterns; although some degree of similarity is to be found in the narrow region near the wall where the eddy stress is nearly constant. Further it will be clear that the theory will present difficulties in cases where d^2u/dy^2 is zero at some point.

6.15.4 Squire's[1] Dimensional Theory and the Inner Velocity Distribution

The agreement between the numerical results predicted by the various mixing length theories in many cases suggests that the underlying dimensional reasoning common to the theories plays a more vital part than the remaining characteristic assumptions. This consideration led Squire to evolve a proof of equations (6.15,6) and (6.15,8) on the basis of dimensional analysis as follows.

We assume that we can write for a region of constant shear stress

$$u_\tau^2 = \frac{\tau_w}{\rho} = \varepsilon' \frac{du}{dy} \qquad (6.15,11)$$

where ε' is the effective or eddy kinematic viscosity, and the viscous stress is neglected.

Now ε' can depend only on u_τ and y, if the effect of viscosity is neglected, and since ε' has the dimensions $(\text{length})^2/\text{time}$ it must be of the form

$$\varepsilon' = k u_\tau y \qquad (6.15,12)$$

where k is a constant.

From equations (6.15,11) and (6.15,12) we immediately obtain

$$u = \frac{u_\tau}{k} \ln y + \text{const.}$$

in agreement with equation (6.15,6).

The argument is readily taken a stage further by noting that ε' can only be related to the distance from the wall as given in equation (6.15,12) if y is large compared with the laminar sub-layer, and more accurately it should be related to the distance from a point y_0, where y_0 is a point simply related to the thickness of the laminar sub-layer and therefore proportional to ν/u_τ. Thus, instead of (6.15,12) we write

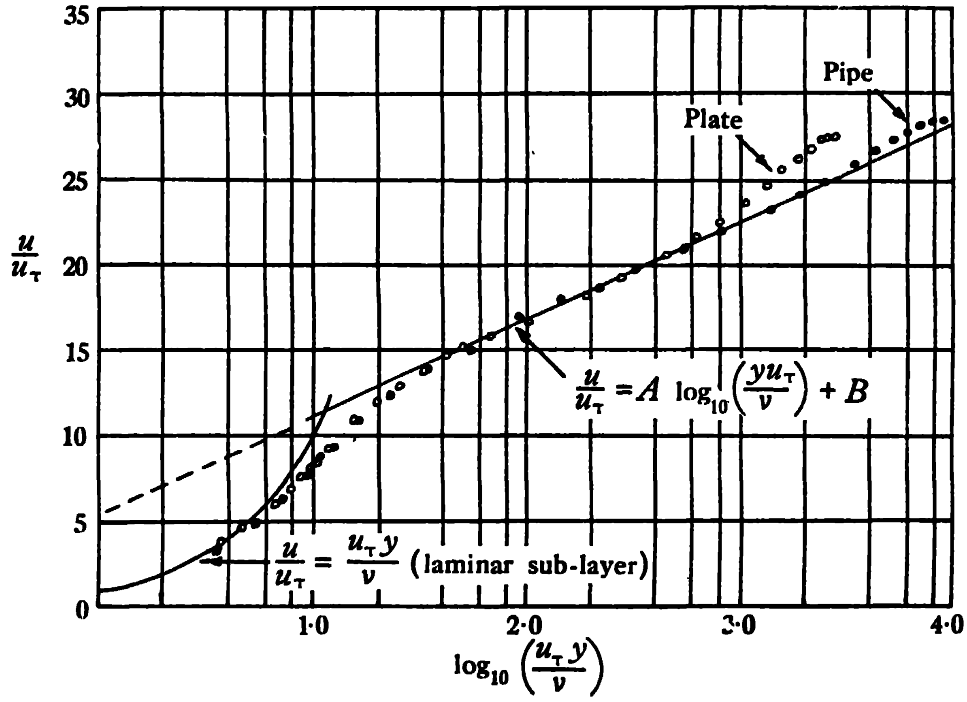


Fig. 6.15,2. Typical distribution of u/u_τ as a function of yu_τ/ν for the flow in a pipe and in the boundary layer on a flat plate.

$$\varepsilon' = ku_\tau(y - y_0) \tag{6.15,13}$$

and we improve (6.15,11) by writing

$$u_\tau^2 = (\nu + \varepsilon')\, du/dy \tag{6.15,14}$$

and so include the viscous stress.

These last two equations now lead to

$$\frac{u}{u_\tau} = \frac{1}{k} \ln \left[\frac{u_\tau(y - y_0)}{\nu} + \frac{1}{k}\right] + B \tag{6.15,15}$$

where B is a constant.

Typically, investigations of the flow in pipes and along plates lead to results for the boundary layer velocity distribution such as are illustrated in Fig. 6.15,2.

We see that for a wide range of values of $u_\tau y/\nu$ greater than about 30 there is very close agreement with the form of the equation 6.15,7 which we can write

$$\frac{u}{u_\tau} = A \log_{10}\left(\frac{yu_\tau}{\nu}\right) + B.$$

Here the constant A differs from that of equation (6.15,7) by the factor ln 10.

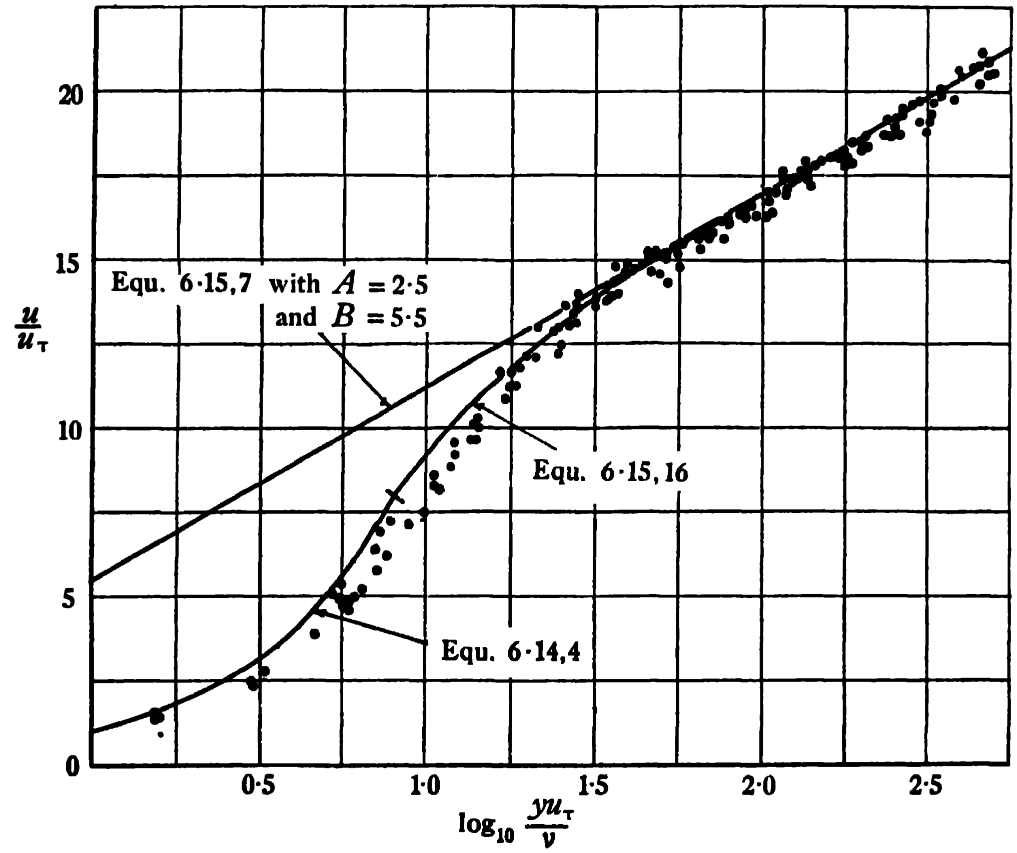


Fig. 6.15,3. Comparison of Squire's formulae and measurements by Reichardt.

However, different experimenters are not in complete agreement as to the values of the constants A and B in this equation, and there appear to be small but significant differences between the results for pipes and those for plates. Values of A are variously quoted between 5·0 and 6·7, and values for B range from 4·0 to 7·15. These differences are in part due to experimental errors, but it also seems likely that A and B are slowly varying functions of Reynolds number, tending to asymptotic constant values only at very high Reynolds numbers.[1] Values that have wide acceptance for normal applications are $A = 5{\cdot}75$, $B = 5{\cdot}5$. If we accept these values, then we see that in equation (6.15,15) we should put $k = 0{\cdot}40$ and $B = 5{\cdot}5$.

Further comparison with experimental results for $u_\tau y/\nu < 30$ suggests that y_0 in equation (6.15,15) should be such that $u_\tau y_0/\nu = 7.8$. Thus equation (6.15,15) with these numerical values becomes

$$\left.\begin{aligned}\frac{u}{u_\tau} &= 2{\cdot}5 \ln\left(\frac{u_\tau y}{\nu} - 5{\cdot}3\right) + 5{\cdot}5\\ &= 5{\cdot}75 \log_{10}\left(\frac{u_\tau y}{\nu} - 5{\cdot}3\right) + 5{\cdot}5\end{aligned}\right\} \qquad (6.15,16)$$

for $u_\tau y/\nu > 7{\cdot}8$.

It is suggested that the laminar sub-layer is in the region defined by $u_\tau y/\nu < 7{\cdot}8$, and there

$$\frac{u}{u_\tau} = \frac{u_\tau y}{\nu}. \qquad (6.14,4)$$

The two expressions for u/u_τ given by equations (6.15,16) and (6.14,4) join smoothly at $u_\tau y/\nu = 7{\cdot}8$. A comparison with some experimental results due to Reichardt[2] is shown in Fig. 6.15,3.

6.15.5 The Outer Velocity Law

Although equation (6.15,7) was derived on the basis of conditions that hold in that part of the inner region of fully turbulent flow where the shear stress is constant, nevertheless it is found in practice to approximate reasonably well to the mean velocity distribution over a somewhat larger part of the turbulent layer on a flat plate and an even larger part of the boundary layer in a pipe. Thus, for some purposes, where detailed accuracy of the velocity profile is not essential, equation (6.15,7) can be taken as describing the velocity distribution over the whole layer. In such cases, to avoid any difficulty that may arise from u tending to minus infinity as y tends to 0, the following slightly modified form can be used:—

$$\frac{u}{u_\tau} = A \ln\left[1 + C\frac{yu_\tau}{\nu}\right] \qquad (6.15,17)$$

where C is a constant. We see that for large y

$$\frac{u}{u_\tau} = A \ln C + A \ln\left(\frac{yu_\tau}{\nu}\right)$$

and from comparison with equation (6.15,7) we deduce that

$$A \ln C = B \qquad \text{or} \qquad C = \exp B/A \approx 9{\cdot}0.$$

However, for certain investigations we require to have a more accurate picture of the velocity distribution over the outer part of the layer. We may expect on physical grounds that in that part of the layer viscosity will play no direct part, the essential parameters being u_1, u_τ, y and δ (or a, the radius of the pipe). Thus we can infer on dimensional grounds that

$$\left.\begin{aligned}\frac{u_1 - u}{u_\tau} &= f\left(\frac{y}{\delta}\right), \text{ on a plate,}\\ &= g\left(\frac{y}{a}\right), \text{ in a pipe,}\end{aligned}\right\} \qquad (6.15,18)$$

where f and g are functions, to be determined by experiments, and which it is hoped would be sensibly independent of Reynolds number over a wide range of Reynolds number.

In view of the wide applicability of equation (6.15,7) referred to above, v. Kármán[1] suggested that the full boundary layer profile was given by

$$\frac{u}{u_\tau} = A \log_{10}\left(\frac{yu_\tau}{\nu}\right) + \phi\left(\frac{y}{\delta}\right) \qquad (6.15,19)$$

where $\phi(y/\delta)$ is a universal function taking the constant value B in the inner region.†

It follows that

$$\frac{u_1}{u_\tau} = A \log_{10}\left(\frac{\delta u_\tau}{\nu}\right) + \phi(1)$$

Therefore

$$\frac{u_1 - u}{u_\tau} = -A \log_{10}\left(\frac{y}{\delta}\right) + \phi(1) - \phi\left(\frac{y}{\delta}\right) = f\left(\frac{y}{\delta}\right). \qquad (6.15,20)$$

For flow in a pipe a similar universal function can be derived, but it is not quite the same as for a flat plate. We see, therefore, that on the basis of v. Kármán's hypothesis (equation 6.15,19) a velocity distribution can be predicted of the form of that given in equation (6.15,18) but applying to the whole of the boundary layer (except of course the laminar sub-layer) and not just to the outer region.

The available experimental data demonstrates that equation (6.15,20) has wide validity, although again small differences exist between the results for $f(y/\delta)$ determined by different experimenters, and there is some evidence that the constants are in fact slowly varying functions of Reynolds number.[2]

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[1] T. von Kármán, *J. Aero. Sci.*, I, 1 (1934).

† It can be argued that if a region exists in which both the general relations represented by equations (6.15,8) and (6.15,18) hold, then the relation (6.15,7) must apply in that region. Thus, this relation can be deduced on purely functional and dimensional grounds without reference to the mechanism of turbulence.

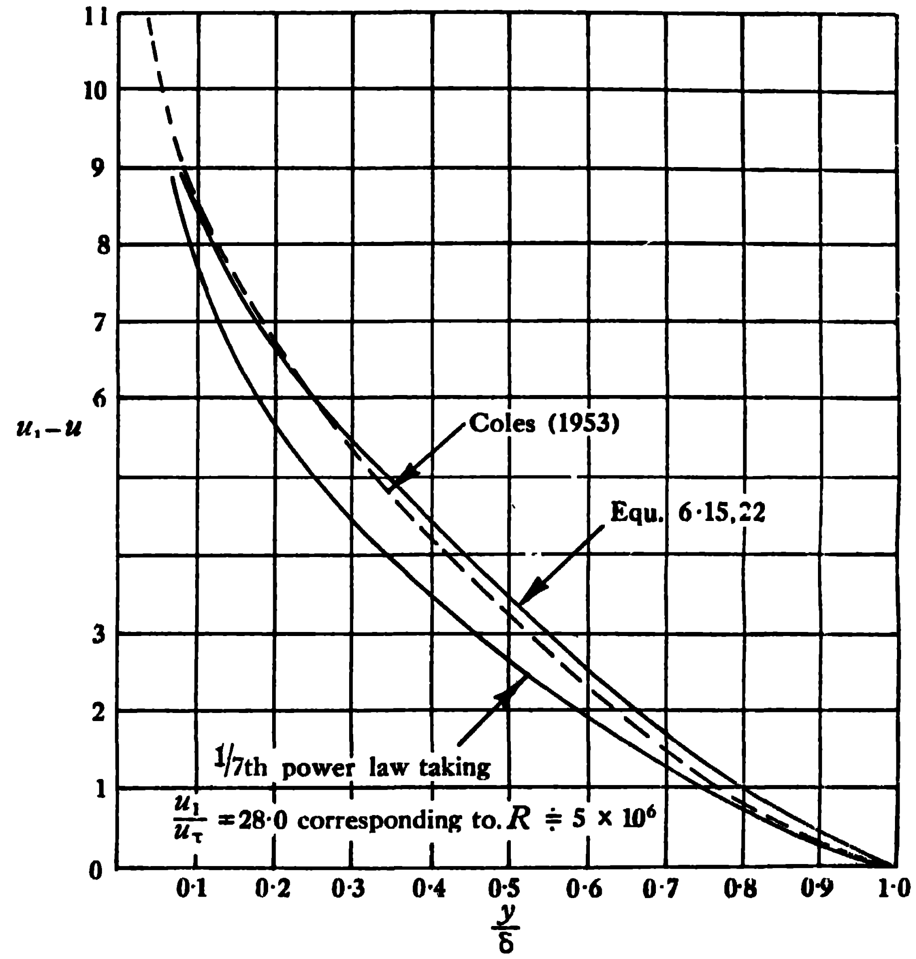

Fig. 6.15,4. Universal velocity distributions for the turbulent boundary layer on a flat plate.

From available results it appears that a reasonably good empirical fit to the function $\phi(y/\delta)$ for the boundary layer on a flat plate is given by

$$\phi\left(\frac{y}{\delta}\right) - \phi(0) = K \sin^3\left(\frac{\pi}{2}\frac{y}{\delta}\right)$$

where $K = 2{\cdot}8$.

If we take $\phi(0) = B = 5{\cdot}5$, then

$$\phi(y/\delta) = 5{\cdot}5 + 2{\cdot}8 \sin^3\left(\frac{\pi}{2}\frac{y}{\delta}\right) \qquad (6.15{,}21)$$

and

$$f(y/\delta) = 2{\cdot}8\left[1 - \sin^3\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right] - A \log_{10}\frac{y}{\delta}.$$

If we put $A = 5{\cdot}75$, we get

$$f(y/\delta) = 2{\cdot}8\left[1 - \sin^3\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right] - 5{\cdot}75 \log_{10}\left(\frac{y}{\delta}\right). \qquad (6.15{,}22)$$

A curve for $f(y/\delta)$ deduced by Coles[1] from an analysis of available experimental data is compared with that given by equation (6.15,22) in Fig. 6.15,4.

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[1] D. Coles, loc. cit. (p. 300).

where η_p is the pump efficiency. We are here assuming for simplicity that all the sucked fluid suffers the same total pressure changes; if this were not the case, as is likely in practice, the expression on the right-hand side of equation (6.24,12) would have to be replaced by a corresponding integral summing the product involved for all the fluid elements.

If power P were supplied to the propulsive unit it would balance the rate of work involved in overcoming a drag corresponding to a drag coefficient

$$C_{Dp} = \frac{P\eta_a}{\frac{1}{2}\rho u_0^3 S}$$

where η_a = efficiency of the propulsive unit. Thus, we can regard C_{Dp} as a drag coefficient equivalent to the power expended in pumping.

We see that

$$\begin{aligned} C_{Dp} &= \frac{(H_0' + \Delta H - H_0)Q}{\frac{1}{2}\rho u_0^3 S} \frac{\eta_a}{\eta_p} \\ &= \frac{(H_0' + \Delta H - H_0)C_Q}{\frac{1}{2}\rho u_0^2} \frac{\eta_a}{\eta_p} \end{aligned} \tag{6.24,13}$$

where C_Q is based on the wing area S.

Now suppose

$$H_0' = p_0 + \tfrac{1}{2}\rho(u_0 + u_e)^2$$

where p_0 is the undisturbed stream static pressure. Then u_e is the velocity of discharge relative to the undisturbed stream and hence the emerging jet reacts on the aircraft with a thrust $\rho Q u_e$.

It follows that the net effect of the suction is equivalent to an increase of drag coefficient

$$C_{DS} = \Delta C_{DW} + \frac{\eta_a}{\eta_p} \frac{(H_0' + \Delta H - H_0)C_Q}{\frac{1}{2}\rho u_0^2} - \frac{2u_e}{u_0} C_Q \tag{6.24,14}$$

where ΔC_{DW} is the change due to the suction in boundary layer drag as indicated by a wake traverse, which is presumed not to include a traverse of the above exhaust jet. It is assumed that the emerging jet is in the direction of the drag axis. We see that

$$\frac{dC_{DS}}{du_e} = \frac{2C_Q}{u_0}\left[\frac{\eta_a}{\eta_p}\left(1 + \frac{u_e}{u_0}\right) - 1\right]$$

and hence C_{DS} is a minimum when

$$\frac{u_e}{u_0} = \frac{\eta_p}{\eta_a} - 1. \tag{6.24,15}$$

If $\eta_p = \eta_a$, then C_{DS} is a minimum when $u_e = 0$ and the jet is discharged with the free stream total pressure. In that case

$$C_{DS} = \Delta C_{DW} + \frac{\Delta H}{\frac{1}{2}\rho u_0^2} C_Q. \tag{6.24,16}$$

If we consider suction at the rear of a flat plate at zero incidence, so that all the air in the boundary layer is removed and then ejected at full free stream total pressure, then since there is no wake,

$$\Delta C_{DW} = -C_{D0} = -2\theta/c,$$

where C_{D0} is the boundary layer drag without suction and θ is the momentum thickness at the trailing edge without suction.

Further, if we neglect entry and duct losses but take account of the variation in total pressure across the boundary layer just prior to entry, we must replace $\Delta HC_Q/\frac{1}{2}\rho u_0^2$ in equation (6.24,16) above by

$$\frac{1}{\frac{1}{2}\rho u_0^3 c}\int_0^\infty (H_0 - H)u\,dy$$

where H is the total pressure in the boundary layer at height y at the suction station.

This becomes

$$\frac{1}{cu_0^3}\int_0^\infty (u_0^2 \quad u^2)u\,dy = \delta_E/c,$$

where δ_E is the energy thickness at the suction station (see equation 6.4,9).

Hence $$C_{DS} = \frac{\delta_E}{c} - \frac{2\theta}{c}\ . \quad \text{But}$$

$$\delta_E = \int_0^\infty \frac{u}{u_0}\left(1 - \frac{u}{u_0}\right)\left(1 + \frac{u}{u_0}\right) dy < 2\int_0^\infty \frac{u}{u_0}\left(1 - \frac{u}{u_0}\right) dy, \text{ since } \frac{u}{u_0} < 1.$$

Therefore $\delta_E < 2\theta$.

It follows that C_{DS} is negative, i.e. it pays to remove all the boundary layer at the rear of the plate by suction and then eject it at full stream total pressure.

It should not be overlooked, however, that this analysis deals with an idealised case. Thus, entry and duct losses have been neglected and we have assumed $\eta_p = \eta_a$. Nevertheless, it is instructive to realise that the overall process of boundary layer suction and ejection can lead to a drag reduction if the power involved is more efficiently expended than in creating the wake that would develop without suction.

EXERCISES. CHAPTER 6.

1. Derive the momentum integral equation (equation 6.6,4) by integrating the boundary layer equation (equation 6.3,5) with respect to y from the surface to the outer edge of the boundary layer.
2. For the laminar boundary layer on a flat plate at zero incidence derive the values of $c_f\sqrt{R_x}$, $C_F\sqrt{R}$, $\frac{\theta}{x}\sqrt{R_x}$, $\frac{\delta^*}{x}\sqrt{R_x}$, $\frac{\delta}{x}\sqrt{R_x}$ and H listed on p. 259 for the various corresponding assumed velocity profiles.

3. By successive differentiation of equation (6.7,7) show that $F'''(0) = 0$, $F^{iv}(0) = 0$, $F^{v}(0) = -\alpha^2$, $F^{vi}(0) = F^{vii}(0) = 0$, $F^{viii}(0) = 11\alpha^3$, and so on, where $\alpha = F^{II}(0)$.

Hence deduce that near the surface (i.e. near $\zeta = 0$) the function F can be expressed by the series

$$F = \frac{\alpha\zeta^2}{2!} - \frac{\alpha^2\zeta^5}{5!} + 11\frac{\alpha^3\zeta^8}{8!} - 375\frac{\alpha^4\zeta^{11}}{11!} + -$$

Show that this series, as well as equation (6.7,7) are consistent with

$$F/\alpha^{1/3} = \text{function of } \xi, \text{ where } \xi = \alpha^{1/3}\zeta.$$

4. You are given that close to a slightly wavy boundary, whose shape is defined by

$$y_w = \varepsilon \sin \alpha x,$$

the local free stream velocity is approximately given by

$$u_1 = u_0(1 + \alpha y_w),$$

where u_0 is the undisturbed stream velocity, ε and $2\pi/\alpha$ are the wave amplitude and wave length respectively.

Assume that the rate of laminar boundary layer growth may be taken to be approximately the same as that given by Pohlhausen's method for a quartic velocity distribution. Find the maximum permissible height of a wave of length 5 cm situated approximately 30 cm downstream from the leading edge if local separation of the laminar boundary layer is to be avoided. Assume the condition for separation is as given by the Pohlhausen method for a quartic velocity distribution, viz. $\lambda = -12$. (Answer 0·0074 cm.)

5. Assume that the velocity distribution in the laminar boundary layer is of the form

$$\frac{u}{u_1} = F(\eta) + \lambda G(\eta)$$

where $\eta = y/\delta$, and $\lambda = \dfrac{\delta^2}{\nu}\dfrac{du_1}{dx}$.

Show that the first four important boundary conditions for $F(\eta)$ and $G(\eta)$ are:—

$$F(0) = 0, F(1) = 1, F''(0) = 0, F'(1) = 0,$$

$$G(0) = 0, G(1) = 0, G''(0) = -1, G'(1) = 0.$$

If $F(\eta) = \sin[(\pi/2)\eta]$, find $G(\eta)$ in the form of a quadratic in $\sin[(\pi/2)\eta]$.

Answer $\dfrac{2}{\pi^2}\sin\left(\dfrac{\pi}{2}\eta\right)\left[1 - \sin\left(\dfrac{\pi}{2}\eta\right)\right]$.

6. For the values of $F(\eta)$ and $G(\eta)$ determined in Exercise 5 deduce the values of λ and H when the boundary layer is on the point of separation. Hence, by means of the momentum integral equation show that if the boundary layer is on the point of separating over some distance along the surface then over that distance the boundary layer momentum thickness is proportional to u_1^{-6}.

$\left(\text{Answer } \lambda = \dfrac{-\pi^2}{2}, H = 4\right)$

7. If in equation (5) the functions $F(\eta)$ and $G(\eta)$ are taken to be cubics show that

$$F(\eta) = \frac{3\eta}{2} - \frac{\eta^3}{2} \quad \text{and} \quad G(\eta) = \frac{\eta}{4} - \frac{\eta^2}{2} + \frac{\eta^3}{4}.$$

Find in this case the value of λ when the flow is on the point of separating and show that then the displacement thickness will be half the boundary layer thickness. (Answer $\lambda = -6{\cdot}0$)

8. From equation (6.12,18) and the associated analysis deduce that if similar velocity profiles are obtained at all stations

$$u_1 = \text{const.}\, x^m \text{ or } u_1 = \text{const.} \exp(\alpha x)$$

where m and α are arbitrary constants.

9. From equation (6.12,18) and the associated analysis show that u_1 must increase linearly with x over a range of x if over that range the boundary layer is to have a constant form of velocity distribution (i.e. u/u_1 is a function of y/δ, only) and a constant thickness. Show that then the parameter $m = -0{\cdot}075$. Hence deduce the corresponding value of H given that

$$c_f R_\theta = 0{\cdot}5, \text{ where } c_f = \tau_w/\tfrac{1}{2}\rho u_1^2, \text{ and } R_\theta = u_1\theta/\nu.$$

(Answer $H = 1{\cdot}33$)

10. You are given that over a wing section the local free stream velocity distribution can be represented by

$$u_1 = 1{\cdot}25\, u_0 \sin\frac{\pi x}{c}$$

where u_0 is the undisturbed stream velocity and c is the wing chord.

From equation (6.12,18) and the associated analysis show that separation will occur when $x/c = 0{\cdot}571$, given that the parameter $l = 0$ when the parameter $m = 0{\cdot}085$.

11. Given a local free stream velocity distribution

$$u_1 = u_0(1 - \beta x)$$

show from equation (6.12,25) and the associated analysis that separation will occur when $\beta x = 0{\cdot}165$.

12. Assume that the velocity distribution in the laminar boundary layer on a flat plate is given by $u/u_1 = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$. You may also assume that if u_k is the velocity in the boundary layer where $y = k$, then a small roughness of height k will not generate eddies to disturb the boundary layer and so cause transition if its Reynolds number $u_k k/\nu$ is less than about 50. Show that at a distance x from the leading edge the maximum permissible roughness height for the boundary layer to remain undisturbed is given approximately by

$$\frac{k}{c} = \frac{12}{R^{3/4}}\left(\frac{x}{c}\right)^{1/4},$$

where $R = u_1 c/\nu$, and c is the chord of the plate.

13. Suppose that the power law approximation for the velocity distribution in the turbulent boundary layer on a flat plate, in the form

$$\frac{u}{u_\tau} = C_1\left(\frac{y u_\tau}{\nu}\right)^{1/n}$$

is chosen so as to have the same value of u and du/dy for some value of y as that given by the inner velocity law

$$\frac{u}{u_\tau} = A \ln\left(\frac{yu_\tau}{\nu}\right) + B.$$

Show that then the value of C_1 and the value of y at the common point are given by

$$C_1 = nA \exp\left[\frac{B}{An} - 1\right]$$

and
$$\frac{yu_\tau}{\nu} = \exp\left[n - \frac{B}{A}\right].$$

Given $A = 2{\cdot}5$ and $B = 5{\cdot}5$, find C_1 for $n = 7$ and 9.

(Answer 8·82, 10·56). It is of interest to note how close these values are to those found to give the best fit with experimental data for flow in pipes (see the table on p. 312, § 6.15).

14. With the value of C_1 determined in the previous exercise for $n = 9$ (i.e. $C_1 = 10{\cdot}56$) deduce that

$$c_f = 0{\cdot}0372\, R_x^{-1/6}, \text{ and } C_F = 0{\cdot}0445\, R^{-1/6}.$$

(It will be noted how close these relations are to those quoted in equations (6.17,12) and (6.17,13) as good fits for the corresponding Prandtl-Schlichting relations.)

15. If we write $C_F(x) = \dfrac{1}{x}\displaystyle\int_0^x c_f\, dx$, deduce that

$$c_f = C_F(x) + \frac{x\, dC_F(x)}{dx}.$$

Hence show that the Prandtl-Schlichting relation between C_F and R (equation 6.17,27) is consistent with the relation

$$c_f = C_F(x)\left[1 - \frac{2{\cdot}58}{\ln R_x}\right].$$

Deduce the value of c_f when $R_x = 10^6$. (Answer 0·00373)

16. From the results of the previous question and with the assumption that the velocity distribution satisfies the power law

$$\frac{u}{u_\tau} = 10{\cdot}56\left(\frac{yu_\tau}{\nu}\right)^{1/9}$$

find the thickness of the turbulent boundary layer at the trailing edge of a flat plate of chord 25 cm given that the Reynolds number of the plate is 10^6. (Answer 0·69 cm.)

17. For the boundary layer of Question 16 find the thickness of the laminar sub-layer at the trailing edge of the plate, given that at the edge of the laminar sub-layer $u_\tau y/\nu = 7{\cdot}8$. (Answer $4{\cdot}5 \times 10^{-3}$ cm.)

18. It is sometimes argued that a roughness element immersed in the laminar sub-layer will not have an eddying wake and therefore the thickness of the sub-layer can be determined from the condition that $u_l l/\nu < 50$, approx. where u_l is the velocity at height l above the surface. Show that this argument leads to a value of l of the right order.

19. Satisfy yourself that equation (6.20,9) leads to the same answer in the case of a boundary layer on a flat plate at zero incidence as that given by equation (6.19,6).

20. The Reynolds number of a flat plate at zero incidence is 5×10^6. Transition on each surface occurs at $0{\cdot}4c$. The velocity distribution in the laminar part of the boundary layer is given by $\frac{u}{u_1} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)$ whilst in the turbulent part of the layer it may be taken to fit a power law of index 1/7. Find the drag coefficient of the plate (taking both surfaces into account). (Answer 0·00476)

21. The stream velocity u_1 just outside the boundary layer on a surface at a chordwise distance x from the leading edge is given by

$$u_1 = u_0\left(1 - 0{\cdot}2\frac{x}{c}\right)$$

and the Reynolds number $R = u_0c/\nu = 10^7$.

On the basis of a power law relation between θ/c and c_f, with $n = 9$, and $H = 1{\cdot}4$, find the values of θ/c and c_f at the trailing edge. It may be assumed that the pressure gradients are small and the boundary layer is turbulent from the leading edge. (Answer $\theta/c = 0{\cdot}00236$, $c_f = 0{\cdot}00156$)

22. Find the value of the boundary layer drag coefficient for a symmetrical wing at zero incidence each surface of which has the same value of θ and u_1 at the trailing edge as the surface of Exercise 21. (Answer 0·00464)

23. A smooth porous plate is at zero incidence in an air stream of velocity 30 m./s and uniform suction is applied to the plate causing a flow into the plate of 0·015 m./s per second per square metre of plate area. Show that far downstream from the leading edge the velocity component u tends to $u = 30[1 - e^{-106y}]$ and the skin friction coefficient is given by $c_f = 0{\cdot}001$. Find the values of the displacement and momentum thicknesses.

Here y is measured in metres and ν is taken as $1{\cdot}46 \times 10^{-5}$ m^2./s. (Answer 0·974 mm, 0·487 mm.)

24. Prove that the momentum integral equation on a porous wing surface with suction is as given in equation (6.24,8).

CHAPTER 7

FLOW IN PIPES

7.1 Introduction

In Chapter 6 we were largely concerned with the flow of a real fluid external to a given surface, whereas in this Chapter and in Chapter 8 we are concerned with the flow of a real fluid internal to prescribed surfaces. These subjects inevitably draw on common fundamental concepts and basic data. However, in view of the special practical importance of the flow in pipes and channels in engineering, it was considered desirable to cover such common ground afresh, so that this Chapter and Chapter 8 can be read without the need for a prior detailed study of Chapter 6.

Pipes are used to convey fluids from place to place under pressure or vacuum, where it is necessary to prevent contamination or loss by evaporation and in heat exchangers. In practice the fluid may be an ordinary viscous liquid such as water, a gas or vapour, or a mixture composed of liquid and gas or liquid and solid. We shall restrict the discussion here to incompressible and homogeneous fluids of constant viscosity. Provided that the fractional range of pressure in the pipe is not too great, the treatment also covers adequately the flow of compressible fluids. The topic of 'choking' in gas flow is discussed in § 9.4, water hammer and surges are treated in § 10.8 and Chapter 8 is devoted to flow in open channels. It is assumed throughout this chapter that the pipe is discharging 'full-bore'; otherwise it is effectively an open channel. Also, unless it is stated otherwise, the pipes considered are untapered and with a circular bore. Pipes of small bore are often called *tubes* but these words will here be treated as synonyms. Measurement of the pressure difference across a length of tube, having a smooth and uniform bore, provides the best means for measuring the viscosity of a fluid, always provided that the flow is laminar.

7.2 Applications of Dimensional Analysis

We shall now apply dimensional analysis to the basic problem of pipe flow, namely, the resistance to the flow in steady and uniform regimes.† Thus the velocity at any point (or its average with respect to time when the flow is turbulent) is independent of time while the distribution of velocity over the section is the same at all sections. The fluid will be treated as incompressible with density ρ and constant viscosity μ. The bore of the pipe is cylindrical (or prismatic) and d is the typical linear dimension of the normal section; in the important case of the circular pipe we shall take d to be the diameter of the bore. Since by hypothesis the velocity distribution is the same at all sections of the pipe, it follows from the consideration of continuity that the velocity is the same at corresponding points of all sections. We might adopt the maximum velocity as the typical velocity to be used in the dimensional analysis but it is convenient and legitimate to take the average axial velocity v which is related to the volume Q of fluid discharged in unit time by the equation

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† The reader is advised to consult Chapter 4 and especially § 4.10 before reading this section.

$$Q = vA \qquad (7.2,1)$$

where A is the area of the bore. We apply dimensional analysis to a length d of the pipe, so the conditions of geometric similarity are satisfied. Let τ_m be the mean value of the frictional stress on the wall in the axial direction, and let w be the perimeter of the bore (equal to πd for the circular pipe). Then the force of resistance on the part considered is

$$F = wd\tau_m. \qquad (7.2,2)$$

It follows from equation (4.10,3) that

$$F = \rho v^2 d^2 f(R) \qquad (7.2,3)$$

where the Reynolds number is†

$$R = \frac{\rho v d}{\mu} = \frac{vd}{\nu} \qquad (7.2,4)$$

and the function $f(R)$ depends only on the shape of the bore. Hence

$$\tau_m = \rho v^2 \left(\frac{d}{w}\right) f(R). \qquad (7.2,5)$$

Now d/w is constant for pipes whose bores are similar and may thus be absorbed into the function of R. Adopting the non-dimensional friction coefficient‡

$$f = \frac{\tau_m}{\frac{1}{2}\rho v^2} \qquad (7.2,6)$$

we may then write

$$f = \phi(R). \qquad (7.2,7)$$

In the very important case of the circular bore the frictional stress is constant round the circumference and is thus equal to τ_m.

The function ϕ in equation (7.2,7) is invariable so long as the shape of the bore is invariable. Now it is observed that the resistance of a pipe of given length and bore and with a given rate of discharge depends on the roughness of the surface and this shows that in applying (7.2,7) we must interpret geometric similarity very strictly, i.e. the similarity must extend to the roughness as well as to the general shape of the bore. In practice, roughness is very variable from specimen to specimen and depends on such things as initial finish, corrosion and the occurrence of deposits.† It is therefore advisable to bring in roughness explicitly as an independent variable and we define the roughness height or diameter k in the following way. Take a pipe whose bore is smooth and of the same size and shape as the one under consideration and let the smooth bore be covered with close-packed smooth spherical grains, all of the same diameter. This diameter is then equal to the roughness diameter k when the two pipes have the same resistance when tested under the same conditions. It must be admitted that this definition is imperfect because it is possible that k will vary with the Reynolds number but we may accept it as a working approximation to the truth. We shall now have k/d as an independent non-dimensional parameter which influences the value of the friction coefficient and we shall generalize equation (7.2,7) thus:

$$f = \phi\left(R, \frac{k}{d}\right). \tag{7.2,8}$$

This equation must be valid when the roughnesses are strictly similar and k is now some linear dimension of a typical rugosity. It is found when k/d is very small and R is not too large that f is independent of k/d. In such conditions the pipe behaves as if it were perfectly smooth (see § 6.18.1). Such exact theoretical solutions of the resistance problem as have been obtained relate to perfectly smooth pipes and to laminar flow (see § 7.6).

Let τ be the local value of the frictional stress in the direction of the axis of the pipe. Then τ/ρ has the dimensions $ML^{-1}T^{-2}/ML^{-3} \mathrel{\underline{\cap}} L^2T^{-2}$ i.e. τ/ρ has the dimensions of the square of a velocity. It is convenient to define a velocity u_τ called the *friction velocity* or *shearing stress velocity* by the equation

$$u_\tau = \sqrt{\frac{\tau}{\rho}}. \tag{7.2,9}$$

This is used in the theory of turbulent flow in pipes (see § 7.7).

In equation (7.2,7) let

$$\phi(R) = \frac{2\psi(R)}{R} \tag{7.2,10}$$

where the last equation serves to define $\psi(R)$. Then (7.2,6) becomes

$$\frac{\tau_m}{\mu v/d} = \psi(R). \tag{7.2,11}$$

This equation has exactly the same validity as (7.2,6) with (7.2,7) but (7.2,11) is more convenient when the flow is laminar (see § 7.6).

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† Deposits and corrosion may also influence the effective bore.

7.3 Transition from Laminar to Turbulent Flow

It is a fact of great importance that flow in a pipe is laminar when the Reynolds number is less than a critical value, but turbulent when it is larger. The change from laminar to turbulent flow is associated with an important change in the manner in which resistance varies with the speed of flow and is accompanied by a great increase in the resistance. These facts were discovered by Osborne Reynolds[1] whose experiments will now be described.†

A diagram of the apparatus used by Reynolds is shown in Fig. 7.3,1.

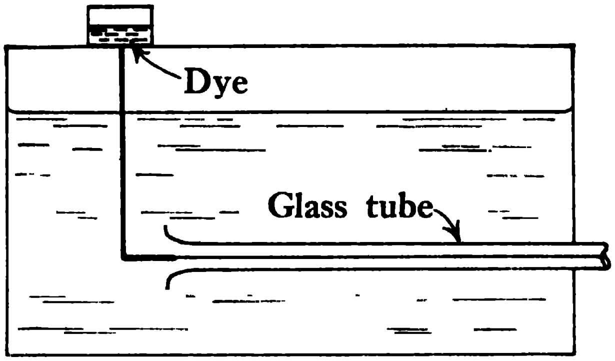

Fig. 7.3,1. Reynolds' apparatus.

Water entered a horizontal glass tube through a faired bell-mouth from a tank where the level at the free surface was kept constant. The nature of the flow in the pipe was made visible by allowing a highly coloured solution to exude slowly from a capillary tube so placed that the filament of coloured fluid entered the tube centrally. Reynolds found that, when the mean velocity of discharge was less than a certain limiting value, the coloured filament extended through the tube in a straight line. However, when the velocity exceeded the critical value, the filament first wavered and then became unsteady and irregularly convoluted so that, to the eye, the colouring

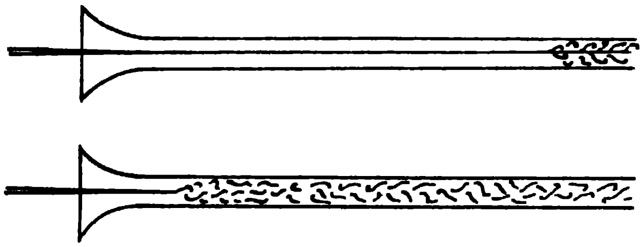

Fig. 7.3,2. Reynolds' experiment.

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[1] 'An Experimental Investigation of the Circumstances which Determine whether the Motion of Water shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels', *Phil. Trans. Roy. Soc.*, 1883, and *Papers on Mechanical and Physical Subjects*, Vol. II, p. 51.

† At an earlier date it had been remarked by Hagen (see § 7.6) that the motion of the liquid leaving a tube ceased to be smooth and regular when the velocity was relatively high. He also observed that the velocity at the transition from smooth to irregular flow depended on the diameter of the tube and on the temperature of the water.

appeared to fill the tube. Reynolds applied dimensional analysis to the phenomenon of 'transition' and concluded that the change from laminar to turbulent flow should occur for a fixed value of the quantity, now in his honour called the Reynolds number, which is defined in equation (7.2,4) and discussed in Chapter 4. Numerous more recent investigations have confirmed the conclusions of Reynolds and have established the following additional facts:

(a) The value of the Reynolds number at transition is sensitive to turbulence in the stream entering the pipe and to mechanical vibration. When special precautions are taken to avoid turbulence and other causes of disturbance, laminar flow is maintained at values of R up to 2×10^4 or beyond, at least for distances from the entry equal to 100 diameters.

(b) Even when the entering stream is highly turbulent, turbulent flow will not be maintained in a pipe at large distances from the entry when R is less than about 2×10^3. This critical value must be regarded as an absolute constant, although it would be difficult to establish its numerical value within close bounds.

7.4 Phenomena near the Entry to a Pipe

The distribution of velocity across the section of a given pipe under conditions of steady discharge is found to be the same for all sections not too near the mouth, whereas there is a progressive change of the velocity distribution as the section considered recedes from the mouth, until the final state is reached. Let us first consider laminar flow and suppose that the entry is well faired, so the velocity may be taken as uniform at the beginning of the cylindrical part of the pipe. However, the fluid is at rest where it is in contact with the wall of the pipe and a boundary layer (see Chapter 6) is set up. Just at the mouth this is exceedingly thin but the thickness increases as we move downstream. Now the volume passing every section of the pipe in unit time is the same, so the thickening of the boundary layer (where the flow is retarded) must be accompanied by an *increase* of the velocity in the region near the axis of the pipe. The development of the velocity profile for laminar flow in a pipe of circular section, as obtained experimentally by Nikuradse, is shown in Fig. 7.4,1. Theories

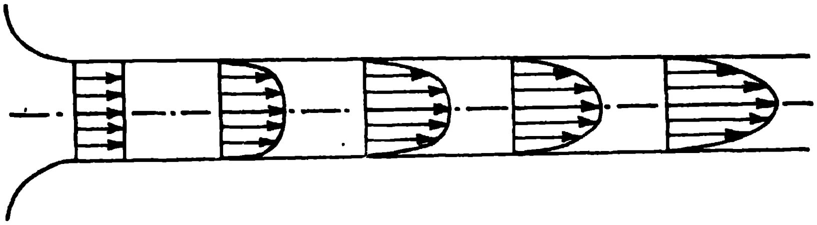

Fig. 7.4,1. Development of velocity profile for laminar flow.

developed by Boussinesq and by Schiller indicate that the velocity profile is the same for all pipes at distances x from the mouth such that the non-dimensional quantity x/Rd is constant and the diagram shows the profiles for five values of this variable. According to the calculations of Boussinesq the 'transition length', defined to be that beyond which the greatest velocity does not depart by more than 1 per cent from its final value (see § 7.6), is such that $x = 0{\cdot}065\ Rd$. The acceleration of the flow at the axis near the mouth implies a fall of static head here, in accordance with Bernoulli's theorem.

Near the mouth of the tube, in the region where a recognisable boundary-layer exists, the velocity gradient at the surface is greater than it is in the region far downstream where the regime is uniform. Consequently, the frictional stress near the mouth is also increased and the total pressure drop over the tube is greater than if the regime were uniform. Allowance for this must be made when a capillary tube is used as a viscometer and the pressure drop corresponding to the velocity head must also be allowed for.† When it is possible to make pressure tappings at points not too near the mouth these complications are avoided.

When the flow is turbulent, the conditions near the mouth are qualitatively similar to those already described but, in general, the final regime is reached nearer the entry, other things being the same. Moreover, turbulence in the stream at entry and roughness of the surface both tend to hasten the establishment of the fully turbulent regime.

In the region extending from the mouth up to the point where the boundary layers from the various parts of the pipe wall meet and merge, and beyond which the velocity profile does not change, the flow is described as *entry flow*. The flow further downstream is called *fully developed pipe flow* or simply *pipe flow*.

7.5 Uniform and Steady Regimes

We shall suppose that the flow is truly steady if laminar or statistically steady if turbulent; in the latter case the values of the velocities, pressures and stresses will be averages taken over an interval of time sufficiently long for sensibly constant values to be obtained. We shall also suppose that the bore is uniform and that the velocity is the same at corresponding points of all cross-sections of the pipe—this is the condition of uniformity of the regime of flow. In such circumstances, as will be shown, there is a simple relation between the negative gradient of total head along the pipe and the frictional stress at the surface of the pipe. This surface need not be smooth but the roughness must be independent of position along the pipe.

Consider the element of fluid cut off by normal planes of section at the points P_1 and P_2 on the axis of the pipe and let P_2 be at the distance l downstream from P_1. Since the motion is steady the acceleration of the fluid is everywhere zero† and the forces on the element are therefore in balance. Consider the forces in the direction of flow. The force of resistance opposing the motion is $lw\tau_m$ where w is the perimeter of the bore and τ_m is the mean frictional stress (see § 7.2). Since the streamlines are straight and parallel to the axis there can be no pressure gradient normal to them. Hence the pressure over the section at P_1 is constant and has the value p_1 say; similarly the pressure over the section at P_2 has the constant value p_2. The net thrust in the direction of flow due to these pressures is accordingly $(p_1 - p_2)A$, where A is the area of the normal section of the bore. Lastly, the gravitational thrust in the direction of flow is the weight of the element, namely $g\rho lA$, multiplied by s, the downward slope of the axis of the pipe, given by

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† For details, see Barr's *Monograph of Viscometry.*

$$s = -\frac{dz}{dx} \tag{7.5,1}$$

where x is measured axially in the direction of flow and z is measured vertically upwards from a fixed horizontal plane. Hence the condition of equilibrium is

$$lw\tau_m = (p_1 - p_2)A + sg\rho lA$$

which may be rewritten as

$$\frac{\tau_m}{m} = \frac{p_1 - p_2}{l} + sg\rho \tag{7.5,2}$$

where

$$m = \frac{A}{w} \tag{7.5,3}$$

is called the *hydraulic mean depth* of the bore. Equation (7.5,2) can be also written in the forms

$$-\frac{dp}{dx} = \frac{\tau_m}{m} - sg\rho \tag{7.5,4}$$

$$-\frac{dH_T}{dx} = \frac{\tau_m}{g\rho m} \tag{7.5,5}$$

where H_T is the 'total head' (see § 3.4) given by

$$H_T = \frac{v^2}{2g} + \frac{p}{g\rho} + z \tag{7.5,6}$$

since v is here independent of x. For a circular bore of diameter d

$$m = \frac{d}{4}. \tag{7.5,6}$$

When we relate the mean stress τ_m to the mean velocity by equation (7.2,6) we derive from (7.5,5)

$$-\frac{dH_T}{dx} = \frac{f}{m}\left(\frac{v^2}{2g}\right) \tag{7.5,7}$$

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† In turbulent flow the time average of the acceleration is zero.

where f is the non-dimensional friction coefficient defined in (7.2,6). It should be noted that gravity merely influences the pressure gradient while the gradient of total pressure is independent of gravity. Equation (7.5,7) is valid whether the flow is laminar or turbulent.

For a uniform pipe of length l and circular bore of diameter d the total loss of head is

$$H = l\left(-\frac{dH_T}{dx}\right) = \frac{lf}{m}\left(\frac{v^2}{2g}\right) = \frac{4lf}{d}\left(\frac{v^2}{2g}\right).$$

The volume of fluid discharged in unit time is

$$Q = \frac{\pi}{4}d^2v$$

and therefore

$$H = \left(\frac{64lf}{\pi^2 2gd^5}\right)Q^2 = kQ^2, \text{ say,} \tag{7.5,8}$$

while

$$Q = \frac{\pi}{8}\sqrt{\frac{2gHd^5}{lf}}. \tag{7.5,9}$$

The estimation of f for turbulent flow is considered in § 7.7.

7.6 Laminar Flow in Pipes

We shall now investigate laminar flow in smooth-walled cylindrical tubes and we shall suppose that the regime of flow is steady and uniform along the tube, which implies that the part of the tube considered is not very near the entry (see § 7.4). For the reason given at the end of § 7.5 gravity will be neglected. We shall begin with the important case where the bore is a circle of diameter d. The streamlines are straight lines parallel to the axis of the tube. Let u be the velocity in the axial direction OX; the other components are zero. For a circular tube u is a function only of the radial distance r measured from the axis. The velocity gradient or rate of shear in the radial direction is du/dr and the shearing stress is therefore $\mu(du/dr)$ (see § 3.10).

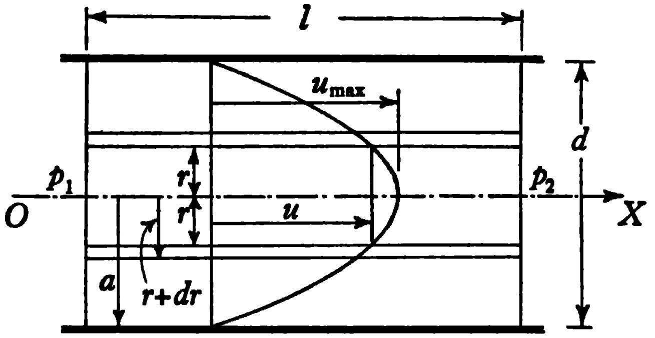

Fig. 7.6,1. Laminar flow in a circular tube.

Consider now the forces on an annular element of fluid of axial length l contained between cylinders of radii r and $r + dr$ (see Fig. 7.6,1). On the inner surface the stress is $\mu(du/dr)$ and this gives rise to a retarding force on the element equal to $2\pi\mu lr(du/dr)$. The stress on the outer surface of the element gives an accelerating force

$$2\pi\mu l\left\{r\frac{du}{dr}+\frac{d}{dr}\left(r\frac{du}{dr}\right)dr\right\}$$

and the net accelerating force is therefore $2\pi\mu l(d/dr)(r\,du/dr)\,dr$. The thrust due to the pressure is $2\pi(p_1-p_2)r\,dr$ where p_1 and p_2 are the pressures at the upstream and downstream ends of the element, respectively (see the discussion in § 7.5). Since the element is not accelerated the total axial force is zero and we derive the equation

$$2\pi\mu l\frac{d}{dr}\left(r\frac{du}{dr}\right)dr+2\pi(p_1-p_2)r\,dr=0$$

or

$$\mu l\frac{d}{dr}\left(r\frac{du}{dr}\right)=-r(p_1-p_2). \qquad (7.6,1)$$

Hence

$$\mu lr\frac{du}{dr}=-\tfrac{1}{2}r^2(p_1-p_2)+A_1$$

since (p_1-p_2) is independent of r; A_1 is a constant of integration. When we divide by r and integrate again we obtain

$$\mu lv=-\tfrac{1}{4}r^2(p_1-p_2)+A_1\ln r+A_2 \qquad (7.6,2)$$

where A_2 is a second constant of integration. Let a be the radius of the bore. Then u is zero when $r=a$ since there is no slip at the surface; further, A_1 must be zero for, otherwise, the velocity would be infinite when $r=0$. Hence

$$A_2=\tfrac{1}{4}a^2(p_1-p_2)$$

and (7.6,2) finally yields

$$u=\frac{(a^2-r^2)(p_1-p_2)}{4\mu l}. \qquad (7.6,3)$$

The volume of fluid passing in unit time is

$$Q=\int_0^a 2\pi ru\,dr$$

$$=\frac{\pi(p_1-p_2)a^4}{8\mu l} \qquad (7.6,4)$$

$$=\frac{\pi(p_1-p_2)d^4}{128\mu l}. \qquad (7.6,5)$$

The mean velocity of flow is

$$v=\bar{u}=\frac{Q}{\pi a^2}=\frac{(p_1-p_2)a^2}{8\mu l} \qquad (7.6,6)$$

while the maximum velocity, which occurs when r is zero, is

$$u_{\max} = \frac{(p_1 - p_2)a^2}{4\mu l}. \tag{7.6,7}$$

Thus we see that when the flow is laminar the velocity distribution is parabolic and the maximum velocity is exactly twice the mean velocity, while the pressure drop is proportional to the rate of discharge. For a given pressure difference $(p_1 - p_2)$ the rate of discharge is:—

Proportional to the *fourth power* of the diameter d.

Inversely proportional to the length l.

Inversely proportional to the viscosity μ.

Equation (7.6,5) is known as the Hagen-Poiseuille formula. It should be noted that consistent units must be used in applying the formula.

The frictional stress on the wall of the tube is

$$\begin{aligned} \tau = \tau_m &= -\mu\left(\frac{du}{dr}\right)_{r=a} \\ &= \frac{(p_1 - p_2)a}{2l} && \text{by} \quad (7.6,3) \\ &= \frac{4\mu v}{a} && \text{by} \quad (7.6,6) \\ &= \frac{8\mu v}{d}. && (7.6,8) \end{aligned}$$

Hence the non-dimensional friction coefficient as defined in equation (7.2,6) is

$$f = \frac{\tau_m}{\frac{1}{2}\rho v^2} = \frac{16\mu}{\rho v d} = \frac{16}{R} \tag{7.6,9}$$

where R is the Reynolds number as defined in equation (7.2,4). On comparison with equation (7.2,11) we see that when the flow is laminar and the regime is uniform

$$\psi(R) = 8. \tag{7.6,10}$$

It is beyond our present scope to enter into the details of the theory of laminar flow in tubes of non-circular section. However, in all cases the rate of volume discharge is inversely proportional to the viscosity and independent of the density of the fluid. It then readily follows by dimensional analysis that the rate of discharge is given by

$$Q = \frac{k(p_1 - p_2)d^4}{\mu l} \tag{7.6,11}$$

where k is a constant depending only on the shape of the bore and d is a linear dimension of the bore, say its greatest diameter. For an elliptic section of major axis d and minor axis nd

$$k = \frac{\pi n^3}{64(1 + n^2)} \tag{7.6,12}$$

which agrees with (7.6,5) when n is unity. For a rectangle whose longer side is d while the shorter is nd

$$k = \frac{n^3}{12}(1 - nS) \tag{7.6,13}$$

where†

$$S = \frac{192}{\pi^5} \sum_{m=0}^{\infty} \frac{1}{(2m + 1)^5} \tanh (2m + 1)\frac{\pi}{2n}. \tag{7.6,14}$$

For a square ($n = 1$)

$$k = 0{\cdot}03516.$$

There is an exact analogy between the laminar flow of a viscous fluid through a tube of arbitrary sectional form and the deflection of a membrane in tension under a uniform difference of pressure. The tension per unit width in the membrane is constant (as in a soap film), and it is stretched across a hole cut in a rigid sheet, the hole being geometrically similar to the bore of the tube. The velocity at any point of the section of the tube is then equal to the deflection at the geometrically corresponding point of the membrane, multiplied by a constant, while the rate of discharge is proportional to the volume between the membrane and the plane of its edge. The same membrane analogy applies to the torsion problem of St. Venant and to various other physical problems.‡

7.7 Turbulent Flow in Pipes

We confine attention here to the statistically steady and uniform regime of turbulent flow which exists everywhere in the pipe except quite near the entry or other discontinuity and shall begin by contrasting the state of things with that existing in laminar flow (see § 7.6). The pipes considered are of circular section and not tapered.

(1) The velocity profile is not altogether independent of the Reynolds number, whereas it is invariably parabolic for laminar flow.

(2) The velocity profile is much 'fuller' than for laminar flow (see Fig. 7.7,1). This implies that the ratio (mean velocity) : (maximum velocity) is substantially higher than for laminar flow. For very large values of the Reynolds number the ratio is about 0·8 to 0·82 whereas it is 0·5 for laminar flow.

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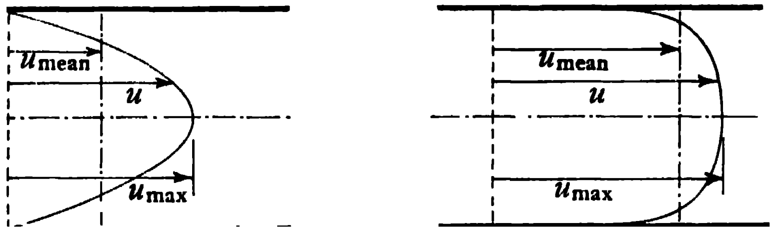
† The function S is tabulated on p. 146 of Barr's *Monograph of Viscometry* where it is designated R.

‡ For further details see p. 147 of *Physical Similarity and Dimensional Analysis* by W. J. Duncan.

(3) The resistance, other things being the same, is generally much higher for turbulent than for laminar flow. This is a consequence of the fact that the slope of the velocity profile at the surface of the pipe is much greater for turbulent than for laminar flow when the bore and mean velocity are the same for both.

(4) The resistance rises with increasing mean velocity much more rapidly than for laminar flow. For a limited range of Reynolds numbers the resistance of a given pipe will be adequately represented as proportional to the *n*th power of the mean velocity, where *n* is greater than unity, the value characteristic of laminar flow. The limiting value of *n* for very large Reynolds numbers is 2 and, in ordinary conditions, the value is only slightly smaller.



(a) (b)

Fig. 7.7,1. Velocity distribution in (a) laminar and (b) turbulent flow in a circular tube.

(5) The resistance is more sensitive to roughness of the pipe wall than for laminar flow and increases substantially with the roughness unless the rugosities lie entirely within the laminar sub-layer (see below).

The investigations of Stanton and his collaborators[1] have established the fundamentally important fact that there is a very thin layer of fluid adjacent to the wall of the pipe where the flow is laminar. They employed a special instrument which they called a *surface Pitot tube* and this is illustrated in Fig. 7.7,2. The opening of the tube, i.e., the distance of the sharp edge of the lip from the wall of the pipe, was controlled by a micrometer screw and, when this opening was zero, the upper surface was accurately flush with the surface of the wall. It will be noted that the whole space between the sharp lip and the wall is open to the stream and the question arises: where is the effective centre of the mouth? Let the dynamic pressure measured by the instrument correspond to a certain velocity *u*. Then the effective centre of the mouth is at such a distance from the wall that the velocity there would be *u* if the instrument were absent. Stanton determined the position of the effective centre of the opening for a range of openings in a preliminary series of experiments in which the flow was laminar and the velocity accordingly accurately known at all distances from the wall.[†] The thickness of the laminar sub-layer, where the velocity is very nearly proportional to the distance from the wall, is of the order of $5\nu/u_\tau$ where u_τ is the 'friction velocity' defined in (7.2,9).

Although the experiments of Stanton have established the existence of a laminar sub-layer when the general flow in the pipe is turbulent, it would be incorrect to suppose that the flow in the laminar sub-layer is steady, even when the turbulent flow is statistically steady. The work of Fage[1] has clearly demonstrated that there are large fluctuations in the direction and magnitude of the velocity (in the plane parallel to the pipe wall) right up to the wall itself, and these fluctuations are impressed on the flow in the sub-layer by the turbulence of the general stream. In his experiments Fage used the *ultramicroscopic technique* in which the small particles which are always present in tap water were made visible against a dark background by extremely intense illumination and the particles were observed through a microscope whose objective could be rotated at a controlled speed about an axis offset from but parallel to the optical axis of the instrument. Since the depth of focus was very small, the particles which were in sharp focus were all situated within a very thin layer of fluid. The technique of measurement of the axial velocity fluctuation consisted in varying the speed of rotation until a bright stationary spot of light appeared in the field of view of the eyepiece: the velocity of the image imposed by the rotation was then equal and opposite to the instantaneous velocity of the particle. Fluctuations of the direction of flow (as projected on a plane parallel to the pipe wall) could

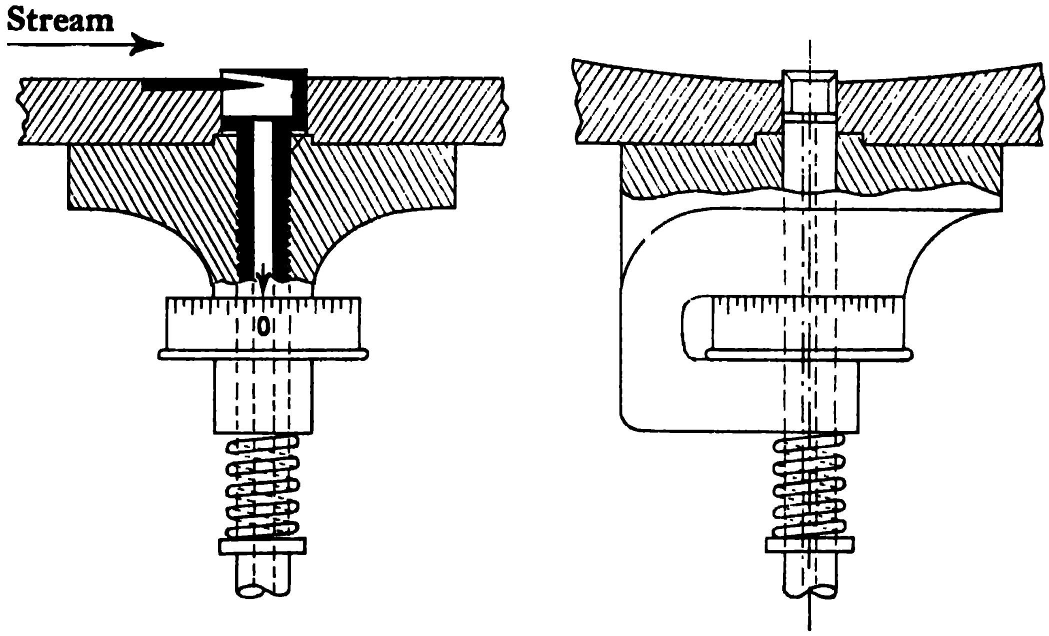


Fig. 7.7,2. Stanton's surface pitot tube.

be determined by observation of the bright tracks of the particles when viewed with the objective stationary.

Three regions of flow in a pipe of circular bore can be distinguished:

(a) the central region, extending to within 0·9 of the radius from the axis.

(b) the region near the wall but outside the laminar sub-layer.

(c) the laminar sub-layer, which is exceedingly thin.

The velocity distributions in the three regions are of different types, as described below, but there is a smooth transition from any region to its neighbours.

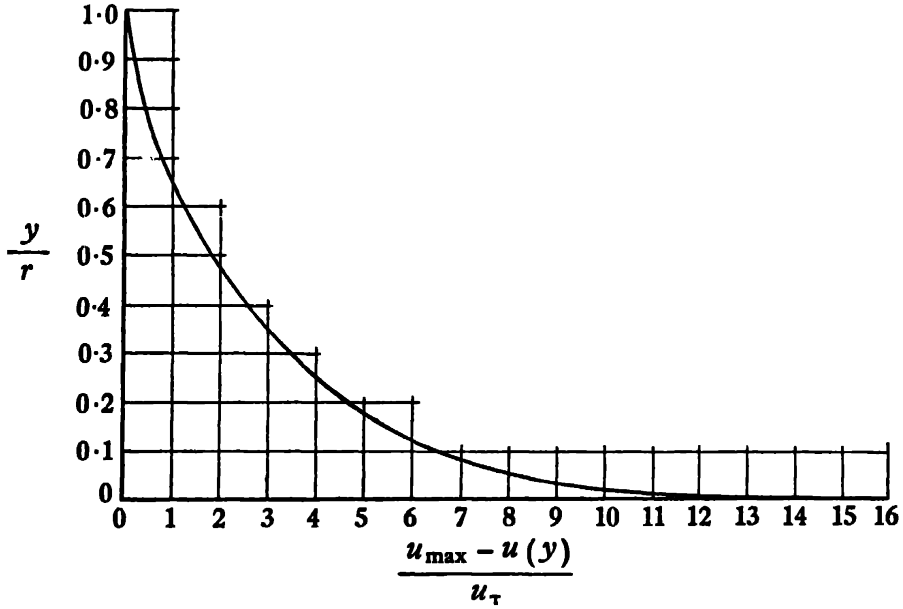


Fig. 7.7,3. Von Kármán's velocity deficiency curve.

(a) It was concluded by von Kármán on the basis of an argument from similarity that the velocity in the central region satisfies the relation†

$$u_{\max} - u(y) = u_\tau f\left(\frac{y}{r}\right) \tag{7.7,1}$$

where y is the distance from the wall, r is the radius of the bore, $u_{\max}$ is the velocity at the axis and u_τ is the friction velocity defined in (7.2,9). The function $f(y/r)$ is the same for all pipes and is defined by the graph in Fig. 7.7,3. Since von Kármán's argument is based on an incomplete theory, the law expressed by equation (7.7,1) must be regarded as semi-empirical; the form of the function $f(y/r)$ is obtained from experiments. Another relation having the same kind of basis is

$$v = u_{\max} - 4{\cdot}07u_\tau \tag{7.7,2}$$

where v is the mean velocity of discharge defined in (7.2,1).

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† This is the 'outer velocity law' of § 6.15.8.

(b) In the region near the wall (but outside the laminar sub-layer)† and for smooth pipes J. Nikuradse[1] has given the semi-empirical formula based on mixing length theory (see § 6.15.2)

$$\frac{u(y)}{u_\tau} = 5{\cdot}75 \log_{10}\left(\frac{yu_\tau}{\nu}\right) + 5{\cdot}5. \tag{7.7,3}$$

The corresponding formula for a pipe with a sufficiently rough wall is

$$\frac{u(y)}{u_\tau} = 5{\cdot}75 \log_{10}\left(\frac{y}{k}\right) + C_2 \tag{7.7,4}$$

where k is the roughness height (see § 7.2) and C_2 is a constant whose value is about 8·5 provided that $u_\tau k/\nu$ is greater than about 70. Since in practice the roughness is itself irregular, k and C_2 should be regarded as empirical constants to be found by experiment. It appears that the pipe behaves as if hydraulically smooth when ku_τ/ν is less than 4.

(c) Within the laminar sub-layer the velocity is proportional to y and

$$\frac{u(y)}{u_\tau} = \frac{yu_\tau}{\nu}. \tag{7.7,5}$$

The exact proportionality of u and y implied by this formula is only strictly true extremely near the wall.

We begin the discussion of resistance with the case of perfectly smooth pipes. All the formulae for resistance are empirical and only a selection from the very numerous ones which have been proposed will be given here. The friction coefficient f defined in (7.2,6) is used throughout.

(a) Blasius proposed the relation

$$f = \frac{0{\cdot}0791}{R^{1/4}} \tag{7.7,6}$$

and this can be used for values of R up to 8 × 10^4.

(b) Lees gave the formula

$$f = 0{\cdot}0018 + \frac{0{\cdot}153}{R^{0{\cdot}35}}. \tag{7.7,7}$$

The upper limit for R is about 4 × 10^5.

(c) The logarithmic resistance formula, which may be used for the highest Reynolds numbers (subject to the pipe being still effectively smooth), is of the form

$$\frac{1}{\sqrt{f}} = A \log_{10} R\sqrt{f} + B. \tag{7.7,8}$$

According to Prandtl†

$$A = 4{\cdot}0, \quad B = -0{\cdot}396$$

or, alternatively,

$$A = 3{\cdot}9, \quad B = +0{\cdot}074$$

where the latter values give the better results for extremely large values of R. The logarithmic formula is a little awkward in use since f appears on both sides of the equation but the value of f for a given R can be found fairly quickly by successive approximation. It will be noted that R is an explicit function of f and, when A and B are known, it is easy to construct a graph or table connecting f with R.

For rough pipes when the value of R is large enough for all the rugosities to be well proud of the laminar sub-layer, so that $u_\tau k/\nu$ is greater than about 70,

$$\frac{1}{\sqrt{f}} = 4 \log_{10}\left(\frac{d}{2k}\right) + 3{\cdot}48 \qquad (7.7,9)$$

where k is the roughness diameter or height (see § 7.2). Since the expression on the right of the last equation is independent of v and of ν we now have the simple result that the friction coefficient f is constant for a given pipe. This is in accordance with the much used Chézy formula (see below). For any rough pipe there is a transition region of R where the resistance does not follow the laws for either smooth or completely rough surfaces and here some, but not all, of the roughnesses have penetrated the laminar sub-layer. Fig. 7.7,4 shows the dependence of f on R for a number of values of d/k for values of R up to 100 million.

The foregoing observed facts can be explained as follows. When the roughnesses lie entirely within the laminar sub-layer, eddies are not shed from them and the additional resistance caused by their presence is negligible. As the height of the roughnesses increases (or as the thickness of the laminar sub-layer decreases), some of them protrude through the sub-layer and produce wakes of eddies while at the same time their drag increases. The extra resistance is thus the sum of the "form drags" (see § 6.2) of those roughnesses which have penetrated through the laminar sub-layer. With roughnesses which are all sufficiently large in relation to the thickness of the sub-layer, the additional form drag altogether swamps the basic skin frictional resistance. Since the non-dimensional coefficient for the form drag is almost independent of Reynolds number we eventually arrive at a stage where the resistance coefficient of the pipe is practically independent of the Reynolds number.

It was very early recognised by experimenters in hydraulics that the resistance of a given pipe varies approximately as the square of the rate of discharge (when this is not very small) and Chézy gave the formula

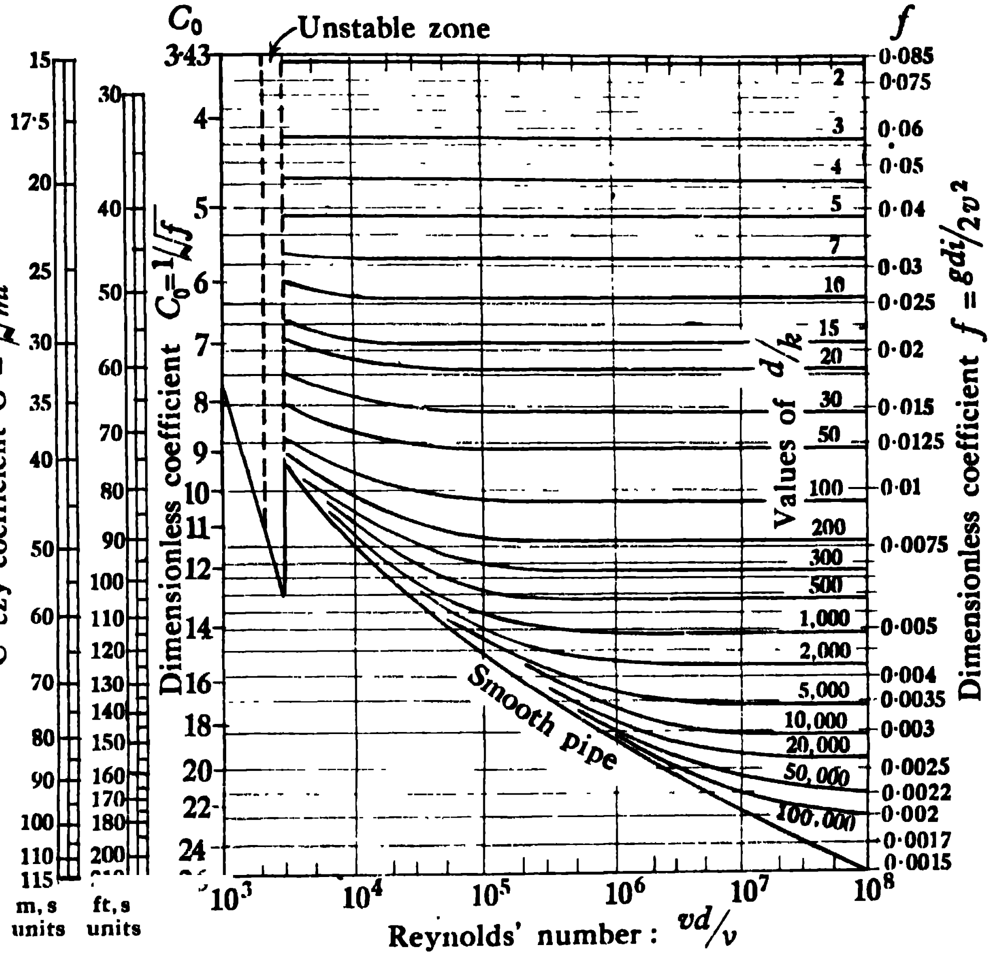
$$v = C\sqrt{mi} \qquad (7.7,10)$$

---

† Prandtl used $\lambda_1 = 4f$ and the constants have been adjusted accordingly.

where v is the mean velocity of discharge, m is the hydraulic mean depth, i is the total head gradient given by

$$i = -\frac{dH_T}{dx} \tag{7.7,11}$$



Refer to § 8.2 for k values

Fig. 7.7,4. Pipe friction chart. (Partly based on diagrams included in a paper by P. A. Lamont, *Proc. Inst. C. E.*, Part 3, No. 3, 1954, p. 248.)

while C is a constant now called the Chézy coefficient. We can derive the Chézy formula as follows when the friction coefficient f is constant. By equations (7.5,7) and (7.7,11)

$$\frac{v^2}{2g} = \frac{mi}{f}$$

or

$$v = C_0\sqrt{2gmi} \tag{7.7,12}$$

where

$$C_0 = \frac{1}{\sqrt{f}}. \tag{7.7,13}$$

Now g varies only slightly over the surface of the earth and if we regard it as constant we may rewrite (7.7,12) as

$$v = C\sqrt{mi}$$

in accordance with (7.7,10). The value of the Chézy coefficient is thus

$$C = C_0\sqrt{2g} = \sqrt{\frac{2g}{f}}. \tag{7.7,14}$$

The coefficient C_0 is non-dimensional, but C is not; the reader should consult § 8.2 for the dependence of C on the choice of units. The great merit of the Chézy formula is its simplicity but there is the difficulty that C is by no means a universal constant. The estimation of C will be facilitated by the use of the scales given at the side of Fig. 7.7,4 *where the units are feet and seconds, metres and seconds.*[†]

7.8 **Bends**

Flow of a viscous fluid in a curved pipe is necessarily more complex than in a straight pipe—for example, the streamlines become twisted curves of a generalised helical form. The region of maximum velocity is no longer at the axis of the pipe while the resistance to the flow is increased.

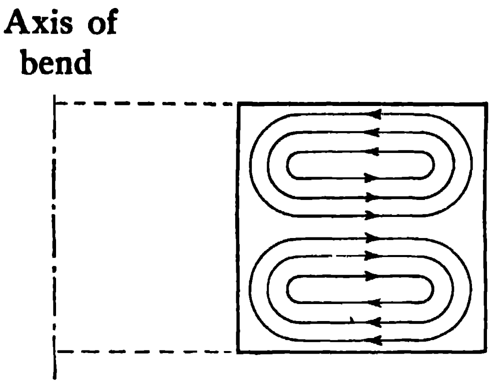


Fig. 7.8,1. Curved streams. Secondary flow.

For simplicity let us consider first a pipe of square section which is bent into a circular arc (see Fig. 7.8,1). Since the streamlines must be curved in the same sense as the pipe itself there is a radial pressure gradient (corresponding to the centrifugal pressure), so the pressure is greater on the outer wall of the pipe than at the corresponding point on the inner wall. The fluid in the boundary layer very near the upper and lower walls is nearly at rest; consequently little or no centrifugal pressure gradient exists in these regions. However, the fluid in these boundary layers is exposed to the pressure difference set up across the central part of the section and it must therefore flow towards the inner wall of the bend. Accordingly there is a double circulation of the fluid in the radial plane as shown in the figure and, since this is compounded with the forward motion, the paths of the particles (identical with the streamlines in steady motion) are of a generalised helical form. Since there is a component of velocity radially outward in the central part of the section, it follows that the fastest moving fluid is situated between the centre of the section and the outer wall. In a straight pipe of square section the velocity distribution is symmetrical about both axes through the centre and normal to the walls. However, in a bend there is no longer symmetry about the axis which is normal to the plane of the bend. All that has been said above about a pipe of square section is also true for a circular section or any other compact section which is symmetrical about a radial and a normal axis. However, affairs are different qualitatively when the ratio of the radial to the normal width of the section is small. Here the region of greatest velocity near the middle of the section is nearer the inner than the outer wall.† The distortion of the flow pattern caused by a bend is accompanied by an increase of resistance which, however, is very slight when the ratio of the pipe diameter to the radius of the bend is small. It is convenient to express the extra resistance caused by the bend as an effective increase in the length of the pipe (supposed straight) equal to *n* pipe diameters. The value of *n* ranges from about 1 for an ordinary elbow to about 0·6 for a gently swept bend and it is to be understood that this is an addition to the actual length of the bend, measured along its centre line. Alternatively, the loss in the bend may be expressed as a fraction of the dynamic pressure; for instance, the loss in a gentle 90° bend is of the order of 20 per cent of the dynamic pressure.

[†] An up-to-date set of design curves suitable for pipes and open channels is given in Hydraulics Research Papers, Nos. 1 and 2, of the Hydraulics Research Station, Wallingford.

7.9 Sudden Changes of Section

Any sudden change of section causes a loss of total head, which may be large. We begin with the case of a sudden expansion since a simple approximate theory of this can be given (see Fig. 7.9,1). The streamlines at *DE* are nearly straight and the pressure at *DE* is therefore uniform and equal to p_1, say. The velocity at the annular wall *BD*, *EC* is very small and we may therefore assume that the pressure is p_1 all over *BDEC*. Hence the thrust on the fluid element *BCGF* is $A_2(p_1 - p_2)$, where A_2 is the area of section at *FG* and p_2 is the pressure there, and we neglect the small frictional force on the pipe wall between *BC* and *FG*. Let us assume for simplicity that the velocity is uniform and equal to v_1 at *DE* while it is uniform and equal to v_2 at *FG*. Then by the Principle of Momentum (see §§ 3.2, 3.6) we have thrust equal to momentum generated in unit time, or

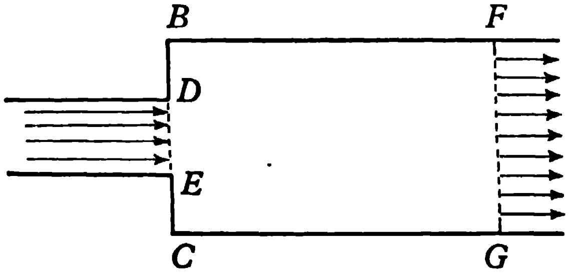


Fig. 7.9,1. Sudden change of section.

$$A_2(p_1 - p_2) = \rho Q(v_2 - v_1) = \rho v_2 A_2(v_2 - v_1).$$

Therefore

$$p_1 - p_2 = \rho v_2(v_2 - v_1) \qquad (7.9,1)$$

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† See *Modern Developments in Fluid Dynamics*, Vol. I., p. 87.

and the loss of total head between DE and FG is

$$H_l = \frac{p_1 - p_2}{\rho g} + \frac{v_1^2 - v_2^2}{2g}$$

$$= \frac{(v_1 - v_2)^2}{2g} \tag{7.9,2}$$

on substitution from (7.9,1). Then if A_1 is the sectional area before the expansion

$$v_2 = v_1 \left(\frac{A_1}{A_2}\right)$$

and

$$H_l = \left(\frac{v_1^2}{2g}\right)\left(1 - \frac{A_1}{A_2}\right)^2. \tag{7.9,3}$$

If there were no pressure recovery the loss of head would be

$$H_l' = \frac{v_1^2 - v_2^2}{2g} = \left(\frac{v_1^2}{2g}\right)\left(1 - \frac{A_1^2}{A_2^2}\right) \tag{7.9,4}$$

and

$$H_l' - H_l = \left(\frac{v_1^2}{2g}\right)\left(\frac{2A_1}{A_2}\right)\left(1 - \frac{A_1}{A_2}\right) \tag{7.9,5}$$

which is positive, so there is some pressure recovery. It must be pointed out that this theory is crudely approximate because the velocity is taken to be uniform at DE and at FG.

There is no correspondingly simple theory for the loss at a sudden contraction. The conventional treatment is to assume that a *vena contracta* of minimum cross-sectional area A_c is formed just beyond the contraction. The loss of total head is then taken to be equal to that for a sudden expansion from A_c to A_2. The velocity at the throat of the vena contracta is

$$v_c = v_2\left(\frac{A_2}{A_c}\right) = \frac{v_2}{C_c} \tag{7.9,6}$$

where C_c is the coefficient of contraction. When we substitute v_c for v_1 in (7.9,3) and replace A_1/A_2 in that equation by C_c we obtain

$$H_l = \frac{v_2^2}{2gC_c^2}(1 - C_c)^2$$

$$= \left(\frac{v_2^2}{2g}\right)\left(\frac{1}{C_c} - 1\right)^2. \tag{7.9,7}$$

The last equation can be written more conveniently as

$$H_l = k\left(\frac{v_2^2}{2g}\right). \tag{7.9,8}$$

The value of k depends acutely on the area ratio A_2/A_1 and to some extent on the Reynolds number. Its value may be estimated with the help of Table 7.9,1 where the ratio of the diameter d_2 downstream to the diameter d_1 upstream is used as variable.

TABLE 7.9,1

Value of the Resistance Coefficient k of a Sudden Contraction in a Pipe
(*Derived from the results of* Merriman *and others*)

Diameter ratio d_2/d_1	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
Resistance coefficient k	0·47	0·46	0·44	0·41	0·37	0·28	0·18	0·09	0·04

More or less sudden changes of effective sectional area may occur at valves in pipe lines. As for the case of bends (see § 7.8) it is convenient to specify the extra resistance bv an equivalent length of pipe equal to n diameters. For a fully open 'globe' stop valve n is about 10 but for fully open gate valves and for well faired valves generally the value is small (of the order of 0·2).

7.10 **Diffusers**

A diffuser or recuperator is a passage which gradually increases in sectional area downstream and its function is to reduce the velocity of flow while, at the same time, preserving the total head as far as possible. Thus a diffuser is a device for converting velocity head or dynamic head into pressure head. Usually a diffuser is of conical form but this is not necessary.

For a perfect fluid and for steady flow in the absence of body forces we have by Bernoulli's theorem (see § 3.4)

$$\frac{q_1^2}{2g}+\frac{p_1}{g\rho}=\frac{q_2^2}{2g}+\frac{p_2}{g\rho}$$

while continuity requires that

$$q_1A_1=q_2A_2.$$

Here quantities with suffix 1 refer to the entry to the diffuser while those with suffix 2 refer to the exit and it is assumed that the flow is uniform at these sections. Hence we obtain

$$p_2-p_1=\tfrac{1}{2}\rho q_1^2\left\{1-\left(\frac{A_1}{A_2}\right)^2\right\}. \tag{7.10,1}$$

This equation gives the pressure recovery in a perfect diffuser but in practice the ideal pressure recovery must be multiplied by a factor which is necessarily less than unity. This factor depends on the nature of the surface but mainly on the nature of the flow at entry and on the angle of the cone. Let α be the total vertical angle of the cone. When α is very small the flow will adhere to the wall of the diffuser but the total resistance will be large because, with a given value of A_2/A_1, the diffuser will be long when α is small. The resistance will again be large when α is large because the flow will break away from the wall of the diffuser and energetic eddies will be formed, leading to much dissipation of mechanical energy. Evidently there is an intermediate value of α for which the pressure recovery is a maximum and it is generally accepted that this is in the neighbourhood of 5 degrees. Experiments by Winternitz and Ramsay[1] indicate that the non-dimensional pressure recovery coefficient is approximately a linear function of $(\theta\alpha/d)$ where θ is the momentum thickness of the boundary layer (see § 6.4) at entry to the diffuser and d is the diameter there. The pressure recovery coefficient is

$$C_{pr} = \frac{p_2 - p_1}{\frac{1}{2}\rho v^2} \tag{7.10,2}$$

where v is the mean velocity at entry. In a series of tests on straight diffusers of 5° and 10° total angle and with various values of θ produced by the use of wire screens to thicken the boundary layer, it was found that C_{pr} fell from about 0·8 to about 0·65 when $(\theta\alpha/d)$ increased from 0·02 to 0·1, where α is measured in degrees. The walls of the diffusers were commercially smooth and the expansion ratio A_2/A_1 was 4.

7.11 Nozzles and Orifices

A convergent nozzle is a passage whose sectional area decreases downstream and which ends in a clear opening from which the fluid can issue freely. A convergent nozzle is thus the converse of a diffuser (see § 7.10) and its function is to convert pressure head into velocity head as efficiently as possible. The efficiency of a short and well faired smooth nozzle may be 98 per cent. or even more, so we have a close approach to perfect conversion of pressure head into velocity head. Equation (7.10,1) can again be applied, but A_2 is now less than A_1, while p_2 is the ambient pressure in the region where the jet issues from the nozzle. Hence the equation may be written conveniently as

$$q_1^2 = \frac{p_1 - p_2}{\frac{1}{2}\rho\{(A_1/A_2)^2 - 1\}} \tag{7.11,1}$$

where p_1 is the pressure at entry to the nozzle.

In the context of flow in pipes an orifice is a hole cut in a plate sandwiched between two lengths of pipe (see also § 5.10). The hole is almost always circular and concentric with the pipe bore; it is sometimes sharp edged. The orifice can be used for measuring the velocity of flow in conjunction with a manometer to measure the drop of pressure across the orifice.

An orifice must be calibrated and is, in general, less accurate than a Venturi meter (see §§ 3.4 and 5.10) but it is cheaper and occupies less space. Orifices of other types are discussed in § 8.6. Instead of an orifice, a nozzle within the pipe may be used in conjunction with a manometer to measure the flow. Essential features of standard nozzles and orifices for flow measurement are given in British Standards Code B.S. 1042: Part I: 1964 where the positions of the tappings upstream and downstream and other dimensions are fully detailed.

7.12 Branched Pipes and Systems

We shall consider here only steady flow in pipe systems but oscillations in such systems are treated in § 10.2. There are three general principles concerning pipe systems:

(a) *Continuity*: the algebraic total volume of fluid arriving at any junction in unit time is zero. This can be stated alternatively: the sum of the rates of discharge towards the junction is equal to the sum of the rates of discharge away from the junction.

(b) *Identity of heads*: The heads at the ends of all pipes meeting at a junction are the same and equal to the head at the junction.

(c) *Summation of losses*: The head lost in two or more pipes in series (with the same direction of flow throughout) is the sum of the heads lost in the individual pipes and this is true whether or not the rate of discharge is the same for all the pipes in series.

In the statement of Principle (b) it is assumed that any entry or exit losses at the junction are included in the resistances of the individual pipes.

Most of the detailed work which follows is based on the quadratic resistance law in accordance with which the head lost in a given pipe is proportional to the square of the rate of discharge but methods for dealing with other laws will be indicated. For laminar flow the resistance is proportional to the rate of discharge and the problem is formally identical with that of a network of electrical resistances. The mathematical solution proceeds in the standard manner for sets of linear equations, with the aid of determinants or otherwise. We quote, but do not prove, the theorem that the flows in a network with prescribed heads at the terminals adjust themselves so that the total rate of dissipation of energy is a minimum;† this is true for laminar flow only.

We shall now assume that the head H lost in a pipe is related to the rate of discharge Q by the equation

$$H = kQ^2 \tag{7.12,1}$$

where, by equation (7.5,8), the resistance coefficient is given by

$$k = \frac{64lf}{\pi^2 2gd^5}. \tag{7.12,2}$$

---

† A proof of the theorem, in electrical notation, will be found in Chapter IX of *Electricity and Magnetism* by J. H. Jeans.

First suppose we have a pair *of pipes connected in series and with the same* rate of discharge in both. Then the total loss of head is the sum of the losses in the individual pipes and it follows that the resistance coefficient k of the combination is given by

$$k = k_1 + k_2 \tag{7.12,3}$$

where k_1, k_2 are the resistance coefficients of the individual pipes. When there are n pipes in series with the same rate of discharge throughout we clearly have

$$k = \sum_{r=1}^{n} k_r. \tag{7.12,4}$$

Next let us take a pair of pipes in parallel so the loss of head H is the same for both. Then

$$k_1Q_1^2 = k_2Q_2^2 = H$$

and

$$Q = Q_1 + Q_2 = \sqrt{H}\left(\frac{1}{\sqrt{k_1}} + \frac{1}{\sqrt{k_2}}\right) = \sqrt{\frac{H}{k}}$$

where k is the effective resistance coefficient of the two pipes in parallel. Hence

$$\frac{1}{\sqrt{k}} = \frac{1}{\sqrt{k_1}} + \frac{1}{\sqrt{k_2}} \tag{7.12,5}$$

where all the square roots must be taken positive. The last equation is equivalent to

$$k = \frac{k_1k_2}{k_1 + k_2 + 2\sqrt{k_1k_2}}. \tag{7.12,6}$$

When there are n pipes in parallel we have, by an obvious generalisation of (7.12,5),

$$\frac{1}{\sqrt{k}} = \sum_{r=1}^{n} \frac{1}{\sqrt{k_r}} \tag{7.12,7}$$

where again all the square roots are to be taken positive. By use of equations (7.12,4) and (7.12,7) the effective resistance coefficients of many pipe systems can be obtained without difficulty.

It must be noted that equation (7.12,1) is only valid for one direction of flow. If we assume that k is independent of the direction of flow and measure the difference of head in the same sense as before, the equation must be rewritten

$$H = -kQ^2$$

when the direction of flow is reversed. This implies that if the solution of a problem does not yield flows in the directions assumed in formulating the equations it is necessary to recast these to accord with the true directions of flow. However, in many simple problems there is never any doubt about the directions of flow.

We shall next consider the system shown diagrammatically in Fig. 7.12,1. The resistance coefficients of AJ, JB and JC are k_1, k_2 and k_3 respectively, while the heads at A, B and C are H_1, H_2 and H_3 respectively. These are supposed known and

$$H_1 > H_2 \geqslant H_3. \qquad (7.12,8)$$

Let Q_2 be the rate of discharge in JB and Q_3 that in JC. Then the rate of discharge in AJ is

$$Q_1 = Q_2 + Q_3. \qquad (7.12,9)$$

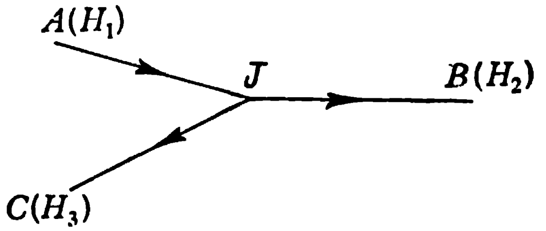

Fig. 7.12,1. Branched pipe system.

Now let H be the head, at present unknown, at the junction J. Then

$$H_1 - H = k_1(Q_2 + Q_3)^2$$
$$H - H_2 = k_2Q_2^2$$
$$H - H_3 = k_3Q_3^2.$$

The first step in the solution consists in obtaining two equations from which H has been eliminated. Add the last equation of the set to the first

$$H_1 - H_3 = k_1(Q_2 + Q_3)^2 + k_3Q_3^2. \qquad (7.12,10)$$

Subtract the second equation from the third

$$H_2 - H_3 = k_3Q_3^2 - k_2Q_2^2. \qquad (7.12,11)$$

We now have two equations† in the two unknowns Q_2 and Q_3. It will be convenient to reduce these to non-dimensional forms as follows: put

$$H_1 - H_3 = H_0 \qquad (7.12,12)$$

$$s = \frac{H_2 - H_3}{H_1 - H_3} \qquad (7.12,13)$$

$$\frac{k_2}{k_1} = c_2 \qquad (7.12,14)$$

$$\frac{k_3}{k_1} = c_3 \qquad (7.12,15)$$

$$x_2 = Q_2\sqrt{\frac{k_1}{H_0}} \qquad (7.12,16)$$

$$x_3 = Q_3\sqrt{\frac{k_1}{H_0}}, \qquad (7.12,17)$$

---

† Equation (7.12,10) follows at once from the fact that the sum of the heads lost in AJ and JC is $H_1 - H_3$ while the difference of the heads lost in JC and JB is $H_2 - H_3$.

where s, c_2, c_3, x_2 and x_3 are all non-dimensional. Then equations (7.12,10) and (7.12,11) become after a little reduction

$$1 = (x_2 + x_3)^2 + c_3 x_3^2 \tag{7.12,18}$$

$$s = c_3 x_3^2 - c_2 x_2^2. \tag{7.12,19}$$

To solve these we shall eliminate x_2. Multiply the first by c_2 and use the second to eliminate x_2^2. We get after a little reduction

$$x_2 = \frac{s + c_2}{2c_2 x_3} - \frac{x_3(c_2 + c_3 + c_2 c_3)}{2c_2}. \tag{7.12,20}$$

Next multiply (7.12,19) by $4c_2 x_3^2$ and use (7.12,20) to eliminate x_2. The final result after reduction may be written

$$Ax_3^4 - 2Bx_3^2 + C = 0 \tag{7.12,21}$$

where

$$A = (c_2 + c_3 + c_2 c_3)^2 - 4c_2 c_3$$

$$B = (s + c_2)(c_2 + c_3 + c_2 c_3) - 2sc_2$$

$$C = (s + c_2)^2.$$

Equation (7.12,21) may be solved for x_3^2 in the usual way and x_3 is finally obtained as the positive square root of this; one of the two possible values of x_3^2 will be rejected because it does not lead to a positive value of x_2. The solution of the problem is then

$$Q_3 = x_3\sqrt{\frac{H_0}{k_1}}$$

$$Q_2 = x_2\sqrt{\frac{H_0}{k_1}} = Q_3\left(\frac{x_2}{x_3}\right)$$

where x_2/x_3 may be obtained from (7.12,20). Alternatively x_2 can be obtained from (7.12,19) or this equation may be used to check the work. In the special case where the heads at B and C are equal ($s = 0$) equation (7.12,19) gives

$$\frac{x_2}{x_3} = \sqrt{\frac{c_3}{c_2}}$$

and (7.12,18) yields at once

$$x_3 = \sqrt{\frac{c_2}{(c_2 + c_3 + c_2 c_3) + 2\sqrt{c_2 c_3}}}. \tag{7.12,22}$$

This can also be obtained by application of (7.12,3) and (7.12,6) since in this special case JB and JC are effectively in parallel.

The foregoing general solution is valid only so long as the flows are in the directions of the arrows in Fig. 7.12,1. Unless the solution yields positive values for both x_2 and x_3 the equations must be recast in accordance with the true directions of flow. The system is in a critical state when x_2 is zero. Then (7.12,19) gives

$$x_3^2 = \frac{s}{c_3}$$

while (7.12,20) yields

$$x_3^2 = \frac{s + c_2}{c_2 + c_3 + c_2c_3}\,.$$

Hence

$$s(c_2 + c_3 + c_2c_3) = c_3(s + c_2)$$

which gives

$$s = \frac{c_3}{1 + c_3}\,. \qquad (7\ 12,23)$$

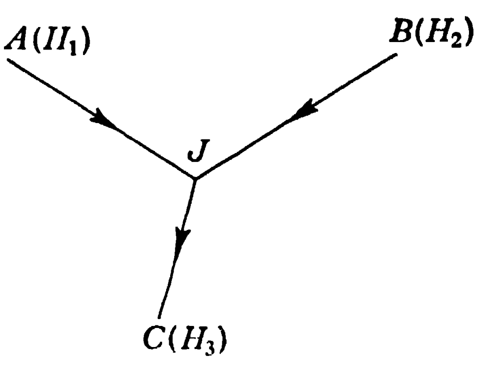

Fig. 7.12,2. Branched pipe system.

When s exceeds this critical value the flow in JB cannot be in the direction assumed and the equations must be recast.

Now let us consider the system shown in Fig. 7.12,2. We can obtain the solution by the following device. Imagine all the flows to be reversed in direction while all the heads are reversed in sign. Then all conditions are satisfied and the system is of the type already discussed in detail, so the formulae already established may be applied, with due care in the interpretation. To assist in avoiding error, C and A may be labelled A' and C' respectively.

When there is only a single junction in the system, the solution can easily be obtained graphically for any law of resistance and for any number of pipes. To illustrate this method, let us consider the system shown in Fig. 7.12,1. Assume any value of the head H at the junction (but this must obviously lie between H_1 and H_3). Then we can find at once the values of the rates of discharge Q_1, Q_2 and Q_3. Repeat this for several values of H and plot graphs of Q_1 and $(Q_2 + Q_3)$ on a base of H. The point of intersection of these graphs gives the true values of Q_1 and of H, so a complete solution is obtained. It will often be possible to make a good guess at the value of H, so much plotting is not needed. Moreover, it is always obvious whether H must be increased or decreased. Thus, if Q_1 is greater than $(Q_2 + Q_3)$, H is too small whereas it is too large if Q_1 is less than $(Q_2 + Q_3)$.

If there are two or more junctions the graphical method already described is not generally applicable and we are driven to use some process of successive approximation to the solution. One such process is 'relaxation'.[1]

We begin by assigning plausible values to the heads at the junctions. Then we keep all but one of these constant and find the value of this required to satisfy the condition of continuity at the corresponding junction. The other unknown heads are adjusted in succession in the same way. However, these adjustments will, in general, upset the balance at the junction first treated and the head there must be further adjusted. The process is continued until a satisfactorily close approximation is obtained. A device which can sometimes be used to obtain a good initial approximation consists in simplifying the system in a suitable manner. For instance, the system may contain some pipes of relatively high resistance which could be supposed to be closed without influencing the principal flows to an important extent. The solution may also be obtained from a suitable analogic computer.

Finally, it should be noted that a direct solution based on equations (7.12,4) and (7.12,7) can always be obtained when the system is effectively an arrangement in series of sets of pipes in parallel. A set of pipes is effectively in parallel when they meet at one junction and are subject to equal heads at their other ends. When dealing with more complex systems it will be of advantage to replace sets of pipes in series, parallel or series—parallel by equivalent single pipes.

EXERCISES. CHAPTER 7

1. A uniform circular tube of bore radius a has a fixed coaxial circular-cylindrical rigid core of radius b. Find the axial velocity u in the steady laminar flow of an incompressible viscous fluid under a pressure gradient $\left(-\dfrac{dp}{dx}\right)$ and the rate of discharge Q.

$$\left(\text{Answer}\quad u = \frac{(-dp/dx)}{4\mu}\left\{a^2 - r^2 - (a^2 - b^2)\frac{\ln(a/r)}{\ln(a/b)}\right\}\right.$$

$$\left. Q = \frac{\pi(-dp/dx)(a^2 - b^2)}{8\mu}\left\{a^2 + b^2 - \frac{a^2 - b^2}{\ln(a/b)}\right\}\right)$$

2. In the last exercise discuss the case where $h = a - b$ is small and compare with flow between parallel planes.

   (Answer The ratio of the rate of discharge from the annulus per unit circumference to Q' for parallel planes differs from unity only by terms in the first and higher powers of h/a)

3. In the problem of Exercise 1 find the frictional stresses f_a and f_b at the outer and inner walls of the tube.

$$\left(\text{Answer}\quad f_a = \frac{1}{4}\left(-\frac{dp}{dx}\right)\left\{2a - \frac{a^2 - b^2}{a\ln(a/b)}\right\}\right.$$

$$\left. f_b = \frac{1}{4}\left(-\frac{dp}{dx}\right)\left\{-2b + \frac{a^2 - b^2}{b\ln(a/b)}\right\}\right)$$

4. In the problem of Exercise 1 the core moves axially downstream with constant velocity u_0. Find the additional terms in the velocity and rate of discharge.

$$\left(\text{Answer}\quad \frac{u_0\ln(a/r)}{\ln(a/b)},\quad \frac{\pi u_0(a^2 - b^2)}{2\ln(a/b)} - \pi u_0 b^2\right)$$

5. *In the Erk viscometer* (whose principle is due to Couette) a pair of capillary tubes of the *same* bore are placed in series. Their lengths are l_1, l_2 with l_1 considerably greater than l_2, and the corresponding pressure drops are p_1, p_2. Show that, provided l_2 is great enough for the final parabolic velocity distribution to be set up before the exit, the true pressure drop per unit length for such a velocity distribution is
$$\Pi = \frac{p_1 - p_2}{l_1 - l_2}.$$
(For details of the instrument, see Barr's 'Monograph of Viscometry'.)

6. In what circumstances will the conclusions of § 7.5 regarding pipes of non-circular section remain valid when the surface roughness is not everywhere the same? (Answer When the roughness is constant along any line parallel to the axis of the pipe)

7. A viscous fluid is in steady laminar motion through a tube whose section is a very narrow rectangle of depth h under the influence of a pressure gradient. Find the rate of discharge Q' per unit width in the part of the section not near its ends. $\left(\text{Answer}\quad Q' = \frac{h^3}{12\mu}\left(-\frac{dp}{dx}\right)\right)$

8. The section of the tube discussed in Exercise 7 is of total width b. Find the rate of discharge approximately by regarding this as the sum of the discharges for a rectangular tube of depth h and width $2h$ and of a layer of depth h and width $(b - 2h)$. (Assume that S of equation (7.6,13) has the value 0·628 when $n = 0{\cdot}5$.) $\left(\text{Answer}\quad Q = \frac{1}{12}h^3\left(\frac{p_1 - p_2}{\mu l}\right)(b - 0{\cdot}628h)\right)$

9. Liquid is in laminar motion through a tube whose section is an ellipse of area A. The pressure difference, length and viscosity are all fixed. Find the ratio of the axes of the ellipse when the rate of discharge is a maximum and give the maximum value in terms of A.

(Answer Maximum occurs when axial ratio is unity (circle). Then
$$Q = \frac{\pi^3 A^2(p_1 - p_2)}{2048\mu l}.\Bigg)$$

10. Verify that for pipes of circular bore, diameter d, the friction coefficient f of equation (7.2,6) is given by
$$f = \frac{gdi}{2v^2}$$
where i is the gradient of total head. If the head lost for a pipe of length l be $H = kQ^2$, show that
$$k = \frac{32lf}{\pi^2 g\, d^5}$$
Show also that
$$Q = \frac{\pi}{4}\sqrt{\left(\frac{gi\,d^5}{2f}\right)} \qquad d = \sqrt[5]{\left(\frac{32fQ^2}{\pi^2 gi}\right)}$$

11. A Venturi meter, throat diameter 0.4 m, is inserted in a water main of 1.2 m internal diameter. A mercury manometer correctly connected to the Venturi shows a difference of level of 112 mm and all connecting tubes are full of water. If the specific gravity of mercury is 13·6 and the coefficient of discharge of the Venturi is 0·97, calculate the discharge in cubic metres per 24 hours. (Answer $5{\cdot}76 \times 10^4$)

12. A cylindrical open-topped tank, 1·5 m internal diameter, with axis vertical,

discharges to atmosphere through a pipe 10 m long and of 80 mm internal diameter. Calculate the time for the water level in the tank to fall from 9 m to 6 m above the level of the pipe outlet. The friction coefficient f (see equation (7.2,6)) for the pipe is 0·008 and the entrance to the pipe is sharp-edged. (Assume loss of total head at entry to be half the dynamic head.) (Answer 51·3 s)

13. The kinematic viscosity of water flowing through a pipe of 125 mm bore is $1{\cdot}14 \times 10^{-6}$ m^2/s. Use Fig. 7.7,4 to find the loss of head in metres across 700 m of pipe when the rate of discharge is 4×10^{-2} m^3/s and the roughness ratio d/k is 200. (Answer 97·1 m)

14. A pipe carrying fresh water is of 75 mm bore diameter. When the rate of discharge is $3{\cdot}7 \times 10^{-2}$ m^3/s the loss of head between pressure tappings 60 m apart is 90 m. Find the friction coefficient f and the friction velocity u_τ. Estimate the thickness of the laminar sub-layer, given that $\nu = 1{\cdot}14 \times 10^{-6}$ m^2/s. (Answer $f = 0{\cdot}0079$ $u_\tau = 0{\cdot}527$ m/s.
Thickness of sub-layer $5{\cdot}4 \times 10^{-3}$ mm)

15. A pipe AB of 150 mm circular bore is 180 m long and has friction coefficient $f = 0{\cdot}005$ when discharging water at the rate of 0·04 m^3/s. At B water is bled off at the rate 0·01 m^3/s and the remaining water is discharged through a pipe BC of 125 mm circular bore, 270 m long and with $f = 0{\cdot}006$. Calculate the total loss of head from A to C (neglect dynamic head). (Answer 22·6 m)

16. The roughness ratio of a pipe of 50 mm bore diameter is known to be 500. Use Fig. 7.4,4 to estimate the kinematic viscosity of the water flowing through the pipe from the following data:
    Loss of head 730 mm over 15 m length of pipe
    Rate of discharge $2{\cdot}68 \times 10^{-3}$ m^3/s (Answer $\nu = 1{\cdot}36 \times 10^{-6}$ m^2/s)

17. Discuss the limitations of the method used in the last exercise in view of the characteristics shown in Fig. 7.4,4.

18. Water ($\nu = 1{\cdot}14 \times 10^{-6}$ m^2/s) flows through a pipe 610 m long and of 200 mm bore diameter. The head lost by friction is 15·5 m when the rate of discharge is 0·0713 m^3/s. Use Fig. 7.7,4 to estimate the roughness ratio d/k. (Answer 1,100)

19. Estimate the internal diameter of a pipe required to pass water at the rate of nine thousand cubic metres per 24 hours with a hydraulic gradient $i = 0{\cdot}002$, given that the friction coefficient f is 0·005. (Answer $d = 389$ mm)

20. A pipe 7·5 km long and 0·9 m internal diameter connects two reservoirs having a difference of level of 60 m. Calculate the discharge in U.S. gallons per day if $f = 0{\cdot}0047$ (consider friction losses only). (Answer $64{\cdot}0 \times 10^6$)

21. A solution of glycerine of specific gravity 1·05 and kinematic viscosity 8×10^{-5}m^2/s is to be pumped through a pipe 24 m long and of 75 mm bore diameter at a rate of 0·024 m^3/s. Calculate the net power required if the roughness factor is 200 (see Fig. 7.7,4). (Answer 5·23 kw)

22. A horizontal water main is of given length and the rate of discharge is specified. The water is driven through the main by a pump and the capital cost of the pumping station is $C_1 = a + bP$, where a and b are constants while P is the hydraulic power wasted in the pipe by friction. On the assumption that the friction coefficient f is fixed, show that $P = c/d^5$, where c is a constant. If the capital cost of the pipe line is $C_2 = m + nd^2$, where m and n are constants, find the bore diameter d so that the total capital cost shall be a minimum. $\left(\text{Answer} \quad d = \sqrt[7]{\left(\frac{5bc}{2n}\right)}\right)$

23. Two sudden enlargements of a pipe of circular section occur between an upstream part of sectional area A_1 and a downstream part of sectional area A_3. Find the sectional area A_2 of the intermediate portion for maximum pressure recovery. (Neglect frictional losses and any interference between the two enlargements.) Compare the maximum pressure recovery with that for a single expansion from A_1 to A_3. $\left(\text{Answer} \quad A_2 = \dfrac{2A_1A_3}{A_1 + A_3}.\right.$

Increase of pressure recovery $\left.\dfrac{1}{4}\rho v_1^2\left(1 - \dfrac{A_1}{A_3}\right)^2\right)$

24. The pipe AJ is 122 m long, 300 mm bore diameter and the friction coefficient f is 0·006. The pipe JB is 36 m long, 200 mm bore diameter and f is 0·0079 while pipe JC is 18 m long but otherwise identical with JB. A and B are 30 m and 12 m respectively above C. Find the rates of discharge in the three pipes. (Neglect entry losses. It may be assumed that the flow in JB is from J to B.)

(Answer. Rates of discharge in m³/s: AJ 0·398, JB 0·087, JC 0·311)

25. Water flows from reservoir A with surface level 30 m above datum through a pipe 300 m long, 0·30 m bore diameter, $f = 0{\cdot}006$, to the junction J. Reservoir B, surface level 6 m higher than A, discharges to junction J through a pipe 450 m long, 0·38 m bore diameter, $f = 0{\cdot}005$. The total discharge from J is taken through a pipe 600 m long, 0·45 m bore diameter, $f = 0{\cdot}004$, to datum level at C (atmospheric pressure). Calculate the rate of discharge in each pipe. (Answer AJ 0·219, BJ 0·436, JC 0·655 m³/s)

CHAPTER 8

FLOW IN OPEN CHANNELS

8.1 Introduction

Flow in an open channel differs from flow in a pipe inasmuch as the liquid has a free surface; on account of the additional freedom of the liquid new and interesting phenomena become possible. The liquid at the free surface is in contact with the atmosphere; the pressure there is the same as that of the atmosphere and is usually assumed to be constant. Let us now consider flow in a channel of rectangular section, the bottom of the rectangle being horizontal, and let h be the depth of the layer of liquid which, for simplicity, we here assume to be constant. As will be shown in § 10.5 any gravitational wave of small amplitude whose wave length is much greater than h is propagated relative to the fluid with velocity $\sqrt{gh}$. Suppose that the velocity of the fluid along the channel is uniform and equal to v. Then no wave of the type mentioned can be stationary relative to the walls of the channel when $\sqrt{gh}$ exceeds v; the wave front will either move upstream with speed $\sqrt{(gh)} - v$ relative to the walls or downstream with speed $\sqrt{(gh)} + v$. In such conditions the flow is said to be *subundal* or *tranquil.* However, when v is equal to or greater than $\sqrt{gh}$ it is possible for a wave front to occupy a fixed position (perhaps oblique) relative to the walls. The flow is called *superundal*† or *shooting* when v exceeds $\sqrt{gh}$. It will be shown in § 8.4 that a hydraulic jump, which is a rather sudden increase in the depth of the flowing layer of liquid, can occur only when the flow is superundal. In a general way it may be said that subundal flow in an open channel differs little from flow in a pipe but in superundal flow quite new phenomena may occur.

There is a close analogy between flow of liquid in an open channel and flow of a compressible fluid. Subundal and superundal flows of the liquid correspond respectively to subsonic and supersonic flows of the compressible fluid while the hydraulic jump corresponds to a shock wave (see § 8.4).

In this chapter we consider only the motion of liquids which are supposed to be incompressible and homogeneous. The gravitational field is assumed to be uniform and of intensity g.

8.2 Steady and Uniform Regimes

We suppose here that the channel is of cylindrical form‡ and the slope of the bed is s, i.e. any generating line of the cylinder makes an angle with the horizontal whose *sine* is equal to s. When the regime is here said to be uniform the velocity is the same at corresponding points of all cross-sections of the channel but it need not be constant across the section. The regime is steady when the velocity at any given point is independent of time but, when the flow is turbulent, it is to be understood that the velocity is averaged over a considerable time interval.

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† The terms subundal and superundal were introduced by Binnie.

‡ This covers cylinders of any sectional shape, also prismatic forms.

Let us consider an element of fluid contained between a pair of normal planes at unit distance apart. As the regime is uniform and steady, every fluid particle is unaccelerated and the forces on the element must therefore be in equilibrium. Since the atmospheric pressure at the free surface is assumed to be constant, it follows that the pressures at corresponding points of any two normal sections are the same, so the pressures on the element are themselves in equilibrium. Consequently the component of the weight of the element in the direction of flow must balance the frictional force on the element at the bed of the channel. Let τ be the frictional stress at the bed in the direction of the generators (the direction of mean flow) and let dw be an element of the wetted perimeter. Then the total frictional resistance per unit length is

$$\int \tau\, dw = w\tau_m \tag{8.2,1}$$

where w is the total wetted perimeter and τ_m is the mean frictional stress. The weight of the element is $\rho g A$, where A is the cross-sectional area of the channel up to the free surface,† and the component of the weight in the direction of the mean flow is $s\rho g A$. Consequently the condition of equilibrium is

$$w\tau_m = s\rho g A$$

or

$$\tau_m = s\rho g m \tag{8.2,2}$$

where

$$m = \frac{A}{w} \tag{8.2,3}$$

is the *hydraulic mean depth*. Now we find by dimensional analysis that

$$\frac{\tau_m}{\frac{1}{2}\rho v^2} = \phi\left(R, \frac{k}{m}\right). \tag{8.2,4}$$

In this equation v is the mean velocity of flow, related to Q, the volume discharged in unit time, by the equation

$$Q = vA, \tag{8.2,5}$$

k is the roughness diameter and R is the Reynolds number defined by

$$R = \frac{vm}{\nu}. \tag{8.2,6}$$

The method of derivation of (8.2,4) is essentially the same as that of (7.2,8) but here the hydraulic mean depth m is adopted as the typical length in place of d. In view of (8.2,4) equation (8.2,2) becomes

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† Thus A is the area of the 'wetted cross-section'.

$$\frac{sgm}{\frac{1}{2}v^2} = \phi\left(R, \frac{k}{m}\right)$$

and this may be rewritten

$$v = C_0\sqrt{2gsm} \tag{8.2,7}$$

where

$$C_0 = \sqrt{\frac{1}{\phi\left(R, \frac{k}{m}\right)}} \tag{8.2,8}$$

is a non-dimensional coefficient. Since g is very nearly constant at the earth's surface it is usual to put

$$C = C_0\sqrt{2g} \tag{8.2,9}$$

where C is called the *Chézy coefficient*,† and (8.2,7) becomes the well-known Chézy formula

$$v = C\sqrt{sm}. \tag{8.2,10}$$

The Chézy coefficient is not non-dimensional and its numerical value depends on the choice of the units of length and time. Since the unit of time is the same (the second) for the British and metric systems it follows that

$$C\,(\text{feet}) = C\,(\text{metres}) \times 1{\cdot}81 \tag{8.2,11}$$

where 1·81 is the square root of the number of feet in a metre. The value of C_0 depends on the shape of the wetted section of the channel, the Reynolds number and the roughness. Natural waters may carry silt in suspension and it is found that the silt tends to increase the rate of flow and therefore the values of the coefficients C_0 and C.

Since the original introduction of the Chézy formula towards the end of the 18th century a very large number of attempts have been made to represent the Chézy coefficient as a function of one or more of the quantities v, m, s and R. While many of these were of some practical utility for restricted ranges of the variables, few if any were well based. We quote here a formula given by Thijsse‡ which is reasonably based and is probably as reliable as any:

$$C = \phi\sqrt{g} \tag{8.2,12}$$

with

$$\phi = 5{\cdot}75\log_{10}\left(\frac{12m}{\frac{\delta}{\Lambda} + k}\right) \tag{8.2,13}$$

where k is a roughness diameter which may be estimated with the help of Table 8.2,1 while δ is the thickness of the laminar sub-layer at a smooth wall as given by

$$\delta = \frac{12\nu}{u_\tau} \tag{8.2,14}$$

where u_τ is the 'friction velocity' defined by the equation

$$u_\tau = \sqrt{\frac{\tau_m}{\rho}} = \sqrt{gms}. \tag{8.2,15}$$

It should be noted that, according to experience, k increases with age.

TABLE 8.2,1.

Magnitude of Surface Irregularity k (in feet and metres) (after Thijsse).

	ft	m		ft	m
Glass	1×10^{-4}	3×10^{-5}	Unfinished concrete	1×10^{-2}	3×10^{-3}
Asphalted steel	1×10^{-4}	3×10^{-5}	Pitched stones, mortar jointed	1×10^{-2}	3×10^{-3}
Vibrated concrete	2×10^{-4}	6×10^{-5}	Smooth earth	2×10^{-2}	6×10^{-3}
Smooth welded steel	3×10^{-4}	1×10^{-4}	Bad brickwork	2×10^{-2}	6×10^{-3}
Very smooth concrete	6×10^{-4}	2×10^{-4}	Pitched stones	$2{\cdot}5\times10^{-2}$	1×10^{-2}
Steel with c.s.k. rivets	1×10^{-3}	3×10^{-4}	Old concrete	5×10^{-2}	2×10^{-2}
Planed wood	1×10^{-3}	3×10^{-4}	Gravel	7×10^{-2}	2×10^{-2}
Cement plaster	$1{\cdot}5\times10^{-3}$	5×10^{-4}	Coarse gravel	2×10^{-1}	5×10^{-2}
Rough wood	2×10^{-3}	7×10^{-4}	Stones	4×10^{-1}	1×10^{-1}
Good brickwork	2×10^{-3}	7×10^{-4}	Channels with vegetation	5×10^{-1}	1.5×10^{-1}
Steel, riveted	$2{\cdot}5\times10^{-3}$	1×10^{-3}	Rocks	$7{\cdot}5\times10^{-1}$	2×10^{-1}
Old wood	5×10^{-3}	2×10^{-3}	Channel with moving sand	0·03—0·30	0·01 to 0·10
Badly rusted steel	5×10^{-3}	2×10^{-3}	Channel with obstacles	1·0—2·0	0·3 to 0·5

For a given channel the Chézy formula (8.2,10) is often written

$$Q = K\sqrt{s} \tag{8.2,16}$$

where K is called the *conveyance factor* of the channel and is given by

$$K = AC\sqrt{m}. \tag{8.2,17}$$

Suppose that Q and s are given and that it is required to find the corresponding depth h_n. Equation (8.2,16) determines K and h_n can be found from a graph in which K, as defined by (8.2,17), is plotted against h.

8.3 General Theoretical Considerations

Any real liquid is viscous and the flow in channels is almost always turbulent. However, it is legitimate as a working approximation to neglect friction and even to treat the flow as irrotational in the theory of certain phenomena which occur within a short length of channel, whereas the influence of friction on the flow throughout a considerable length of channel is never negligible. The theory must be developed so as to cover adequately both these aspects of the matter.

Let z be the vertical ordinate measured upwards from a horizontal datum plane to the point considered and let z_0 refer to the lowest point of the bed at the section considered. For convenience we shall put

$$y = z - z_0 \tag{8.3,1}$$

so y is the vertical ordinate referred to the lowest point of the section as local origin, see Fig. 8.3,1. The horizontal breadth of the section at the height y above this origin is $b(y)$ but this will be written b when it is constant or where it is unnecessary to emphasise its dependence on y. Distance along the channel is designated x and the slope of the bed is

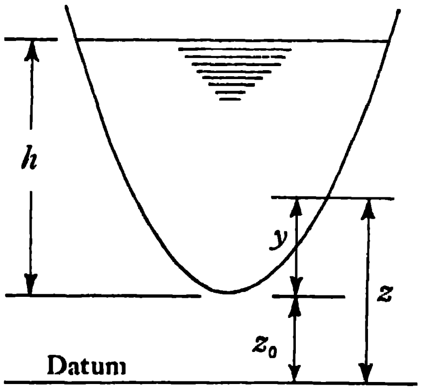

Fig. 8.3,1. Typical section of channel.

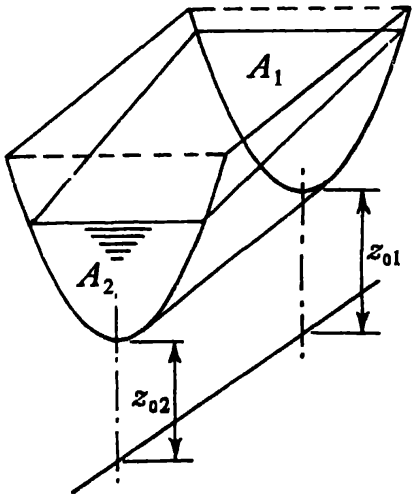

Fig. 8.3,2. Perspective view of channel.

$$s = -\frac{dz_0}{dx} \tag{8.3,2}$$

The pressure at any point is p while the atmospheric pressure, which is assumed to be constant, is p_0. The local value of the velocity of the fluid parallel to the generators of the channel is u while v is the mean value of u taken over a normal section of the channel (see equation (8.2,5)). Since the other components of the velocity are usually small, it is almost always legitimate to take the local resultant speed of the fluid to be u.

We shall begin by considering the application of the Principle of Linear Momentum (see §§ 3.2, 3.6) to steady states of flow and as a first step we must find the thrust T on the fluid at any section of the channel due to the pressure. Let h (assumed constant at the section) be the value of y at the free surface. Then the pressure at a current point of the wetted section is†

$$p = g\rho(h - y) + p_0 \tag{8.3,3}$$

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† This expression for the pressure is valid only when the vertical curvature of the streamlines is very small and the free surface is horizontal across the channel.

and, if dA is an element of area of the wetted section,

$$\begin{aligned} T &= \iint p\, dA \\ &= g\rho\iint (h-y)\, dA + p_0 A \\ &= \{g\rho(h-\bar{y}) + p_0\}A \end{aligned} \tag{8.3,4}$$

where $\bar{y}$ is the ordinate of the centroid of the wetted section (see § 1.8). The mass of fluid passing through an element of area dA in unit time is $\rho u\, dA$ and the momentum of this along the channel is $\rho u^2\, dA$. Hence the total flux of momentum across the section in unit time is $\iint \rho u^2\, dA$. Now let us consider the part of the channel between any pair of normal sections; we shall use the suffix 1 for the upstream section and the suffix 2 for the downstream section. Then, since the flow under consideration is steady, the equation of linear momentum can be written

$$\rho\iint_2 u^2\, dA - \rho\iint_1 u^2\, dA = sW_{12} + T_1 - T_2 - F_{12} - p_0(A_1 - A_2) \tag{8.3,5}$$

where W_{12} is the weight of liquid in the channel between the sections considered, F_{12} is the frictional force, opposing the flow, on the wetted surface between the sections while the final term on the right hand side of the equation is the component in the direction of flow of the atmospheric thrust on the free surface of the liquid.† When we use (8.3,4) to express T_1 and T_2 we find that (8.3,5) becomes

$$\begin{aligned} \rho\iint_2 u^2\, dA - \rho\iint_1 u^2\, dA &= sW_{12} \\ &\quad + g\rho[A_1(h_1 - \bar{y}_1) - A_2(h_2 - \bar{y}_2)] - F_{12}. \end{aligned} \tag{8.3,6}$$

In addition we have the equation of continuity

$$Q = \iint_1 u\, dA = \iint_2 u\, dA \tag{8.3,7}$$

where Q is the volume discharged in unit time. The foregoing equations (8.3,5) and (8.3,6) are valid only when the channel is of cylindrical or prismatic form. If it is tapered there will be additional terms representing the thrust, along the channel, of the pressures on the walls.

We shall next investigate the rate at which mechanical energy is being dissipated by friction in the part of the channel between the normal sections 1 and 2. The mechanical energy of the liquid is the sum of its kinetic and gravitational potential energies; as it is assumed to be incompressible its intrinsic pressure energy is zero. Since the regime of flow considered is steady, the amount of energy E_{12} dissipated in unit time in the region between the sections 1 and 2 is equal to the excess of the flux of energy across section 1 over that across section 2, augmented by the net work done in unit time by the thrusts at sections 1 and 2 (see § 3.2). Hence we find that

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† The total thrust on the liquid between sections 1 and 2 due to p_0 is zero in accordance with the theorem that a uniform pressure gives zero resultant thrust on any closed surface. Hence p_0 does not appear in (8.3,6).

$$\begin{aligned}E_{12} &= \rho\iint_1 u(\tfrac{1}{2}u^2 + gz)\,dA - \rho\iint_2 u(\tfrac{1}{2}u^2 + gz)\,dA \\ &\quad + \iint_1 u\{p_0 + g\rho(h_1 - y)\}\,dA - \iint_2 u\{p_0 + g\rho(h_2 - y)\}\,dA \\ &= \rho\iint_1 u(\tfrac{1}{2}u^2 + gz)\,dA - \rho\iint_2 u(\tfrac{1}{2}u^2 + gz)\,dA \\ &\quad + g\rho\iint_1 u(h_1 - y)\,dA - g\rho\iint_2 u(h_2 - y)\,dA\end{aligned}$$

on account of the equation of continuity (8.3,7). Now use equation (8.3,1) and let z_{01}, z_{02} be the values of z_0 at sections 1 and 2 respectively. The last equation can then be written†

$$\begin{aligned}\frac{E_{12}}{g\rho} &= \iint_1 u\left(z_{01} + h_1 + \frac{u^2}{2g}\right)dA - \iint_2 u\left(z_{02} + h_2 + \frac{u^2}{2g}\right)dA \\ &= Q(z_{01} + h_1 - z_{02} - h_2) + \iint_1 \frac{u^3}{2g}\,dA - \iint_2 \frac{u^3}{2g}\,dA. \qquad (8.3,8)\end{aligned}$$

It will be noted that the multiplier of Q in the last equation is the total fall of level at the free surface in passing from section 1 to section 2. Suppose now that the velocity is uniform at each of these sections, i.e. $u = v_1$ at section 1 while $u = v_2$ at section 2. Then (8.3,8) becomes

$$\frac{E_{12}}{g\rho Q} = z_{01} + h_1 - z_{02} - h_2 + \frac{v_1^2 - v_2^2}{2g}. \qquad (8.3,9)$$

If no energy were dissipated this would show that $[z_0 + h + (v^2/2g)]$ was constant, i.e. that the total head at the free surface was constant, since the pressure there has the constant value p_0. This is in accordance with Bernoulli's theorem (see below). Equation (8.3,9) can be interpreted as follows: The energy dissipated per unit length of channel in unit time is equal to the gradient of total head multiplied by $g\rho Q$. Let i be the gradient of the total head,‡ and let e be the energy dissipated in unit length of channel in unit time. Then (8.3,9) is equivalent to

$$e = g\rho Qi. \qquad (8.3,10)$$

---

† If the lateral and vertical components of velocity are not so small that their squares can be neglected, then u^3 in (8.3,8) must be replaced by uq^2 where q is the resultant speed.

‡ It may be noted that i is a non-dimensional quantity. It is positive when the total head falls in the direction of flow.

Now when the velocity is constant across the section e is equal to v multiplied by the resistance of unit length of channel, i.e.

$$e = v\tau_m w. \tag{8.3,11}$$

Hence (8.3,10) yields

$$\tau_m = g\rho mi. \tag{8.3,12}$$

It is to be understood, however, that it is often a crude approximation to treat the velocity as constant over the section. Equation (8.3,8) is valid even when the channel is tapered, provided that the obliquity of flow is never great.

We shall next consider the flow of a perfect fluid in an open channel. It is not to be expected that the theory will yield results in exact quantitative agreement with experiments made with real viscous liquids but there is ample justification for the development of the theory since it provides completely adequate explanations of many of the important phenomena of flow in open channels. Attention will be confined to steady flow, so Bernoulli's theorem (see § 3.4) can be applied. Accordingly the total head

$$\frac{q^2}{2g} + \frac{p}{g\rho} + z \tag{8.3,13}$$

is constant along any streamline, where q is the resultant speed of the fluid. For a streamline in the free surface the pressure has the constant value p_0 and $[(q^2/2g) + z]$ is therefore constant; thus the resultant speed and height are functionally related.

Now let us suppose that the liquid flows into the channel from a large reservoir where the velocity is so small that its square is negligible. The free surface in the reservoir will then be a horizontal plane at a height h_1 above the datum plane. We shall take the channel to have a horizontal bed, which will be adopted as the datum plane, and to be of rectangular section of breadth b; the entry to the channel is well faired so that the flow is smooth. In the reservoir the total head has the constant value $h_1 + (p_0/g\rho)$ and this is therefore the value of the total head for *all* the streamlines; it follows that the flow is everywhere irrotational.† At any section of the channel not too near the entry the flow will be parallel to the bed and sides and, being irrotational, the velocity must be uniform. Let v be the constant velocity and h the depth. Then Bernoulli's equation for a streamline in the surface is

$$\frac{v^2}{2g} + \frac{p_0}{g\rho} + h = \frac{p_0}{g\rho} + h_1$$

or

$$v = \sqrt{2g(h_1 - h)} \tag{8.3,14}$$

and the volume of fluid discharged in unit time is

$$Q = bh\sqrt{2g(h_1 - h)}. \tag{8.3,15}$$

---

† This also follows at once from Kelvin's theorem (see § 3.7) since the circulation in all circuits is zero in the reservoir.

Let h_1, b and g be constant and consider the dependence of Q on h. We see that Q is zero when $h = 0$ and when $h = h_1$ while

$$\frac{dQ}{dh} = b\sqrt{(2g)}\,\frac{(h_1 - \frac{3}{2}h)}{\sqrt{(h_1 - h)}}. \tag{8.3,16}$$

Hence Q is a maximum when

$$h = h_c = \tfrac{2}{3}h_1 \tag{8.3,17}$$

and this is called the *critical depth.* By (8.3,15) the maximum rate of discharge is

$$Q_m = b\sqrt{\left(\frac{8}{27}gh_1^3\right)}. \tag{8.3,18}$$

Suppose that Q is given and that h is to be found. Then from (8.3,15)

$$h^3 - h_1h^2 + \frac{Q^2}{2gb^2} = 0 \tag{8.3,19}$$

which can also be written

$$\left(\frac{h}{h_1}\right)^3 - \left(\frac{h}{h_1}\right)^2 + \frac{4}{27}\left(\frac{Q}{Q_m}\right)^2 = 0 \tag{8.3,20}$$

or

$$2\left(\frac{h}{h_c}\right)^3 - 3\left(\frac{h}{h_c}\right)^2 + \left(\frac{Q}{Q_m}\right)^2 = 0. \tag{8.3,21}$$

For any value of Q between zero and Q_m these cubic equations yield two real and positive values of h; the third root is real and negative and has no physical significance. Hence there are two alternative values of the depth corresponding to any possible value of the rate of discharge.

Suppose that the channel is provided with a sluice gate downstream for regulating the flow and let the gate be closed initially. Then the fluid will be at rest and the depth at the section considered will be h_1. Now let the gate be opened so slowly that the state of affairs at any instant does not differ appreciably from a steady state. Then h will fall continuously as the discharge increases until finally it reaches the critical value h_c when the flow is quite unrestricted. In this process h never attains any value lying between h_c and zero (see further § 8.5).

When $h = h_c$ and Q is a maximum the velocity as given by equation (8.3,14) is the critical velocity

$$v_c = \sqrt{\tfrac{2}{3}gh_1} = \sqrt{gh_c} \tag{8.3,22}$$

and the Froude number v^2/gh is equal to unity. In general the Froude number is by (8.3,14)

$$\frac{v^2}{gh} = \frac{2(h_1 - h)}{h}. \tag{8.3,23}$$

It follows from this that the Froude number increases steadily from zero towards infinity as h falls from h_1 towards zero. When $h > h_c$ the Froude number is less than unity; the flow is then called tranquil or subundal. However, when $h < h_c$ the Froude number is greater than unity; the flow is now called shooting or superundal. It is proved in § 8.4 that a hydraulic jump cannot occur in a rectangular channel unless the Froude number exceeds unity. It follows from the argument given above that superundal flow will only be attained when the channel has a downward slope or bottom-opening sluices are used. This is further considered in § 8.5.

Now let us suppose that the channel, while still rectangular in section and having a horizontal bed, varies gradually in width from section to section; the flow will still be supposed steady so that Q is the same at all sections. When Q, h_1 and g are constant we get from (8.3,15)

$$\begin{aligned}\frac{db}{dh} &= \frac{Q}{\sqrt{2g}}\frac{3h - 2h_1}{2h^2(h_1 - h)^{3/2}}\\ &= \frac{b(3h - 2h_1)}{2h(h_1 - h)}. \qquad (8.3,24)\end{aligned}$$

This shows that dh/db is positive when $h > h_c$ but negative when $h < h_c$. Suppose now that there is a constriction in the width of the channel (db negative). Then when the flow is subundal ($h > h_c$), dh is also negative, i.e. the level of the free surface falls at the constriction. But when the flow is superundal ($h < h_c$) the level of the free surface rises at the constriction.

ILLUSTRATIVE EXAMPLES

Example 1. *Investigate the change in level of the free surface associated with a smooth change of level of the bed*

We assume the fluid to be perfect and the channel to be of constant breadth. The flow is uniform at sections 1 and 2. Between these sections there is a faired step in the bed so that the bottom at section 2 is at a height h above that at section 1. Accordingly we have

$$h_1 + \frac{v_1^2}{2g} = h + h_2 + \frac{v_2^2}{2g}$$

and

$$h_1 v_1 = h_2 v_2.$$

When v_2 is eliminated we get

$$h_2^3 - h_2^2\left(h_1 - h + \frac{v_1^2}{2g}\right) + \frac{v_1^2 h_1^2}{2g} = 0.$$

The rise in level of the free surface is

$$x = h + h_2 - h_1.$$

When we substitute for h_2 in terms of x and reduce, we obtain

$$x^3 + 2x^2\left(h_1 - h - \frac{v_1^2}{4g}\right) + x(h_1 - h)\left(h_1 - h - \frac{v_1^2}{g}\right) + \frac{v_1^2}{2g}h(2h_1 - h) = 0$$

When $h = 0$ this obviously has the solution $x = 0$. Now let h and x be small quantities and neglect squares and products of these. We get

$$xh_1\left(h_1 - \frac{v_1^2}{g}\right) + \frac{v_1^2hh_1}{g} = 0$$

or

$$x = \frac{v_1^2h}{v_1^2 - gh_1}.$$

When the flow before the step is subundal x is negative, i.e. the free surface falls. The reverse is true when the flow is superundal.

8.4 Hydraulic Jump

It is observed that when water is flowing fast, and especially when it is shallow, a rather sudden increase of depth may occur at a well-marked front. This is called the hydraulic jump and it may be demonstrated very easily in a laboratory experiment in which a jet of water falls vertically into a shallow layer of water. The jet is surrounded by a very shallow layer of water moving radially at high speed and this is bounded by the jump, which is roughly circular in plan; beyond the jump the layer is deeper and the velocity lower. The phenomenon, as it occurs in a channel, can be described as follows (see Fig. 8.4,1). At section 1 the layer of liquid is shallow and the velocity high while at section 2 the layer is deeper and the velocity correspondingly lower. The steep wave front at A has a fixed position relative to the channel (though a slight wavering or oscillation may be present). The upper layer of liquid ABC moves backwards relative to the lower layer, the front at A being in the form of a 'roller'. At BC there is a thrust (due to fluid pressure) urging the fluid ABC towards the left but this is balanced by fluid friction and the conveyance of momentum by turbulent mixing in the region AB. In other words, the upper layer tends to spread itself over the fast moving lower layer but is kept in equilibrium by the frictional drag and turbulent mixing in the region of contact of the layers.

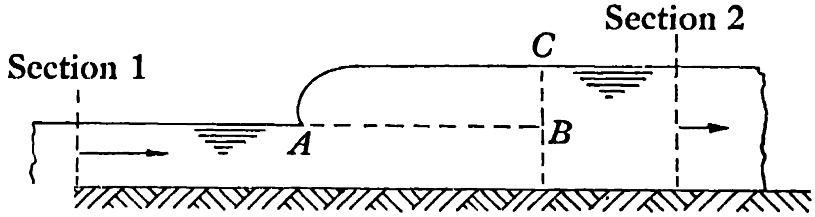

Fig. 8.4,1. Idealized diagram of hydraulic jump.

We shall now develop a theory of the hydraulic jump in which the essentials of the phenomenon are presented as simply as possible. In the first discussion we shall suppose that the normal section of the channel is a rectangle with the base horizontal but other shapes will be treated later; in all cases the channel is taken to be untapered and horizontal. We shall make the following simplifying assumptions:

(a) Friction at the bed and walls is absent. This is justifiable since the phenomena of the jump are completed within a short length of channel.

(b) The flow is uniform and steady at section 1 (upstream from the jump) and at section 2 (downstream from the jump).

(c) The depth is uniform at sections 1 and 2.

Quantities appropriate to sections 1 and 2 will be distinguished by the use of a corresponding suffix. Since the slope of the bed is zero and in view of assumptions (a), (b) and (c), the equation of momentum (8.3,6) can be written

$$v_2^2 A_2 - v_1^2 A_1 = \tfrac{1}{2} g(h_1 A_1 - h_2 A_2)$$

since $\bar{y}$ is equal to half the depth, while the equation of continuity is

$$v_1 A_1 = v_2 A_2$$

with

$$A_1 = bh_1, \quad A_2 = bh_2.$$

Hence

$$v_2 = \frac{v_1 h_1}{h_2} \tag{8.4,1}$$

and we obtain

$$h_1 v_1 (v_2 - v_1) = \tfrac{1}{2} g(h_1^2 - h_2^2)$$

or

$$v_1^2 (h_2 - h_1) h_1 = \tfrac{1}{2} g h_2 (h_2 - h_1)(h_2 + h_1). \tag{8.4,2}$$

Equation (8.4,2) will be satisfied when $h_2 = h_1$ but then there is no jump. However, if $h_2 \neq h_1$ we may remove the factor $(h_2 - h_1)$ from equation (8.4,2) and obtain

$$v_1^2 = g\left(\frac{h_2}{h_1}\right)\left(\frac{h_1 + h_2}{2}\right). \tag{8.4,3}$$

The last equation is a quadratic for h_2 and yields

$$h_2 = -\tfrac{1}{2} h_1 + \tfrac{1}{2} h_1 \sqrt{\left(1 + \frac{8 v_1^2}{g h_1}\right)} \tag{8.4,4}$$

where we have given the radical the positive sign since h_2 cannot be negative.

Hitherto there is nothing to show whether h_2 is greater or less than h_1. However, it will be shown that the mechanical energy dissipated in the region of the jump would be negative if h_2 were less than h_1; this must be ruled out since it would be in conflict with the Second Law of Thermodynamics. In the present instance equation (8.3,9) shows that the energy dissipated in unit time between the sections 1 and 2 is given by

$$\frac{E_{12}}{g\rho Q} = (h_1 - h_2) + \tfrac{1}{2} g(v_1^2 - v_2^2) \tag{8.4,5}$$

since $z_{02} = z_{01}$. But equations (8.4,1) and (8.4,3) yield

$$v_1^2 - v_2^2 = g\left(\frac{h_1 + h_2}{2 h_1 h_2}\right)(h_2^2 - h_1^2) \tag{8.4,6}$$

and (8.4,5) becomes after reduction

$$\frac{E_{12}}{g\rho Q} = \frac{(h_2 - h_1)^3}{4h_1h_2}. \qquad (8.4,7)$$

Thus, as we have stated, the energy dissipated can only be positive† when $h_2 > h_1$. We reach the important conclusion that in a hydraulic jump the greater depth must be downstream. This implies also that the velocity downstream is less than that upstream.

Suppose next that h_2 exceeds h_1 by a very small quantity δ whose square can be neglected. Equation (8.4,4) gives

$$h_1\sqrt{\left(1 + \frac{8v_1^2}{gh_1}\right)} = 3h_1 + 2\delta$$

and when this is squared and reduced we get

$$v_1^2 = gh_1 + \tfrac{3}{2}g\delta. \qquad (8.4,8)$$

Since δ cannot be negative, as proved above, we see that the minimum velocity for the occurrence of a hydraulic jump in a channel of rectangular section is given by

$$v^2 = gh. \qquad (8.4,9)$$

This minimum velocity is the critical velocity as defined in § 8.3 and it is the velocity of propagation of any gravity wave of small amplitude whose wave length is much greater than h (see § 10.5). We have arrived at the following important conclusion: *a hydraulic jump cannot occur in a channel of rectangular section unless the velocity is superundal.* From the general equations (8.4,1) and (8.4,3) we find that

$$v_1v_2 = g\left(\frac{h_1 + h_2}{2}\right). \qquad (8.4,10)$$

Hence *the product of the velocities before and after the jump is equal to the square of the critical velocity corresponding to the mean of the depths before and after the jump.* It also follows from (8.4,1) and (8.4,3) that

$$v_2^2 = g\left(\frac{h_1}{h_2}\right)\left(\frac{h_1 + h_2}{2}\right).$$

But the critical velocity corresponding to the depth h_2 is v_{c2} where

$$v_{c2}^2 = gh_2.$$

Accordingly

$$\left(\frac{v_2}{v_{c2}}\right)^2 = \left(\frac{h_1}{h_2}\right)\left(\frac{h_1 + h_2}{2h_2}\right) < 1 \qquad (8.4,11)$$

for $h_1 < h_2$ and $h_1 + h_2 < 2h_2$. Therefore *the velocity after the jump is subundal.* Similarly we can show that, if v_{c1} is the critical velocity corresponding to the depth h_1, then

$$\left(\frac{v_1}{v_{c1}}\right)^2 = \left(\frac{h_2}{h_1}\right)\left(\frac{h_1 + h_2}{2h_1}\right) > 1. \qquad (8.4,12)$$

---

† This energy appears as turbulence and ultimately as heat and there is a proportionate increase of entropy (see § 9.2).

This confirms that the velocity before the jump is superundal. It may be emphasised that the hydraulic jump is an irreversible process.

There is a close analogy between the hydraulic jump and a shock wave in gas (see § 9.8). This is shown in the following tabular statement.

Analogy of the Hydraulic Jump with a Shock Wave in a Gas

Hydraulic jump in rectangular channel	Plane shock wave normal to stream of gas
1. There is a critical velocity below which the jump cannot occur.	There is a critical velocity below which the shock wave cannot occur.
2. The depth is greater after the jump than before it. Or, mean pressure is greater after the jump than before it.	Pressure is greater after the shock wave than before it.
3. The product of the velocities before and after the jump is equal to the square of the critical velocity corresponding to the arithmetical mean depth.	The product of the velocities before and after the shock wave is equal to the square of the critical velocity.
4. The velocity after the jump is subcritical (subundal).	The velocity after the shock wave is subcritical (subsonic).
5. The process involves increase of entropy.	The process involves increase of entropy.
6. Increase of entropy per unit mass is proportional to cube of difference of depth or mean pressure.	Increase of entropy per unit mass is proportional to cube of pressure difference (when this is small).

We must now consider the hydraulic jump in a channel whose section is not rectangular, but all the assumptions are as before. For convenience put

$$J = A(h - \bar{y}) = \int_0^h (h-y)b(y)\,dy. \tag{8.4,13}$$

Then equation (8.3,6) can here be written

$$v_2{}^2A_2 - v_1{}^2A_1 = g(J_1 - J_2) \tag{8.4,14}$$

while the equation of continuity is

$$v_2A_2 = v_1A_1. \tag{8.4,15}$$

Accordingly (8.4,14) yields

$$v_1{}^2 = g\left(\frac{A_2}{A_1}\right)\left(\frac{J_2 - J_1}{A_2 - A_1}\right). \tag{8.4,16}$$

To find the critical velocity v_c we suppose h_2 to exceed h_1 by a very small quantity. Then

$$J_2 - J_1 = \frac{dJ}{dh}\,dh \quad \text{and} \quad A_2 - A_1 = \frac{dA}{dh}\,dh.$$

Accordingly (8.4,16) becomes in the limit

$$v_c^2 = g\frac{dJ}{dh}\bigg/\frac{dA}{dh}. \tag{8.4,17}$$

By the rules for the differentiation of definite integrals we find, since the integrand vanishes at the upper limit,

$$\frac{dJ}{dh} = \int_0^h b(y)\,dy = A. \tag{8.4,18}$$

Also

$$\frac{dA}{dh} = b(h) \tag{8.4,19}$$

where $b(h)$ is the width of the channel at the free surface. Accordingly (8.4,17) becomes

$$v_c^2 = \frac{gA}{b(h)}. \tag{8.4,20}$$

This is in agreement with the result already obtained for a rectangular channel (see also equation (8.5,15)). We may note that the critical velocity will be small for a channel that is wide at the free surface and high for a channel that is narrow at the free surface.

A hydraulic jump cannot occur unless the velocity exceeds the critical velocity. However, it does not follow that a jump must occur when the velocity is above the critical. The occurrence and location of the jump depend on conditions downstream, as discussed at the end of § 8.5.

8.5 Steady Flow of Variable Depth in Channels of Arbitrary Sectional Form

The mean velocity of flow v is related to the volume Q of fluid passing in unit time by the equation

$$Q = vA \tag{8.5,1}$$

where the area A is a function of the depth h, measured from the lowest point of the bed to the free surface, which is supposed to be horizontal across the channel. On account of viscosity the velocity is not uniform over the section but we neglect this at present and define the *total head* by the equation

$$H_T = z_0 + h + \frac{v^2}{2g} \tag{8.5,2}$$

where z_0 is the height of the lowest point of the bed (at the section considered) above a fixed horizontal datum plane. We also define the *specific head* by the equation

$$H = h + \frac{v^2}{2g} \tag{8.5,3}$$

and then we have from (8.5,2)

$$H_T = H + z_0. \tag{8.5,4}$$

When this is differentiated with respect to x, the distance along the bed measured from a fixed datum and in the direction of flow, we get

$$\frac{dH}{dx} = s - i \tag{8.5,5}$$

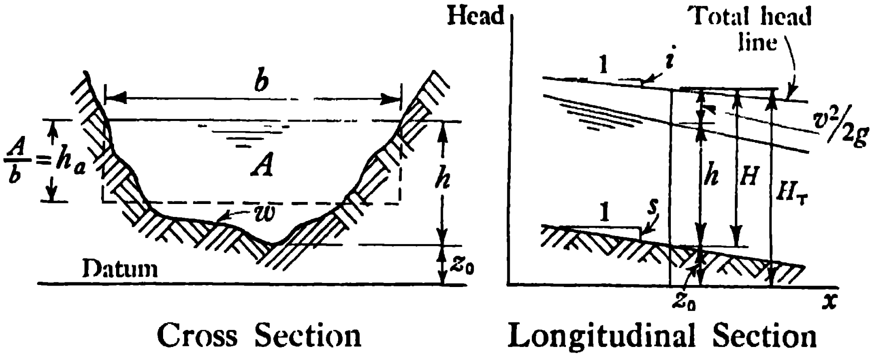

Fig. 8.5,1. Varying flow in channels. Suffix n is used to denote properties when depth is normal; c indicates critical conditions.

where s and i are the slopes of the bed and of the total head line respectively, so that

$$s = -\frac{dz_0}{dx} \tag{8.5,6}$$

$$i = -\frac{dH_T}{dx} \tag{8.5,7}$$

If fluid friction were absent i would be zero and we should have $dH/dx = s$ from (8.5,5). However, on account of the presence of friction, i is positive and for a given channel and values of Q, s and g there is a value of h such that $dH/dx = 0$. This is called the *normal depth* corresponding to Q and is denoted by h_n. Fig. 8.5,1 shows cross and longitudinal sections of the channel and illustrates the definitions of the quantities already mentioned. The average depth h_a of the channel is defined by the equation

$$h_a = \frac{A}{b} \tag{8.5,8}$$

where b is the breadth at the free surface.[†] Since we are restricting the discussion to steady flow, Q is the same at all sections of the channel.

When Q, H and g are given the depth h is obtained by solving the equation

$$H = h + \frac{Q^2}{2gA^2} \tag{8.5,9}$$

[†] The *average depth* must not be confused with the *hydraulic mean depth*.

which follows at once from (8.5,1) and (8.5,3); it is to be remembered that A is a known function of h depending on the shape of the channel section. For the particular case where the section is a rectangle with the bottom horizontal the equation (8.5,9) yields the following cubic equation for h (cp. equation (8.3,19))

$$h^3 - Hh^2 + \frac{Q^2}{2gb^2} = 0. \tag{8.5,10}$$

This has two real and positive roots provided that Q is not incompatible with the value of H; there is also a real negative root which has no physical significance. In general, equation (8.5,9) yields two real and positive values of h, with the same proviso as before. Fig. 8.5,2 shows a plot of h on a base of H for a channel whose section is a 90°V while $Q = 1$ and $g = 9{\cdot}81$. The diagram exemplifies the general fact that for a given channel and rate of discharge there is a value of the depth such that the specific head is a minimum. This depth is called the *critical depth* h_c. We can determine h_c for a channel of arbitrary section shape as follows. From (8.5,3) we derive

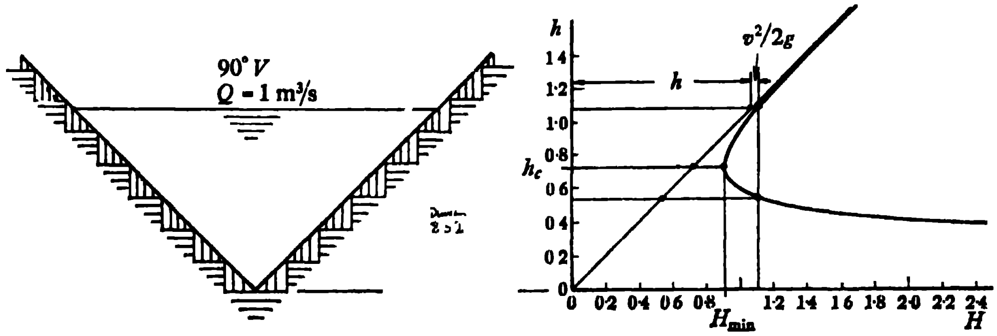

Fig. 8.5,2. Specific head diagram, Q constant.

$$\frac{dH}{dh} = 1 + \frac{v}{g}\frac{dv}{dh}. \tag{8.5,11}$$

when Q is constant (8.5,1) yields

$$\frac{dv}{dA} = -\frac{Q}{A^2}$$

and

$$\frac{dv}{dh} = \frac{dv}{dA}\frac{dA}{dh} = -\frac{bQ}{A^2} = -\frac{bv}{A} \tag{8.5,12}$$

since

$$\frac{dA}{dh} = b. \tag{8.5,13}$$

Hence (8.5,11) becomes

$$\frac{dH}{dh} = 1 - \frac{v^2}{g}\left(\frac{b}{A}\right). \tag{8.5,14}$$

When $h = h_c$ we have dH/dh zero and (8.5,14) gives for the corresponding critical velocity

$$v_c = \sqrt{\frac{gA}{b}} \tag{8.5,15}$$

$$= \sqrt{gh_{ac}} \tag{8.5,16}$$

where h_{ac} is the average depth in the critical condition (see equation (8.5,8)). On comparison with equation (8.4,20) it will be seen that the present definition of the critical velocity yields a result in agreement with that which arises from consideration of the hydraulic jump. Equation (8.5,16) can be put in the form

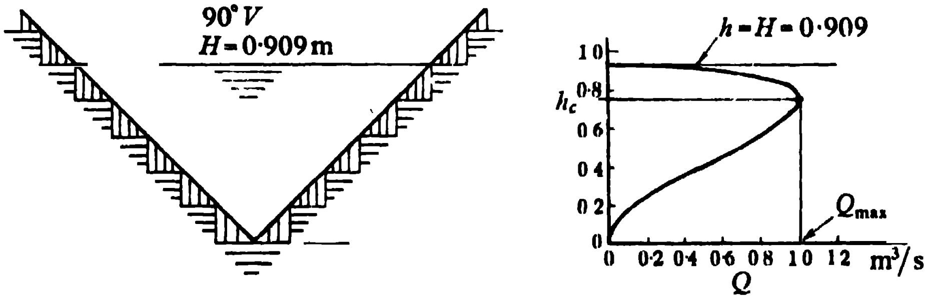

Fig. 8.5,3. Variation of Q with h when H is constant.

$$\frac{v_c^2}{2g} = \tfrac{1}{2}h_{ac}. \tag{8.5,17}$$

Hence *the minimum specific head for a given rate of discharge occurs when the velocity head is equal to half the average depth.* In view of (8.5,1) equation (8.5,15) yields

$$\frac{Q^2}{g} = \frac{A_c^3}{b_c} \tag{8.5,18}$$

where A_c and b_c are the sectional area and the breadth of free surface in the critical condition. The last equation leads to a simple graphical method for determining h_c. The quantity A^3/b is a function of h only and a graph on a base of h can easily be drawn. Then h_c is the abscissa of the point where the graph cuts a straight line parallel to the base whose constant ordinate is Q^2/g. It should be noted that h_c is independent of the slope of the bed and of the roughness of its surface.

So far we have discussed the case where Q is constant and H variable but now we shall suppose that H is given and enquire what value of h makes Q a maximum. We find at once from (8.5,9) that

$$Q = A\sqrt{(2g)}\sqrt{(H - h)} \tag{8.5,19}$$

and the variation of Q with h for the channel sketched in Fig. 8.5,2 is shown in Fig. 8.5,3. When Q is a maximum $Q^2/2g$ is also a maximum and we find from (8.5,19) that the condition for this is

$$2A(H-h)\frac{dA}{dh} - A^2 = 0$$

or, since

$$\frac{dA}{dh} = b,$$

$$H - h_c = \frac{1}{2}\frac{A_c}{b_c}. \tag{8.5,20}$$

By (8.5,3) this reduces to

$$\frac{v_c^2}{2g} = \frac{1}{2}\frac{A_c}{b_c} \tag{8.5,21}$$

which is equivalent to (8.5,15). *Thus, when the specific head is given, maximum discharge occurs when the velocity head has the critical value equal to half the average depth.* The critical condition can thus be defined in the following three ways which are concordant:

1. For a given discharge, the specific head is a minimum.
2. For a given specific head, the discharge is a maximum.
3. For a given depth, the critical velocity is the least for which a hydraulic jump is possible (See the end of § 8.4). This is equivalent to the statement that, for given values of H and Q, a hydraulic jump cannot occur unless h is less than h_c.

The two depths h_n and h_c are fixed by quite independent conditions. Thus when Q and g are given, h_n depends on the slope s of the bed and on the roughness of the bed and walls whereas h_c depends only on the geometry of the cross-section. Moreover, the actual depth h at any section is not necessarily related to either h_n or h_c. For convenience the slope of a channel is said to be *gradual* when h_n is *greater* than h_c and *steep* when h_n is *less* than h_c. The slope is called *critical* when h_n and h_c are *equal.* For normal flow, i.e. when $h = h_n$, any long gravitational wave can move upstream when the slope is gradual but will be swept downstream when the slope is steep. Since h is independent of h_n and h_c it follows that six distinct cases arise:

(a) *Slope gradual*	(b) *Slope steep*
(1) $h > h_n$	(4) $h > h_c$
(2) $h_n > h > h_c$	(5) $h_c > h > h_n$
(3) $h < h_c$	(6) $h < h_n$

The characteristics of the flow in these six cases are discussed below.

We shall now investigate the manner in which the level of the free surface changes along the channel in various circumstances. Equation (8.5,3) yields

$$dh = dH - v\,dv/g$$

and, when Q is constant, we derive

$$dv = -\frac{Q}{A^2}\,dA$$

from (8.5,1). Accordingly

$$dh = dH + \frac{vQ}{gA^2}\,dA = dH + \frac{Q^2}{gA^3}\,dA.$$

But $dH = (s - i)\,dx$ by (8.5,5) and the last equation becomes

$$dh = (s - i)\,dx + \frac{Q^2}{gA^3}\,dA. \qquad (8.5,22)$$

When the sectional form of the channel is variable along the channel dA will not be constant even when h is constant so we have in general

$$\begin{aligned} dA &= \frac{\partial A}{\partial x}\,dx + \frac{\partial A}{\partial h}\,dh \\ &= \frac{\partial A}{\partial x}\,dx + b\,dh. \end{aligned} \qquad (8.5,23)$$

When this is substituted in (8.5,22) we get after rearrangement

$$\frac{dh}{dx} = \frac{s - i + (Q^2/gA^3)\,\partial A/\partial x}{1 - bQ^2/gA^3}. \qquad (8.5,24)$$

For a rectangular channel of variable width we have

$$\frac{\partial A}{\partial x} = h\frac{db}{dx} \qquad (8.5,25)$$

but $\partial A/\partial x$ is zero when b is constant. In the further development of the theory we shall suppose that $\partial A/\partial x$ is zero. Then (8.5,24) may be written

$$\frac{dh}{dx} = \frac{Ns}{D}, \qquad (8.5,26)$$

where

$$D = 1 - \frac{bQ^2}{gA^3} \qquad (8.5,27)$$

and

$$N = 1 - \frac{i}{s}. \qquad (8.5,28)$$

When the depth is normal $(h = h_n)N$ is zero and h is therefore constant. We shall now indicate all quantities appropriate to the normal condition by adding the suffix n. Equation (8.3,12) gives

$$i = \frac{\tau_m}{g\rho m}$$

while equations (8.2,4) and (8.2,8) yield $\tau_m = \dfrac{\rho v^2}{2C_0^{\,2}}$.

Hence

$$i = \frac{1}{mC_0^2}\left(\frac{v^2}{2g}\right)$$

$$= \frac{1}{2mC_0^2}\left(\frac{Q^2}{gA^2}\right). \qquad (8.5,29)$$

If we now assume C_0 to be independent of depth we can apply the last equation to the normal condition, when $i = s$. Accordingly

$$s = \frac{1}{2m_nC_0^2}\left(\frac{Q^2}{gA_n^2}\right) \qquad (8.5,30)$$

and (8.5,28) yields

$$N = 1 - \frac{m_nA_n^2}{mA^2}. \qquad (8.5,31)$$

Also, in view of equation (8.5,18), (8.5,27) becomes

$$D = 1 - \frac{bA_c^3}{b_cA^3}$$

$$= 1 - \frac{h_{ac}A_c^2}{h_aA^2}. \qquad (8.5,32)$$

Finally (8.5,26) becomes

$$\frac{1}{s}\frac{dh}{dx} = \frac{\left(1 - \dfrac{m_nA_n^2}{mA^2}\right)}{1 - \dfrac{h_{ac}A_c^2}{h_aA^2}}. \qquad (8.5,33)$$

For a rectangular channel $h_a = h$ while $A = bh$. Accordingly we now obtain

$$\frac{1}{s}\frac{dh}{dx} = \frac{\left(1 - \dfrac{m_nh_n^2}{mh^2}\right)}{1 - \dfrac{h_c^3}{h^3}}. \qquad (8.5,34)$$

Moreover m and h increase together, so

$$\frac{m_nh_n^2}{mh^2} > 1 \quad \text{when} \quad h < h_n$$

and

$$\frac{m_nh_n^2}{mh^2} < 1 \quad \text{when} \quad h > h_n.$$

Accordingly we find in the six cases already defined:—

(1) *Slope gradual* ($h_n > h_c$)

(a) $h > h_n$	(b) $h_n > h > h_c$	(c) $h < h_c$
$\frac{1}{s}\frac{dh}{dx}$ is positive.	$\frac{1}{s}\frac{dh}{dx}$ is negative.	$\frac{1}{s}\frac{dh}{dx}$ is positive.

(2) *Slope Steep* ($h_c > h_n$)

(a) $h > h_c$

$\frac{1}{s}\frac{dh}{dx}$ is positive.

(b) $h_c > h > h_n$

$\frac{1}{s}\frac{dh}{dx}$ is negative.

(c) $h < h_n$

$\frac{1}{s}\frac{dh}{dx}$ is positive.

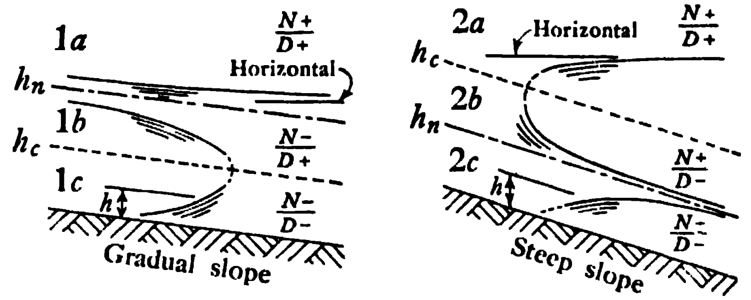

Fig. 8.5,4. Six variations of *dh*/*dx* in a short channel.

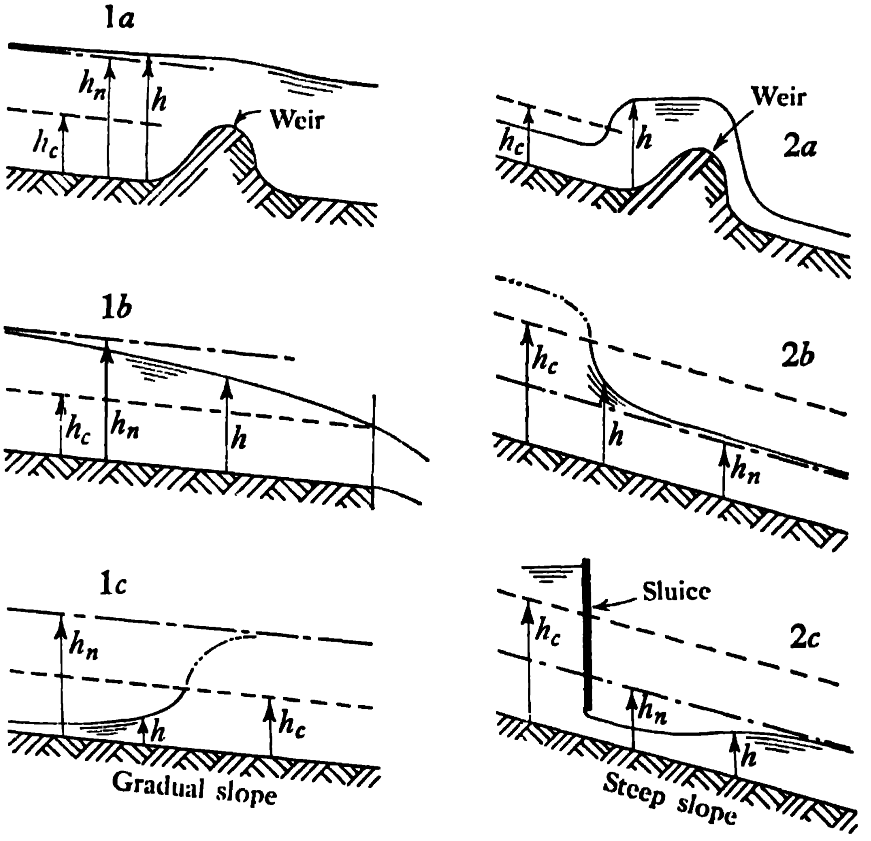

Fig. 8.5,5. Diagram of some practical cases.

The six cases are shown diagrammatically in Fig. 8.5,4 while some practical instances are shown in Fig. 8.5,5. In these figures the free surface is shown by a full line and the critical depth by a dotted line.

We shall now consider the problem of calculating the *surface profile* in an open channel; this is the relation between the depth h and the distance x along the channel. This may be done by numerical step-by-step integration of equation (8.5,24) and the value of i must be estimated for each position along the channel, say by the application of (8.5,29). We also require to know the value of h at some value x_0 of the abscissa. For instance, suppose that part of the length of the channel is uniform as regards section, roughness and bed slope. Then we may assume that $h = h_n$ at the downstream end of this part. In Fig. 8.5,5 we may find h from our knowledge of the flow over weirs in cases 1(a) and 2(a) (see further § 8.6) while in case 2(c) the depth just downstream from the sluice may be estimated from a knowledge of the appropriate coefficient of contraction.

A special problem arises when a hydraulic jump is present (see, for example, Fig. 8.5,5 . . 1(c) and 2(a)) and we shall restrict the discussion to the case where the channel is rectangular. Here we require to know the depth at a point upstream from the jump and at a point downstream from it; the principal problem is to find the situation of the jump. As in § 8.4 we shall use h_1 for the depth just upstream of the jump while h_2 is the depth immediately downstream. Then equation (8.4,3) can be written in the symmetrical form

$$h_1 h_2(h_1 + h_2) = \frac{2Q^2}{gb^2} \tag{8.5,35}$$

since $Q = bh_1v_1$. When Q, b and g are known, the last equation enables us to calculate h_1 when h_2 is known, or h_2 when h_1 is known. The procedure in determining the position of the jump is then as follows. From the data for the flow upstream determine h_1 as a function of x and let this be shown as a graph. Similarly determine h_2 from the data of the flow downstream and derive a second graph of h_1 by means of equation (8.5,35). The two graphs of h_1 intersect at the value of x corresponding to the situation of the hydraulic jump.

8.6 Notches, Weirs and Orifices

A notch is an opening in a wall through which liquid streams. In most instances the tail race lies well below the notch so the liquid issues freely as a jet called the *nappe*. Notches submerged on the downstream side are sometimes used but will not be considered here. It will be supposed that the wall is plane and vertical where it is in contact with the liquid and that the notch is sharp edged. The bed and sides upstream are supposed not to be close to the notch. As usual, the gravitational field is uniform of intensity g and the pressure at the free surface is constant (atmospheric). The velocity of approach at some distance from the notch is supposed to be so small that the velocity head is negligible (see later for the more general case). The liquid is uniform and incompressible while the flow is steady.

We shall begin by applying dimensional analysis (see § 4.8) to the problem of determining the volume Q of liquid discharged through the notch in unit time. All the notches considered are geometrically similar and similarly situated with respect to the plane of the undisturbed free surface of the liquid upstream from the notch. We shall assume that Q depends on the following quantities:

l a typical linear dimension of the notch
g acceleration of gravity
ρ density of the liquid
μ viscosity of the liquid.

The measure formulae of the quantities are

$$Q \circeq L^3T^{-1} \qquad l \circeq L \qquad g \circeq LT^{-2}$$
$$\rho \circeq ML^{-3} \qquad \mu \circeq ML^{-1}T^{-1}$$

We assume that

$$Q = kl^a g^b \rho^c \mu^d \tag{8.6,1}$$

which leads to the dimensional relation

$$L^3T^{-1} \circeq L^a(LT^{-2})^b(ML^{-3})^c(ML^{-1}T^{-1})^d.$$

Accordingly the indicial equations are

$$\begin{aligned} &(M) \quad 0 = c + d \\ &(L) \quad 3 = a + b - 3c - d \\ &(T) \; -1 = -2b - d. \end{aligned}$$

Since there are 4 unknowns and 3 equations there is an indeterminacy; it is convenient to put $d = -n$ and to express the other unknowns in terms of n. We find that

$$a = \tfrac{5}{2} + \tfrac{3}{2}n, \qquad b = \tfrac{1}{2} + \tfrac{1}{2}n \quad \text{and} \quad c = n.$$

Accordingly (8.6,1) can be written

$$Q = kg^{1/2}l^{5/2}\left(\frac{\rho g^{1/2}l^{3/2}}{\mu}\right)^n \tag{8.6,2}$$

We may add any number of terms similar to (8.6,2) with arbitrary values of k and n and the result is still dimensionally correct. This signifies that the most general dimensionally correct relation is

$$Q = g^{1/2}l^{5/2}\,\mathrm{f}\left(\frac{\rho g^{1/2}l^{3/2}}{\mu}\right) \tag{8.6,3}$$

where the form of the function f cannot be determined by dimensional analysis alone. Let

$$R' = \frac{\rho g^{1/2}l^{3/2}}{\mu} = \frac{g^{1/2}l^{3/2}}{\nu} \tag{8.6,4}$$

Then R' is a modified Reynolds number (see § 4.10) for $g^{\frac{1}{2}}l^{\frac{1}{2}}$ has the dimensions of velocity. If viscosity were absent or negligible we should have, from (8.6,3),

$$Q = \text{constant}\sqrt{gl^5}, \tag{8.6,5}$$

always subject to the conditions of geometrical similarity being satisfied. In general, however, the 'constant' in the last formula is a function of R'.

The foregoing investigation can be applied immediately to a V notch of arbitrary apical angle 2θ and it is convenient to take $l = h_0$, the depth of the vertex of the notch below the level of the undisturbed free surface upstream. The condition of geometric similarity is here satisfied for all values of h_0 and we derive

$$\frac{Q}{\sqrt{g h_0^5}} = f(R') \tag{8.6,6}$$

$$\text{where now} \quad R' = \frac{\sqrt{g h_0^3}}{\nu}. \tag{8.6,7}$$

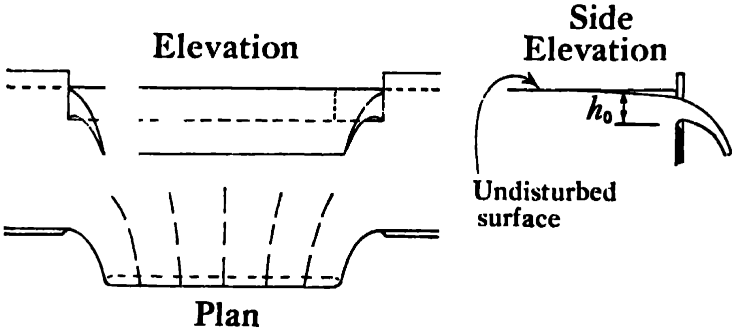

Fig. 8.6,1. Rectangular notch showing end contractions.

The function f will depend on the angle θ and, if it is desired to show this explicitly, (8.6,6) must be written

$$\frac{Q}{\sqrt{g h_0^5}} = f(\theta, R'). \tag{8.6,8}$$

For a notch which is rectangular (with the bottom of the rectangle horizontal) the conditions for similarity are not satisfied unless h_0/b is constant, where h_0 is the depth of the sill below the free surface and b is the breadth. In general, we have for the rectangular notch.

$$\frac{Q}{\sqrt{g h_0^5}} = f\left(\frac{h_0}{b}, R'\right). \tag{8.6,9}$$

Suppose, however, that the notch is wide. Then, except near the ends of the notch, the flow will be two-dimensional in planes perpendicular to the plane of the notch. Hence the rate of discharge Q' per unit width will be constant over the central part of the notch and we shall have

$$\frac{Q'}{\sqrt{g h_0^3}} = f_0(R') \tag{8.6,10}$$

where $f_0(R')$ is some function of the modified Reynolds number. Now (see Fig. 8.6,1) a complete notch with two 'end contractions' may be supposed divided into three parts:

(i) A right hand end of width kh_0

(ii) A central part of width $(b - 2kh_0)$ where the flow is two-dimensional

(iii) A left hand end of width kh_0.

Parts (i) and (iii) may be supposed fitted together and then become a rectangular notch of *constant proportions*, the depth being h_0 and the width $2kh_0$, where k is a constant. The discharge Q_1 from this can be obtained by applying (8.6,3). We shall put $l = h_0$ and write

$$Q_1 = g^{1/2}h_0^{5/2}f_1(R')$$

where R' is given by (8.6,7). For the central part (ii) of the notch the rate of discharge is

$$\begin{aligned} Q_2 &= (b - 2kh_0)Q' \\ &= (b - 2kh_0)g^{1/2}h_0^{3/2}f_0(R'). \end{aligned}$$

Hence the total rate of discharge is

$$\begin{aligned} Q &= Q_1 + Q_2 \\ &= g^{1/2}h_0^{3/2}\{bf_0(R') - h_0f_2(R')\} \qquad (8.6,11) \end{aligned}$$

where $$f_2(R') = 2kf_0(R') - f_1(R'). \qquad (8.6,12)$$

It will be seen that (8.6,11) is a particular case of (8.6,9). If constants are substituted for the two functions of R' in (8.6,11) we arrive at an expression of the same form as the Francis formula for the rate of discharge from a rectangular notch (see below). This agrees well with the results of experiments so long as b/h_0 is not less than 3.

We shall now consider the theory of the flow of perfect fluids through notches. Even for notches of the simplest shapes the exact theory is beyond the scope of an elementary treatise and we shall content ourselves with the usual approximate theory. For any real notch the flow of the liquid at the plane of the notch is not, in general, normal to the plane. However, we may suppose that a honeycomb of slender tubes is fitted into the notch so as to guide the liquid in the normal direction and we shall suppose the walls of these tubes to be exceedingly thin so that there is no blockage. The velocity of flow in any tube can then be found by applying Bernoulli's theorem (see § 3.4) since the liquid is supposed to be frictionless and the flow is steady. Let dA be the cross-sectional area of the tube and h its depth below the plane of the undisturbed free surface upstream. Since the pressure on the downstream side of the notch is atmospheric the velocity is $\sqrt{2gh}$. Consequently the rate of discharge through the tube is

$$dQ = \sqrt{2gh}\, dA.$$

Let $b(h)$ be the breadth of the notch at the depth h and take a strip of vertical width dh. The rate of discharge through this strip is $b(h)\sqrt{2gh}\, dh$ and the total for the notch is

$$Q = \sqrt{2g}\int_0^{h_0} b(h)\sqrt{h}\, dh \qquad (8.6,13)$$

where h_0 is the depth of the apex or sill of the notch. For a rectangular notch $b(h) = b$, a constant, and we get

$$Q = \tfrac{2}{3}b\sqrt{2gh_0^3} \tag{8.6,14}$$

Next, for a triangular or V notch of total apical angle 2θ, whose axis of symmetry is vertical,

$$b(h) = 2(h_0 - h)\tan\theta$$

and

$$Q = 2\sqrt{2g}\tan\theta\int_0^h (h_0 - h)\sqrt{h}\,dh$$

$$= \frac{8}{15}\tan\theta\sqrt{2gh_0^5} \tag{8.6,15}$$

while the breadth is $2h_0\tan\theta$ at the level of the free surface. For a parabolic notch with its vertex at a depth h_0 and of breadth b at the level of the free surface

$$b(h) = b\sqrt{\frac{h_0 - h}{h_0}}\,.$$

Accordingly
$$Q = b\sqrt{\frac{2g}{h_0}}\int_0^{h_0}\sqrt{\{h(h_0 - h)\}}\,dh$$

and the integral may be evaluated by use of the substitution $h = h_0\sin^2\phi/2$. We finally obtain

$$Q = \frac{\pi}{8}b\sqrt{2gh_0^3}. \tag{8.6,16}$$

In all cases we have

$$Q = kb\sqrt{2gh_0^3} \tag{8.6,17}$$

where b is the breadth at the free surface and the value of the coefficient k depends on the shape of the notch. Thus for

V notch $k = \dfrac{4}{15} = 0{\cdot}267$

Parabolic notch $k = \dfrac{\pi}{8} = 0{\cdot}393$

Rectangular notch $k = \dfrac{2}{3} = 0{\cdot}667.$

When lengths are measured in feet and $g = 32$, equation (8.6,17) becomes

$$Q = 8kb\sqrt{h_0^3}\ \text{(cusecs)}$$

or, in SI units (8.6,18)

$$Q = 4.43\ kb\sqrt{h_0^3}\ (\text{m}^3/\text{s})$$

On account of obliquity of flow and the influence of viscosity the actual discharge is less than that given by (8.6,18). The ratio of the actual to the theoretical[†] discharge is called the *coefficient of discharge* C_d and some values are quoted in Table 8.6,1. Although the formulae

$$Q = 8kC_d b\sqrt{h_0^3}\ (\text{cusecs})$$

or

$$Q = 4{\cdot}43kC_d b\sqrt{h_0^3}\ (\text{m}^3/\text{s}) \tag{8.6,19}$$

can be applied to a notch of any shape, the manner in which Q depends on h_0 for a given notch depends on the shape of the notch since the way in which b varies with h_0 is characteristic of the shape. According to the approximate theory and for some simple shapes we shall have Q proportional to h_0^n where for

V notch	$n = 2{\cdot}5$
Parabolic notch	$n = 2$
Rectangular notch without end contractions	$n = 1{\cdot}5$.

A rectangular notch may have two, one or no 'end contractions' according to the manner in which the liquid is guided. Let r be the number of end contractions. Then an improved formula for the rate of discharge, which may be derived by an argument similar to that used in obtaining equation (8.6,11), is

$$Q = \tfrac{2}{3}C_d\sqrt{(2gh_0^3)}\,(b - rch_0) \tag{8.6,20}$$

where c is a constant. This is a generalisation of the formula first put forward by Francis and later advocated by James Thomson.

In the foregoing we have assumed that the velocity of approach v_0 at a considerable distance from the notch is so small that the corresponding dynamic head $\dfrac{v_0^2}{2g}$ is negligible. When this is not so the value of h_0 to be used in the formula is

$$h_0 = h_0' + \frac{v_0^2}{2g} \tag{8.6,21}$$

where h_0' is the measured depth of the apex or sill of the notch below the free surface upstream. This is an immediate deduction from Bernoulli's theorem.

The following empirical formulae are specified in the British Standards publication 'Pump Tests' (BS 599 : 1939) and the foot is the unit of length:

$$[‡]\ 90^\circ \text{ notch} \quad Q = 2{\cdot}48h_0^{2\cdot48}. \tag{8.6,22}$$

Rectangular notch with contractions at both ends $Q = 2{\cdot}465(b - 0{\cdot}1h_0)h_0^{1\cdot5}$ (8.6,23)

Any of the foregoing formulae for discharge from a notch is valid only so long as the head is large enough to ensure that the nappe does not adhere on the downstream face. If the head is too small, adherence occurs and the discharge will be greater than that given by the formula.

[†] This should be called the *very crude* theoretical discharge.

[‡] Transferred to SI units this empirical formula is $Q = 1{\cdot}337h_0^{2\cdot48}$ m^3/s.

TABLE 8.6,1

Coefficients of Discharge of Sharp-Edged Notches

All linear dimensions are in feet. Fluid: water.
H is height of still water surface upstream above apex or sill

90° Vee Notch ($\theta = 45°$). (Barr)

H	0·167	0·208	0·250	0·292	0·333	0·584	0·833
C_d	0·604	0·598	0·595	0·593	0·591	0·585	0·581

Rectangular Notch with Two End Contractions. (Rafter)
Width 6·53.

H	0·5	1·0	2·0	3·0	4·0
C_d	0·633	0·625	0·615	0·613	0·616

Circular Notch. (A. S. Thom and A. W. Babister)
Radius 0·667 (8 inches)

H	0·1	0·2	0·3	0·4	0·5	0·6
C_d	0·613	0·590	0·578	0·578	0·581	0·584

These fit the empirical formula $Q = 3{\cdot}586\, H^{1{\cdot}916}$.†

A weir is in essence a wall built across a river in order to raise the water level upstream; the crest is usually level across the river and extends to its full width. The two main types of weir are called *narrow-crested* and *broad-crested* and are shown in profile in Fig. 8.6,2. The narrow-crested weir only differs from the rectangular notch without end contractions in not always being truly sharp edged and the rate of discharge can be calculated from equation (8.6,20) with $r = 0$ and a suitable value of C_d. In the case of the broad-crested weir the water is theoretically at the critical depth h_c in the region where the flow is sensibly uniform. The specific head referred to the crest as datum level is h_0. Since the channel is rectangular $h_c = \frac{2}{3}h_0$ and by equation (8.3,18) the theoretical rate of discharge is

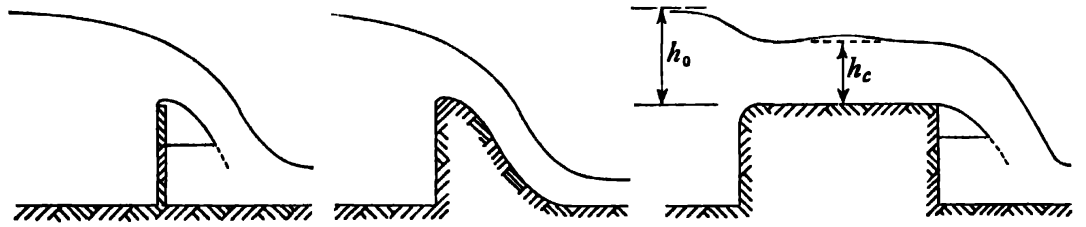

Fig. 8.6,2. Narrow and broad-crested weirs.

$$Q \quad \frac{2b}{\sqrt{27}}\sqrt{2gh_0^3}. \tag{8.6,24}$$

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† In SI units this is $Q = 0{\cdot}989\, H^{1{\cdot}916}$ m³/s over the range $0{\cdot}030 < H < 0{\cdot}183$ m.

When lengths are measured in feet and $g = 32$ the actual rate of discharge will be

$$Q = \frac{16bC_d}{\sqrt{27}}\sqrt{h_0^3}$$

$$= 3{\cdot}08C_d b\sqrt{h_0^3}\ \text{ft}^3/\text{s}, \qquad (8.6,25)$$

or, in S.I. units

$$Q = 1{\cdot}70C_d b\sqrt{h_0^3}\ \text{m}^3/\text{s}.$$

The value of C_d ranges from about 0·9 up to 0·97 and a broad-crested weir is not suitable for metering purposes unless it has been accurately calibrated.

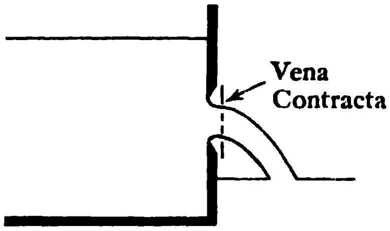

Fig. 8.6,3. The *vena contracta.*

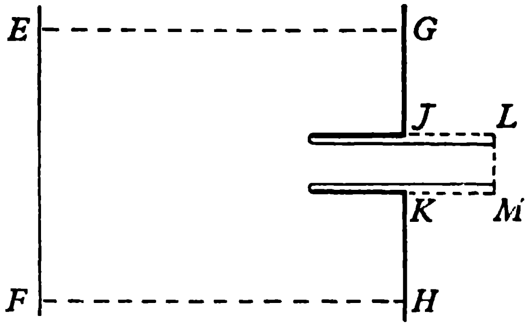

Fig. 8.6,4. The Borda mouthpiece.

An orifice differs from a notch merely in being closed above and totally submerged on the upstream side; we shall still assume that the tail race is low enough for the liquid to issue freely as a jet. Unless the orifice is very efficiently faired the jet will not be cylindrical where it emerges but will become of smaller sectional area downstream; this is the *vena contracta* (see Fig. 8.6,3). The form of the section of the jet may show striking differences at varying distances from the orifice and this is essentially due to the lack of uniformity of speed and obliquity over the plane of the orifice. Theoretical solutions to the problem of determining the shape of the jet and the value of the *coefficient of contraction* (i.e. the ratio of the minimum sectional area of the jet to the area of the orifice) have only been obtained in cases of special simplicity. We shall briefly consider some of these and begin with the theory of the *Borda mouthpiece.*

The Borda mouthpiece is sharp-edged and reëntrant as shown in Fig. 8.6,4. It is either circular in section or rectangular and very wide (two-dimensional Borda mouthpiece); the figure can be interpreted as a diametral section of the circular orifice or as a normal section of the two-dimensional orifice. We assume the fluid to be perfect, body forces are absent and the fluid issues from a chamber where the pressure is maintained at the constant value p_0 (where the fluid is stagnant) into a region of zero pressure. The reëntrant part of the orifice is so long that the velocity head of the fluid in contact with the walls of the vessel is negligible everywhere; consequently the pressure is p_0 everywhere on the walls. We also assume that the jet becomes uniform at some distance from the orifice, so its form is a circular cylinder (or a parallel-sided slab in the two-dimensional case). Where the jet is uniform the streamlines are parallel straight lines and there can be no pressure gradient across them. Since the pressure outside the fluid is zero it follows that it is also zero right across the jet. Now let A_0 be the sectional area of the orifice while A is that of the parallel part of the jet and consider the fluid within the closed 'control surface' *EFGHJKLM*. Since the pressure has the value p_0 everywhere on *EF*, *GJ* and *KH* while it is zero on *LM*, the net thrust on the fluid is p_0A_0, for A_0 is the area of *EF* less that of *GJ* and *KH*. This thrust is equal to the momentum gained by the fluid in unit time (§§ 3.2 and 3.6). Let v be the uniform velocity in the parallel part of the jet. Then the mass of fluid passing in unit time is ρAv and its momentum is ρAv^2. Hence

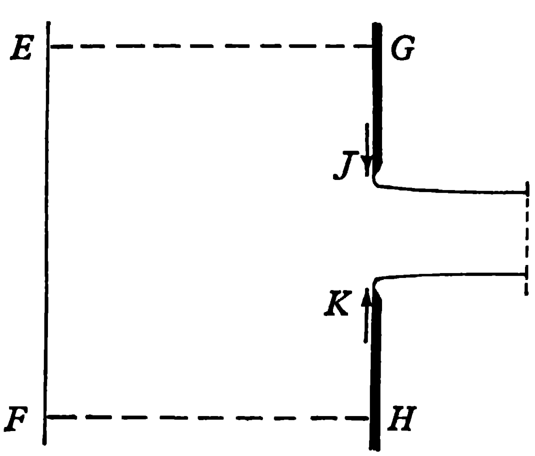

Fig. 8.6,5. Flush sharp-edged orifice.

$$\rho Av^2 = p_0A_0.$$

But we have

$$p_0 = \tfrac{1}{2}\rho v^2$$

by Bernoulli's theorem and we obtain

$$A = \tfrac{1}{2}A_0. \tag{8.6,26}$$

Thus the coefficient of contraction is 0·5. This theory was given by Borda himself but it was rediscovered by Hanlon and William Froude. The only thing not proved is that the jet does become uniform.

We next consider the non-reëntrant or flush sharp-edged orifice (see Fig. 8.6,5) and apply the same argument as for the Borda orifice. Here the fluid near J and K is moving fast and therefore, by Bernoulli's theorem, the pressure on the wall in this region is reduced. Hence the thrust T on the fluid is greater than for the Borda mouthpiece and we shall have

$$T = kp_0A_0 \tag{8.6,27}$$

where $k > 1$. The momentum gained in unit time is still

$$\rho Av^2 = 2p_0A$$

by Bernoulli's theorem. Thus

$$2p_0A = kp_0A_0$$

and the coefficient of contraction is

$$C_c = \frac{A}{A_0} = \frac{k}{2} > 0{\cdot}5. \tag{8.6,28}$$

The theory has been worked out in detail for the two-dimensional sharp and flush edged orifice† and it is found that

$$C_c = \frac{\pi}{\pi + 2} = 0{\cdot}611.$$

This confirms the foregoing argument. The boundary of the jet is shown in Fig. 8.6,6 and it can be shown that the curve is a *tractrix*. It is the symmetrical involute of a common catenary of parameter $\frac{w}{\pi + 2}$, where w is the width of the orifice. The field of flow is symmetrical about the centre line OX; this is itself a streamline which we may suppose replaced by a fixed plane boundary. We thus have the solution to the two-dimensional problem of the escape of liquid through a slot which is adjacent to a plane wall. This corresponds to the practical case of discharge beneath a sluice gate when end contractions are absent. The theoretical coefficient of contraction is of course again 0·611.

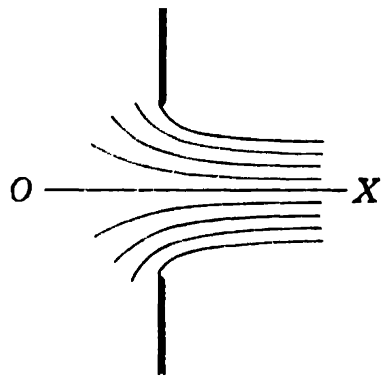

Fig. 8.6,6. Two-dimensional sharp- and flush-edged orifice.

For perfect fluids the coefficient of discharge C_d is equal to the coefficient of contraction C_c in all cases where the jet becomes uniform. For viscous fluids C_d will be less than C_c.

EXERCISES. CHAPTER 8.

1. The section of a channel has a horizontal bottom 1·5 m wide and the sides slope outward at 45° to the horizontal. Find the rate of discharge (uniform steady regime) when the depth of water is 0·4 m and the bed slope is 0·008. Assume $C = 50\ \mathrm{m}^{\frac{1}{2}}/\mathrm{s}$. (Answer 1·83 m^3/s)
2. The section of a channel is a rectangle 2·4 m wide and the slope of the bed is 0·02. Use Table 8.2,1 and Thijsse's formulae to estimate the rate of discharge when the constant depth of the water is 0·4 m. The surface is 'old wood' and the value of the kinematic viscosity is $1{\cdot}16 \times 10^{-6}\ \mathrm{m}^2/\mathrm{s}$. (Answer 4·39 m^3/s)
3. The data are the same as in the last Exercise but the surface is 'asphalted steel'. (Answer 6·58 m^3/s)
4. The data are the same as for Exercise 2 but the surface is 'gravel'. (Answer 3·03 m^3/s).
5. *Hydraulic mean depth of a circular pipe not running full.* The free surface of the liquid subtends the angle 2θ at the centre of the section. Find the hydraulic mean depth and the value of θ for which it is a maximum. (Answer $m = \dfrac{d(\pi - \theta + \cos\theta \sin\theta)}{4(\pi - \theta)}$. Maximum when $\theta = 51°$ (to nearest degree))

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† See Lamb's *Hydrodynamics*, Chapter IV, or Ramsey's *Hydromechanics*, Part II; Chapter VI.

6. Estimate the critical depth in a channel whose section is a 60° V when the rate of discharge is 0·2 m^3/s. If f (the friction coefficient $\tau_m/\frac{1}{2}\rho v^2$) is 0·005, calculate the critical slope of the channel.
(Answer $h_c = 0{\cdot}48$ m, $i_c = 0{\cdot}005$)
7. A channel has a V section with vertical angle 2θ. Show that $i_c/f = \frac{1}{2}\operatorname{cosec}\theta$.
8. The surface level in the lower branch of a canal is constant. Find the time to empty a lock of plan area 90 m^2 through a sluice of area 0·36 m^2 (sluice always submerged, $C_d = 0{\cdot}59$) when the initial surface level in the lock is 3 m above that in the canal. (Answer 5 min. 31 sec.)
9. The section of a channel is the parabola (with vertical axis) $b(y) = \sqrt{cy}$ where c is constant. Find the critical velocity corresponding to the depth h.
$\left(\text{Answer } v_c^2 = \frac{2}{3}gh\right)$
10. The section of a channel consists of a rectangle with horizontal base, width B and height H, continued upwards by vertical lines distant b apart. Find the critical velocity when the depth of flowing liquid is $h > H$.
$\left(\text{Answer } v_c^2 = g\left\{h + H\left(B - b\right.\right.\right.$
11. A jet (rate of discharge Q) falls vertically on a smooth horizontal surface and the liquid flows out symmetrically from the centre of the jet. The velocity and depth are related by $v^2 + 2gh = v_o^2$ where v_o is constant (friction is neglected). Liquid of depth H stands on the plate beyond the hydraulic jump. Find the depth h_c just before the jump, the velocity there and the radial distance from the centre of the jet (neglect the curvature in plan of the jump). *See also next Exercise.*

$$\left(\text{Answer} \quad h_c = \frac{1}{8}\left[\frac{2v_o^2}{g} - H \pm \sqrt{\left\{\left(\frac{2v_o^2}{g} - H\right)^2 - 16H^2\right\}}\right]\right.$$

$$\text{then} \quad v_c^2 = v_o^2 - 2gh_c$$

$$\left.\text{and} \quad r_c = \frac{Q}{2\pi v_c h_c}\right)$$

12. Find the condition that the depth h_c of Exercise 11 may be real and show that, when real, the values of h_c are both positive. How is the relevant value to be selected? $\left(\text{Answer } v_o^2 > \frac{5}{2}gH. \qquad h_c < H\right)$
13. Discuss qualitatively the influence of friction on the situation of the hydraulic jump in Exercise 11 on the assumption that the critical relation (8.4,3) is uninfluenced by friction.
(Answer. Friction reduces velocity at given distance from jet and critical depth h_c is increased. Product vh is increased and therefore radius to jump is reduced)
14. Show that the slope of the water surface at any point along a horizontal rectangular open channel is given by

$$\frac{dh}{dl} = \frac{fhQ^2}{2m(Q^2 - gb^2h^3)}.$$

Water from a spillway enters a horizontal rectangular channel 3 m wide with a velocity of 4 m/s and a depth of 0·9 m. A hydraulic jump forms immediately after the water enters the channel. Find the distance from the jump to the point where the depth of water is 1·2 m. Take $f = 0{\cdot}008$.
(Answer $h_2 = 1{\cdot}322$ m, $l = 9{\cdot}65$ m)

15. A channel is of rectangular section 3 m wide. The profile of the bed consists of a horizontal part *ABC*, a sloping part *CD* with a rise of 0·9 m and a horizontal part *DE*. There is a sluice gate at *B* and the water behind it stands at 2·2 m above the bed; the gate is adjusted till the depth in *BC* is 0.3 m. Find the depth in *DE* (neglect friction) and the rate of discharge.
(Answer. Depth 0·448 m, 5·49 m^3/s)
16. A channel is of rectangular section 2·1 m wide. The profile of the bed consists of a horizontal part *AB*, a (steep) sloping part *BC* with a fall of 2·1 m and a horizontal part *CD*. Estimate the depth of flow in *CD* immediately downstream of the fall when the discharge is 3·18 m^3/s. If downstream conditions are such that a standing wave may occur in *CD*, estimate the depth after the jump. Neglect friction. (Answer 0·204 m, 1·416 m)
17. A tunnel of square section, 2·4 m wide carries a flow of 12 m^3/s. Find the critical slope. If downstream conditions cause a hydraulic jump to form, estimate the maximum slope of the tunnel bed to ensure that the tunnel will not run full. Take $C = 65\ m^{\frac{1}{2}}/s$. (Answer 0·00497, 0·0288)
18. A rectangular channel, 1 m wide and 53 m long, has a bed gradient of 1 in 500, and carries a flow of 0·11 m^3/s from a reservoir. The water discharges freely over a broad-crested weir at the lower end, the sill being 150 mm above the bed of the channel, and extending over the whole breadth. If no hydraulic jump forms in the channel, and losses at the weir are neglected, estimate the still water level in the reservoir above the channel bed at entrance. Take $f = 0{\cdot}010$ for the channel. (Answer 0·238 m)
19. A tunnel of square cross section, 1·2 m wide, $f = 0{\cdot}01$, bed slope 1 in 70, carries a flow of 2·83 m^3/s. Where the tunnel enters the penstock of a turbine, the pressure head of the water is measured to be 15 m above atmosphere at the top of the tunnel. Estimate the length over which the tunnel is running full. If the water enters the tunnel directly from a reservoir, estimate the maximum length of the tunnel for no hydraulic jump to form, and the level of the reservoir above the tunnel bed at entry.
(Answer 1944 m; 1966 m; 1·242 m)
20. An orifice (*ABCD*) in a vertical dam has the form of a *trapezium* whose parallel sides *AB*, *CD* are parallel to the free surface of the liquid in the reservoir. The upper side *AB* is of length b_1 and is at depth h_1 while *CD* is of length b_2 and at depth h_2. Find the 'crude theoretical' rate of discharge Q (see § 8.6).
(Answer $Q = \sqrt{(2g)}J$, where
$$J = \frac{2}{5}(b_2h_2^{3/2} - b_1h_1^{3/2}) + \frac{4}{15}(b_1h_2 - b_2h_1)\left(\frac{h_1 + h_2 + \sqrt{h_1h_2}}{\sqrt{h_1} + \sqrt{h_2}}\right)\Bigg)$$
21. Discuss the following special cases of Exercise 20:—
    (*a*) Truncated triangle with vertex in free surface.
    (*b*) Case where $b_2h_2^{3/2} = b_1h_1^{3/2}$.
    (*c*) Triangle with vertex above base.
    (*d*) Triangle with base above vertex.
    (*e*) Rectangle and parallelogram.
22. Apply Exercise 20 to obtain the rate of discharge through an orifice of *general* triangular form.
(Answer. The general triangle can be considered to be the sum of triangles with base parallel to the free surface and with vertices above and below the base.)

23. Apply Exercise 20 to obtain the rate of discharge through an orifice of *general* quadrilateral form.

    (Answer. The general quadrilateral can be considered to be the sum of a trapezium and of two triangles with their vertices above and below their bases.)

24. The "crude theoretical" rate of discharge from an orifice in the form of a general polygon can be obtained by supposing it to be divided into triangles. In special cases it may be possible to divide the polygon into trapezia of the type considered in Exercise 20, e.g. regular hexagon with one side parallel to free surface.

CHAPTER 9

DYNAMICS OF COMPRESSIBLE FLUIDS

9.1 Introduction. Importance of Mach Number

By a compressible fluid we mean simply one for which, in general, changes of density accompany changes of pressure. Strictly all fluids are compressible, but usually we use the term 'compressible' when the circumstances are such that the change of density as a fraction of the original density accompanying a given change of pressure cannot be regarded as negligible. As a quantitative measure of the compressibility of a fluid we can use the bulk modulus defined in § 1.3. However, whether we wish to take special note of the compressibility of a fluid or not will depend not only on the bulk modulus but also on the actual magnitude of the pressure change to which the fluid is subjected in the process under consideration. It is for this reason that the non-dimensional parameter called the Mach number is of such great importance in describing the effects of the compressibility of a fluid on the forces that arise on a body moving through the fluid.

The Mach number (generally denoted by M) can be defined in general terms as the ratio of a speed to a related speed of sound. Thus, the Mach number of a body moving through a fluid is generally defined as the ratio of the speed of the body relative to the undisturbed fluid to the speed of sound in the undisturbed fluid. The local Mach number at any point in the fluid is likewise defined as the ratio of the local speed of the fluid relative to the body to the local speed of sound. We have seen in § 4.5 and § 4.11 that on dimensional grounds we can deduce that the Mach number can be used as a parameter to describe compressibility effects. It is, however, instructive to see its importance demonstrated on simple physical grounds.

Any disturbance introduced at a point into a fluid medium does not instantaneously take effect throughout the medium, but its effect is propagated through the medium at a finite speed by means of waves spreading from the disturbance origin. If the disturbance in a fluid is small, i.e. the pressure changes introduced are small compared with the undisturbed pressure, then the speed of propagation can be shown to be independent of the magnitude of the disturbance (see § 10.3) and we speak of this unique speed as the speed of propagation of small disturbances. Since the sources of sound normally constitute disturbances which, in this sense, can be described as small, we refer more shortly to this speed as the *speed of sound.*

A small change in pressure at a point will induce corresponding changes of density and velocity there, and these in turn will influence the flow conditions at neighbouring points, resulting in changes of pressure there, and so on. Thus in the propagation of a disturbance the rate of change of density with pressure plays a vital part; the larger this quantity is the longer it takes for changes at one point to influence a neighbouring point, i.e. the slower is the speed of sound. Indeed, as we shall see (§ 10.3), if we write a to denote the speed of sound then

$$a^2 = (\partial p/\partial \rho)_s \tag{9.1,1}$$

where the suffix s denotes that the gradient of pressure with respect to density applies to a condition of constant entropy, i.e. one in which there is no dissipation by friction of mechanical energy into heat and negligible heat conduction.† This latter condition is implicit in the assumption that the disturbance is small, so that dissipative factors, e.g. viscous forces, and heat conduction are negligible. To anticipate further, we shall see later that for a perfect gas with constant specific heats the above formula leads to

$$a^2 = \gamma p/\rho \tag{9.1,2}$$

where γ is a constant for the gas and is the ratio of the specific heat at constant pressure (c_p) to the specific heat at constant volume (c_v).‡ Dry air over a wide range of working conditions can be regarded as a perfect gas with $\gamma = 1{\cdot}40$.

Now consider at any instant a *small* disturbance source past which a fluid is moving with a uniform speed that is less than the speed of sound ($M < 1$). A series of waves will be propagated away from the source at the speed of sound relative to the fluid, and hence the wave emitted at any time t earlier than the instant under consideration will be spherical of radius at but with its centre at the point Ut downstream of the disturbance source, where U is the speed of the stream relative to it. In so far as $U < a$, part of this wave front will lie upstream of the disturbance source and at its nearest its distance from the source will be $(a - U)t$ upstream of it. Hence this distance will increase with time t, and therefore the influence of the disturbance source will eventually spread within a finite time to all points in the fluid at a finite distance from the source. This is indeed typical of subsonic ($M < 1$) flow conditions, where the influence of boundaries (which may be regarded as an aggregate of disturbance sources) is diffused throughout the medium. However, as U increases and approaches a the upstream influence of the disturbance source becomes more concentrated into a narrow region ahead of it, and the time required for it to propagate upstream against the flow increases. These points are illustrated in Fig. 9.1,1.

Now consider what happens when the fluid is moving at a *supersonic speed* ($M > 1$) relative to the disturbance source. This is illustrated in

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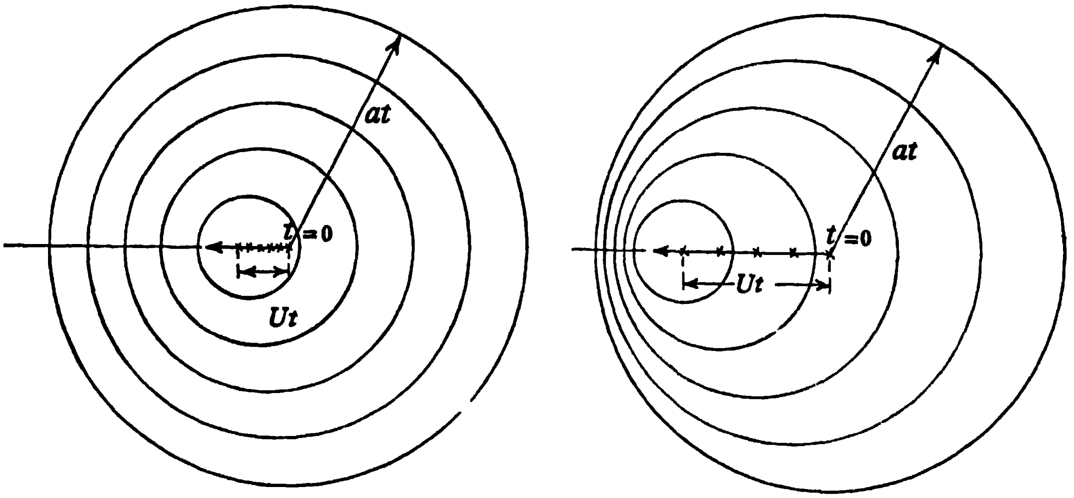
† A fuller discussion of the concept of entropy is given in § 9.2.5.

‡ See § 9.2.4.

Fig. 9.1,2. We see that at any instant the spherical waves generated previously are all swept downstream of the source and, if the flow is assumed to be steady, they all lie within and touch a circular cone with the disturbance source as vertex and with axis parallel to the direction of flow.



(a) $M = 0{\cdot}2$ (b) $M = 0{\cdot}8$

Fig. 9.1,1. Disturbance waves propagated by a small disturbance source at subsonic speeds.

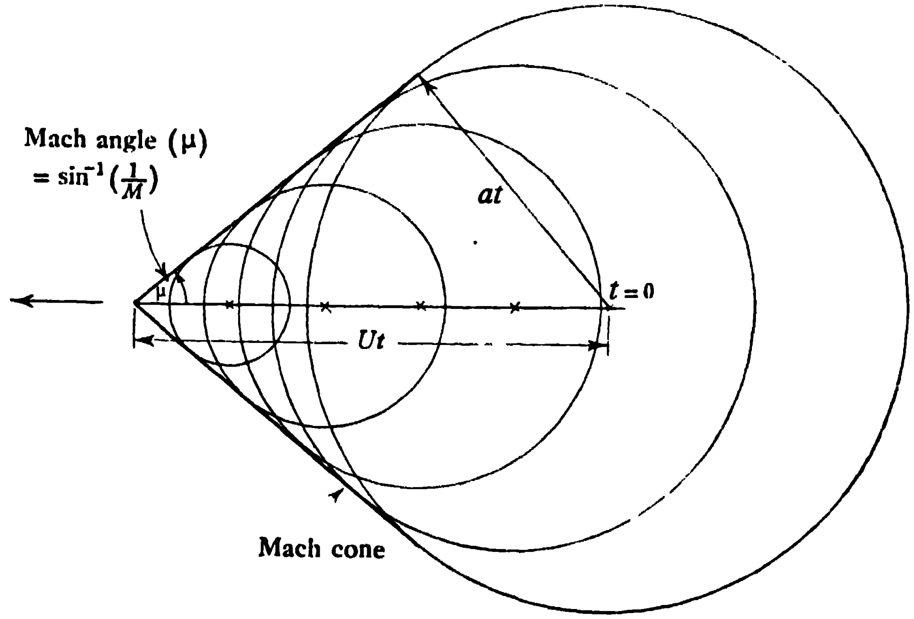


Fig. 9.1,2. Disturbance waves propagated by a small disturbance source at supersonic speeds.

This cone is referred to as the *Mach cone* and its semi-vertex angle

$$\mu = \sin^{-1} 1/M \qquad (9.1,3)$$

is referred to as the *Mach angle*.

Ahead of this cone the fluid remains completely uninfluenced by the disturbance, and an approaching particle of fluid would be unaware of the presence of the disturbance until the particle crossed the Mach cone. It is found in fact that the major changes introduced in the flow by the disturbance are concentrated on the surface of the Mach cone; the subsequent changes inside the Mach cone are small. The region upstream of the Mach cones is sometimes referred to as the 'zone of silence'.

In two dimensions we have a wedge instead of a Mach cone, and the two faces of the wedge are referred to as *Mach waves.* The disturbance effect is practically completely concentrated along the Mach waves, and subsequent changes can for most purposes be regarded as negligible. However, it is important to note that, since the disturbance source is assumed small, the Mach waves are wave fronts on which only small disturbances are introduced, and hence they are 'sound' waves. Therefore, their velocity relative to the oncoming stream has a component normal to themselves equal to the speed of sound, as can be readily verified from the fact that the streamline through a point bisects the angle between the Mach waves through the point and makes the Mach angle with them.

It can therefore be seen that there are essential differences between subsonic and supersonic flows. In the former the influence of disturbances and boundaries are diffused throughout the fluid, in the latter the influence of each disturbance or boundary element is confined to within, and largely on, its downstream Mach cone or Mach waves. These essential differences are reflected in marked changes in the flow characteristics about a body as its Mach number changes from subsonic to supersonic.

Now let us examine this effect of Mach number on the flow past a body from a somewhat different angle. The changes in pressure produced by a streamlined body, such as an aeroplane, moving through a fluid at velocity U are at most of the order of ρU^2, where ρ is the density of the undisturbed fluid. Hence these changes as a fraction of the pressure of the undisturbed fluid (p) are at most of the order of $\rho U^2/p$. The corresponding fractional changes of density are of much the same order and, since $a^2 = \gamma p/\rho$ (equation 9.1,2), we see that these fractional density changes can be written

$$\Delta\rho/\rho = 0(M^2)$$

where $M = U/a$ (the body Mach number). Hence, these density changes can be ignored as long as M is small compared with unity, and then the fluid can be treated as incompressible. With increase of M, however, the accompanying density changes will produce increasing changes in the flow characteristics, and if M is not small compared with unity the effects of compressibility cannot be ignored.

To take this argument a stage further, consider the steady flow in a stream tube of small cross-section in which one-dimensional conditions may be assumed. Then continuity of mass flow along the tube requires

$$\rho U A = \text{const.} \tag{9.1,4}$$

where A is the cross-sectional area of the tube. Hence, if s denotes distance along the tube

$$\frac{1}{\rho}\frac{d\rho}{ds}+\frac{1}{U}\frac{dU}{ds}+\frac{1}{A}\frac{dA}{ds}=0. \tag{9.1,5}$$

But

$$\frac{1}{\rho}\frac{d\rho}{ds}=\frac{1}{\rho}\frac{d\rho}{dp}\frac{dp}{ds}=-\frac{d\rho}{dp}U\frac{dU}{ds},$$

using Bernoulli's equation (see § 3.4). If we assume that the flow in the tube is isentropic, then it follows from equation (9.1,1) that

$$\frac{1}{\rho}\frac{d\rho}{ds}=-\frac{U}{a^2}\frac{dU}{ds}$$

and hence equation (9.1,5) becomes

$$\frac{1}{A}\frac{dA}{ds}=-\frac{1}{U}\frac{dU}{ds}(1-M^2) \tag{9.1,6}$$

where $M = U/a$ = local Mach number in stream tube. Simple as is equation (9.1,6), it is of very great importance. We see that as long as $M < 1$ an increase of velocity U accompanies a decrease of A with distance s and vice versa, but when $M > 1$ an increase of velocity U accompanies an increase of A with distance s and vice versa. Thus, for the flow to change smoothly from subsonic to supersonic the tube must first contract in cross-section until locally the speed becomes equal to the speed of sound, and thereafter the tube must expand in cross-section. Hence the necessity for a convergent-divergent nozzle just upstream of the working section of a supersonic wind-tunnel.

From the above it will be readily appreciated why the Mach number is the essential parameter for describing compressibility effects. As long as the Mach number is small compared with unity we may expect these effects to be negligible, but they will increase roughly as M^2 and hence become rapidly significant with increase of Mach number. At supersonic speeds the effects of compressibility are such that the flow characteristics are very different in most essentials from those of subsonic speeds.

9.2 Some Elementary Thermodynamic Concepts

9.2.1 The Equation of State of a Gas. Functions of State

It is an experimental result that the pressure (p), density (ρ) and temperature (T) of a gas at rest are not independent, but it is found that there exists a relation of the form

$$F(p, \rho, T) = 0. \tag{9.2,1}$$

Hence the values of any two of p, ρ and T determine the value of the third.

The state of a gas is determined by its pressure, density and temperature, and an equation of the form of (9.2,1) is called an *Equation of State.*

At temperatures and pressures well away from liquefaction or dissociation gases satisfy with good approximation the relation

$$\frac{p}{\rho} = \frac{\bar{R}}{m} T \tag{9.2,2}$$

where $\bar{R}$ is the so-called universal gas constant, m is the molecular weight of the gas, and T is the absolute temperature. $\bar{R}$ is $8{\cdot}95 \times 10^4$ ft^2/s^2 °K or $8{\cdot}315 \times 10^3$ m^2/s^2 °K. For air $\bar{R}/m = 3094$ ft^2/s^2 °K or $287{\cdot}4$ m^2/s^2 °K (or J/kg °K).

A gas which obeys equation (9.2,2) is called a *perfect gas.* This law can be readily derived on the basis of the simple kinetic theory of gases, which postulates a gas in which the molecules are small, perfectly elastic spheres in random motion and the total volume of the molecules is negligible compared with the volume that they occupy. It is also assumed that the molecules do not influence each other except when they collide. The pressure can then be identified with the net rate of change of momentum at the containing walls due to incident molecules bouncing off them and is therefore proportional to the number of molecules per unit volume, their mass, and their mean squared velocity. The density is simply the number of molecules per unit volume times the mass of a molecule, whilst the temperature is proportional to the mean kinetic energy of a molecule. The perfect gas law then follows. In so far as a real gas may deviate from the perfect gas law, it does so mainly because the conditions of pressure and temperature are such that the interaction between molecules is not negligible and their total volume is not negligible compared with the space that they occupy.

It is assumed that the equation of state of a gas holds when the gas is in motion as well as at rest. There is no direct proof of this but deductions based on this assumption have been found to be in good accord with experimental results.

Any property of a gas which is completely determined, except for an arbitrary constant, by any two of the state variables p, ρ and T is called a *function of state.* Thus if F denotes a function of state, then the change in F in going from state 1 to state 2, say, is independent of the way in which the change is brought about and depends only on the values of the state variables at the beginning and end of the change.

9.2.2 The First Law of Thermodynamics

Experiments have demonstrated conclusively that heat energy and mechanical energy are interchangeable and equivalent in the sense that a given quantity of heat is found to be always equivalent to the same quantity of mechanical energy. Thus heat energy can be expressed in terms of mechanical units, and conversely mechanical energy can be measured in terms of heat units. The First Law of Thermodynamics then follows if we apply the principle of conservation of energy to processes involving the performance of mechanical work and the exchange of heat.

However, to apply the principle we must introduce the concept of *internal energy*. Consider a quantity of gas in a standard state at rest and suppose we subject the gas to some process as a result of which its state is changed to some given state and it acquires some mean motion. To effect the change a quantity of heat Q_e, say, is absorbed from the external medium, and a certain amount of work W, say, will be done on the gas by the external medium. The internal energy, E, of the gas is then defined as the algebraic sum of the mechanical equivalents of Q_e and W, minus the accompanying increase of the kinetic energy of the mean motion of the gas, K, say. Thus, we can write

$$E = Q_e + W - K. \tag{9.2,3}$$

The First Law of Thermodynamics then says that E is a function of state, independent of the details of the process whereby the change is made. Clearly, if this were not so it would be possible to pass from the standard state to the given state by one process and return by another, and we would have finally some residual change of energy of the external medium, but with no net change of energy of the gas. Thus energy would have been either created or destroyed by this cyclic process contrary to the principle of conservation of energy.

For a small change from one state to a neighbouring state we can write

$$\delta E + \delta K = \delta Q_e + \delta W \tag{9.2,4}$$

where δ denotes the small increments in the quantities concerned in the process.

9.2.3 Reversible and Irreversible Processes

A process whereby a change of state and mean motion of a quantity of gas is brought about is defined as *reversible* if the gas and the external medium can be completely restored to their initial condition. It is a deduction from observation that all natural processes are more or less irreversible and the degree of irreversibility is related to the lack of equilibrium during the process and in particular to the amount of heat that is generated when mechanical energy is dissipated by viscous effects and solid friction, and also to the heat that is transferred by conduction from hotter to colder parts of the system. Thus we find that, when the heat content of a body is increased by dissipation or conduction, the heat addition cannot be wholly converted back into useful mechanical work without further changes taking place. Hence the irreversibility of the process.

For example, consider some gas in a cylinder and suppose the gas to be expanding against a piston (Fig. 9.2,1). Then with a small increase of volume δV the work done by the gas on the piston is $p\ \delta V$, where p is the pressure of the gas. If there were no friction between the piston and the cylinder, and the piston were moving sufficiently slowly for the inertia forces due to its own motion and that of the gas to be negligible, then the pressure of the external medium would be p and so the work done on the external medium would be $p\ \delta V$. In such a case the process would be reversible, at every stage the gas and external medium would be as near equilibrium with each other as we would care to make them. Now suppose that there were some friction between the piston and the cylinder. In this case the pressure p would have to be greater than the pressure of the external medium in order to balance the frictional forces. Thus, the work done on the external medium would be less than $p\ \delta V$, which represents the maximum useful work which the gas can do on the external medium in the expansion. The energy of any motion of the gas or of the external medium produced during the expansion would not all be converted back to mechanical energy, but some would eventually be dissipated as heat and in consequence the work done on the external medium would be reduced.

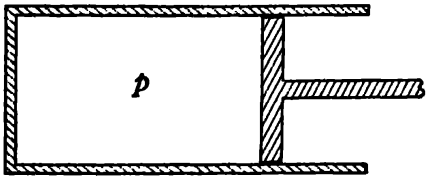


Fig. 9.2,1.

In general, we find that a reversible process between two states is characterised by the fact that the gas undergoing it performs the greatest possible amount of useful work on the external medium, i.e. $\int p\, dV$; in all other processes between the same states the amount of useful work done on the external medium is less. Conversely we can say that the amount of work done by the external medium is never less than $-\int p\, dV$. In particular, the expansion of a gas without the performance of external work is irreversible.

Again, we observe experimentally that, whilst heat passes readily from a hot body to a cold body, we cannot devise a way of making heat pass from a cold body to a hot body without producing some further changes externally. This is another way of saying that the process of heat conduction is irreversible.

These observations anticipate the Second Law of Thermodynamics, about which we shall say more later. However, we must here note that the First Law can tell us nothing about the reversibility or otherwise of a process. This latter concept is not concerned with the conservation of total energy as is the First Law, but with the possible direction in which a change can proceed involving the conversion of mechanical energy into heat, or vice versa, or the transfer of heat energy.

Since dissipation of mechanical energy into heat due to viscous or frictional forces and heat conduction contribute to the irreversibility of a process, and since viscous forces depend on velocity gradients whilst heat conduction depends on temperature gradients, it follows that the smaller these gradients are the less irreversible is the process. This illustrates the general statement that a reversible process must be such that at all stages the gas and its surroundings are in equilibrium. This would imply that strictly the process would require an infinite time to be completed; in practice processes can proceed quite quickly and still be classified as reversible if the coefficients of viscosity and heat conduction are sufficiently small or the gradients of temperature and velocity are sufficiently small.

Consider now the application of the First Law to an element of gas undergoing a small reversible change. Then in equation (9.2,4) we can write

$$\delta W = -p\,\delta V$$

if the gas is at rest, and

$$\delta W = -p\,\delta V + \delta K$$

if the gas is in motion.

Hence equation (9.2,4) becomes

$$\delta E = \delta Q_e - p\,\delta V. \tag{9.2,5}$$

If the process is irreversible, however, then we have that

$$\delta W > -p\,\delta V$$

if the gas is at rest, and

$$\delta W > -p\,\delta V + \delta K$$

if the gas is in motion.

In this case write

$$\delta W = -p\,\delta V + \delta K + \delta Q_i,$$

then $\delta Q_i > 0$, and we have

$$\delta E = (\delta Q_e + \delta Q_i) - p\,\delta V$$

where

$$\left.\begin{aligned} &= \delta Q - p\,\delta V \\ \delta Q &= \delta Q_e + \delta Q_i. \end{aligned}\right\} \tag{9.2,6}$$

We see that δQ represents the total change in heat content of the gas in the process. Part of δQ, viz. δQ_e, is the heat absorbed from outside, but the remainder, δQ_i, is the heat developed internally because of the irreversibility of the process.

A process for which the heat absorbed from outside is zero, i.e. $\delta Q_e = 0$, is called an *adiabatic process*. It will be clear that an adiabatic process is not necessarily reversible. For example, the flow of viscous fluid in an insulated pipe is an adiabatic process, since the fluid cannot absorb heat from or give up heat to its surroundings, but the viscous forces produce changes in the fluid which are irreversible.

9.2.4 Internal Energy and Enthalpy. Specific Quantities

Since E is a function of state it follows from the Equation of State that the quantity

$$I = E + pV \tag{9.2,7}$$

is also a function of state. The quantity I is called the *Enthalpy*, and it has also sometimes been referred to as the *Total Heat*.

It is frequently convenient to talk about the values of thermodynamic quantities per unit mass; such values are distinguished by the adjective 'specific'. We shall denote specific quantities by small letters. Thus, e denotes specific internal energy, i is specific enthalpy, and v is specific volume $= 1/\rho$. In terms of specific quantities equations (9.2,6) and (9.2,7) become

$$\left.\begin{aligned}\delta e &= \delta q - p\,\delta v\\ \text{where}\quad \delta q &= \delta q_e + \delta q_i\end{aligned}\right\} \tag{9.2,8}$$

and

$$i = e + pv. \tag{9.2,9}$$

The term *specific heat* is more particularly defined to mean the quantity of heat required to raise unit mass of the solid or fluid in question through a unit of temperature. For a gas the specific heat depends on the conditions of heating, since if the gas is allowed to expand whilst being heated it will do work against the external medium which will be at the expense of the heat input. However, of particular importance to us are the specific heats determined with the volume of the gas kept constant or with the pressure kept constant. These specific heats are usually written as c_v and c_p, respectively. Thus

$$c_v = \left(\frac{\partial q}{\partial T}\right)_v \quad \text{and} \quad c_p = \left(\frac{\partial q}{\partial T}\right)_p. \tag{9.2,10}$$

From (9.2,8) and (9.2,9) we have

$$\delta q = \delta e + p\,\delta v = \delta i - v\,\delta p$$

and hence

$$c_v = \left(\frac{\partial e}{\partial T}\right)_v \quad \text{and} \quad c_p = \left(\frac{\partial i}{\partial T}\right)_p. \tag{9.2,11}$$

Since the molecules of a perfect gas do not react on each other, the internal energy of a perfect gas does not depend on the spacing of the molecules and therefore does not depend on its density. Hence, for a perfect gas e is a function of T, only, and therefore from (9.2,11) we have

$$e = \int_0^T c_v\,dT, \tag{9.2,12}$$

where we have chosen $T = 0$ when $e = 0$. If the specific heat c_v can be taken as constant between temperatures T_1 and T_2, say, then

$$e_2 - e_1 = c_v T_2 - c_v T_1. \tag{9.2,13}$$

Further, the equation of state for a perfect gas, equation (9.2,2), can be written

$$pv = \frac{\bar{R}}{m} T$$

and hence the second equation of (9.2,11) and equation (9.2,9) lead to

$$i = \int_0^T c_p \, dT \tag{9.2,14}$$

and

$$c_p = c_v + \frac{\bar{R}}{m}. \tag{9.2,15}$$

Thus, over a range of temperature for which c_v may be taken as constant c_p may also be taken as constant and then

$$i_2 - i_1 = c_p T_2 - c_p T_1. \tag{9.2,16}$$

For real gases the quantities c_p and c_v are in fact very slowly varying functions of temperature and pressure, but for air at normal atmospheric pressure they may be taken as constant from 0°C to about 300°C.

We denote the ratio c_p/c_v by the symbol γ, so that (9.2,15) can be written

$$\bar{R}/m = c_p\left(\frac{\gamma - 1}{\gamma}\right) = c_v(\gamma - 1). \tag{9.2,17}$$

The value of γ for a gas depends on its molecular structure, and like the specific heats it varies slowly with temperature and pressure. For air at normal atmospheric temperature and pressure the value of γ is closely enough given by the value predicted for diatomic gases by the kinetic theory of gases, namely 1.40. Under those conditions the corresponding value of c_p is $1{\cdot}0818 \times 10^4$ ft. lbf./slug °C or 1·005 kJ/kg °C.

9.2.5 Entropy and the Second Law of Thermodynamics

Consider a small reversible change for a perfect gas for which we have from equations 9.2,2, 9.2,8 and 9.2,12

$$\delta q_e = c_v \, \delta T + \frac{\bar{R}}{m}\frac{T}{v} \, \delta v.$$

Then, if the process is also adiabatic $\delta q_e = 0$, and we have

$$c_v \frac{\delta T}{T} + \frac{\bar{R}}{m}\frac{\delta v}{v} = 0.$$

It follows that for such a process the function of state

$$s \equiv \int^T \frac{c_v \, dT}{T} + \frac{\bar{R}}{m} \log v \tag{9.2,18}$$

is a constant.

We call the function s the *specific entropy.* We see that it satisfies the differential relation

$$T\,\delta s = \delta e + p\,\delta v. \tag{9.2,19}$$

An alternative form for s, differing only by a constant, is readily found with the aid of the equation of state (equation 9.2,2) to be

$$s \equiv \int^T c_p \frac{dT}{T} - \frac{\bar{R}}{m}\log p, \tag{9.2,20}$$

and if c_p and c_v can be taken as constant then another equivalent form is

$$s \equiv c_v(\log p + \gamma \log v). \tag{9.2,21}$$

From this last form it follows that for a perfect gas with constant specific heats a reversible, adiabatic process, i.e. one at constant entropy, obeys the law

$$pv^\gamma \equiv p/\rho^\gamma = \text{const.} \tag{9.2,22}$$

Such a process is referred to as *isentropic.*

More generally it can be shown that for any gas, perfect or not, there exists a function of state defined by

$$s = \int_0 \frac{dq}{T} = \int_0 \frac{de}{T} + \int_0 \frac{p\,dv}{T} \tag{9.2,23}$$

where 0 refers to a standard state and the integration can be made following any reversible path. We see that s is constant for a reversible, adiabatic process. It follows that for a gas undergoing a small irreversible process from state 1 to state 2, say,

$$\delta s = s_2 - s_1 = (\delta e + p\,\delta v)/T = \frac{\delta q}{T} = \frac{\delta q_e + \delta q_i}{T}$$

$$> \frac{\delta q_e}{T}, \tag{9.2,24}$$

since q_i is necessarily positive.

We come now to the statement of Second Law of Thermodynamics. We have already noted that heat flows naturally from a hot body to a cold body with which it is in contact, without requiring further external change, but the reverse process never occurs. The Second Law enlarges on this observation by stating that it is impossible to construct an engine which will work in a complete cycle and produce no effect except the raising of a weight and the extraction of heat from a reservoir.† Were such an engine capable of being made, it would enable us to exploit directly the practically infinite heat resources of the earth and atmosphere and would relieve us of the necessity to mine fuel and harness water power.

An implication of the Second Law and indeed of our general observations on irreversible processes, is that there exists a property of a system and of its surroundings which is a function of state, such that its total value remains constant for reversible processes but changes for irreversible processes. Now we have seen that for unit mass of gas undergoing a small reversible change

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† It can be readily confirmed that these two statements are equivalent by coupling the engine of the second statement with an engine working in a Carnot cycle in reverse.

$$\delta s = \delta q_e/T.$$

During this change the external medium will lose a quantity of heat δq_e and its change of entropy will likewise be

$$\delta s = -\delta q_e/T.$$

Thus the *total* change of entropy of the gas and its surroundings in a reversible process is zero, and hence in such a process the total entropy is constant. If the process were irreversible however, then we have seen that for the gas

$$\delta s > \delta q_e/T,$$

and for the external medium we must also have

$$\delta s \geqslant -\delta q_e/T.$$

Hence, in this case the *total* change of entropy is positive. It follows that all possible processes must proceed in the direction of increasing total entropy, except in the case of reversible processes for which the total entropy remains constant. In no possible process can the total entropy decrease. This can be regarded as another statement of the Second Law; it provides us with a criterion for assessing the possible direction of a process and a quantitative description in the form of the entropy change for the degree of irreversibility of the process.

9.3 The Dynamical and Energy Equations†

We have seen (§ 3.5) that the equations of continuity and motion for the steady flow of a compressible inviscid fluid are

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0, \tag{9.3,1}$$

and

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + X, \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + Y, \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + Z, \end{aligned}\right\} \tag{9.3,2}$$

where X, Y, Z are components of a body force.

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† The reader is warned that unavoidably the symbol v is here used to denote specific volume and also the velocity component parallel to the y axis. However, in all cases the intended meaning is clear and confusion should not arise. Further we now use V to denote resultant velocity instead of q as in § 3.5.

By a little rearrangement the equations (9.3,2) can be written

$$-v\zeta + w\eta = -\frac{\partial}{\partial x}\left[\varpi + \frac{V^2}{2}\right] + X,$$

$$-w\xi + u\zeta = -\frac{\partial}{\partial y}\left[\varpi + \frac{V^2}{2}\right] + Y,$$

$$-u\eta + v\xi = -\frac{\partial}{\partial z}\left[\varpi + \frac{V^2}{2}\right] + Z,$$

where ξ, η, ζ are the components of vorticity:

$$\xi = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad \eta = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad \zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \quad \varpi = \int \frac{dp}{\rho}$$

and $$V^2 = u^2 + v^2 + w^2.$$

Multiplication of these equations by u, v, w, respectively, leads to

$$\frac{Dh_M}{Dt} \equiv \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\left(\varpi + \frac{V^2}{2} + \chi\right) = 0$$

where χ is the potential of the body forces, and

$$h_M = \varpi + V^2/2 + \chi. \tag{9.3,3}$$

It follows that h_M is constant for a particle moving with the fluid, i.e. h_M is constant along a streamline. As we have already seen (§ 3.4) this is generally referred to as Bernoulli's equation. It will also be clear from the above that if the motion is irrotational

$$h_M = \text{const. everywhere,} \tag{9.3,4}$$

as was earlier proved in § 3.5.

From equations (9.2,9) and (9.2,19) it follows that

$$\begin{aligned}\varpi = \int \frac{dp}{\rho} = \int v\,dp &= i - e - \int p\,dv \\ &= i - \int T\,ds.\end{aligned} \tag{9.3,5}$$

Hence along a streamline with constant entropy

$$h_M = i + \frac{V^2}{2} + \chi = \text{const.}$$

and if the gas is perfect with constant specific heats

$$\begin{aligned}h_M \equiv c_pT + \frac{V^2}{2} + \chi &\equiv c_p\frac{pm}{\rho\bar{R}} + \frac{V^2}{2} + \chi \\ &\equiv \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{V^2}{2} + \chi = \text{const. along a streamline.}\end{aligned} \tag{9.3,6}$$

We have already anticipated the formula for the speed of sound (§ 10.3)

$$a = \left(\frac{dp}{d\rho}\right)_s^{\frac{1}{2}}$$

and since, for a perfect gas with constant specific heats under conditions of constant entropy (equation 9.2,22),

$$p/\rho^\gamma = \text{const.},$$

it follows that

$$a^2 = \gamma p/\rho. \tag{9.3,7}$$

Hence equation (9.3,6) can also be written

$$h_M \equiv \frac{a^2}{\gamma - 1} + \frac{V^2}{2} + \chi = \text{const. along a streamline.} \tag{9.3,8}$$

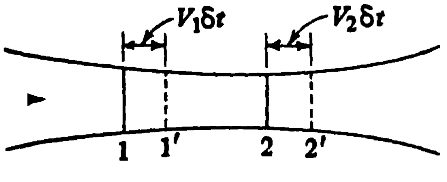

Fig. 9.3,1.

Now let us consider the steady flow of an inviscid non-conducting fluid along a narrow stream tube in which the motion can be regarded as one-dimensional.

Suppose that at the beginning of a small time interval of duration δt a portion of fluid is between sections 1 and 2, as illustrated in Fig. 9.3,1, and at the end of the time interval it is between sections 1′ and 2′. Let suffix 1 denote the values at section 1 of the pressure (p), velocity (V), density (ρ), specific internal energy (e) and cross-sectional area (A), and let suffix 2 denote corresponding quantities at section 2. We will now apply the First Law of Thermodynamics to the changes undergone during the time interval δt by the fluid initially contained between the sections 1 and 2.

The work done by the normal pressures acting on the end section in the interval $\delta t = p_1 A_1 V_1\,\delta t - p_2 A_2 V_2\,\delta t$.

Continuity of rate of mass flow along the tube requires that

$$\rho_1 A_1 V_1 = \rho_2 A_2 V_2 = f, \text{ say.}$$

Hence, the work done by the pressures on the end sections can be written

$$f\,\delta t\left[\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right]. \tag{9.3,9}$$

The work done by the body forces (assumed conservative) is

$$-\left[\int_1^2 f\frac{\partial \chi}{\partial s}\,ds\right]\delta t$$

where s represents distance along the tube. This work is therefore

$$-f(\chi_2 - \chi_1)\,\delta t. \tag{9.3,10}$$

The change of kinetic energy of the fluid in time δt is the difference between that of the fluid contained between sections 2 and 2′ and that of the fluid between sections 1 and 1′. It is therefore given by

$$\tfrac{1}{2}[\rho_2 A_2 V_2^3\,\delta t - \rho_1 A_1 V_1^3\,\delta t] = \frac{f}{2}\,\delta t(V_2^2 - V_1^2). \tag{9.3,11}$$

Similarly the change of internal energy of the fluid

$$\begin{aligned} &= \rho_2 A_2 V_2 e_2\,\delta t - \rho_1 A_1 V_1 e_1\,\delta t \\ &= f\,\delta t(e_2 - e_1). \end{aligned} \tag{9.3,12}$$

Hence, applying the First Law of Thermodynamics, we have

$$\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} + \chi_1 - \chi_2 = \frac{V_2^2}{2} - \frac{V_1^2}{2} + e_2 - e_1$$

or

$$h_{E2} = h_{E1},$$

where

$$\begin{aligned} h_E &\equiv e + \frac{V^2}{2} + \frac{p}{\rho} + \chi \\ &\equiv i + \frac{V^2}{2} + \chi. \end{aligned} \tag{9.3,13}$$

Thus, we see that for the adiabatic flow of an inviscid fluid

$$h_E = \text{const. along a streamline,} \tag{9.3,14}$$

and this equation is sometimes referred to as the energy equation. The quantity h_E is sometimes called the *total energy* per unit mass, although it must be noted that the aptness of this name is limited to steady flow conditions.

It is important to note that, apart from the condition that χ must be continuous between sections 1 and 2, there is nothing in the above discussion leading to equation (9.3,14) which precludes the existence of discontinuities between these sections in any of the properties of the fluid (e.g. pressure, velocity, etc.) such as would occur across shock waves.

For a tube of fluid of large cross-section and bounded by streamlines it readily follows that for any cross-section

$$\iint \rho V h_E\,dA = \text{const.} \tag{9.3,15}$$

where the constant is the same for all cross-sections.

If the fluid were viscous and conducting, then (9.3,13) would be replaced by an equation of the form

$$f h_{E2} = f h_{E1} + W_f + Q$$

where W_f represents the rate of work done on the fluid in the streamtube between sections 1 and 2 by frictional forces, and Q represents the rate of heat transfer into that portion of tube. In so far as equal but opposite contributions must appear in the corresponding energy equations of neighbouring streamtubes, it follows that equation (9.3,15) would still hold for a tube of large cross-section if the external surface of the tube were insulated and if the frictional force on that surface did no work. It holds in particular for flow in an insulated pipe.

We see that the momentum or Bernoulli equation (equation 9.3,3) and the energy equation (9.3,14) are similar but are not in general the same. The former holds for the steady, continuous flow of an inviscid fluid, whilst the latter holds for the steady, adiabatic flow of an inviscid fluid. Thus, for the steady, continuous, adiabatic flow of an inviscid fluid we deduce that

$$h_M = h_E + \text{const}\,,\ \text{along a streamline,}$$

and hence

$$\int v\,dp = i + \text{const.},\ \text{along a streamline.}$$

But, since (equation 9.2,9)

$$\delta i = \delta e + p\,\delta v + v\,\delta p,$$

it follows that

$$\delta s = (\delta e + p\,\delta v)/T = 0,\ \text{along a streamline,}$$

i.e. the entropy is constant along a streamline. This is not surprising, since in specifying that the fluid must be inviscid and adiabatic and that the flow must be continuous and steady we have excluded all the possible factors that could make the change undergone an irreversible one, and hence the entropy must be constant.

We have already noted that if the flow is irrotational then h_M is constant everywhere; it can also be deduced that if in addition the entropy is constant everywhere then so is h_E. These deductions are of considerable practical importance because for many problems of aerodynamic interest the conditions are such that the effects of conductivity and viscosity can be neglected except in the thin boundary layers adjacent to surfaces moving in the fluid, the wakes that trail behind the surfaces, and in shock waves. The flow of an inviscid, adiabatic fluid therefore describes with adequate accuracy the motion in extensive and important regions of the flow. Further, for many problems the conditions far ahead of the disturbing body can be regarded as uniform, and in the absence of discontinuities such as shock waves, or where such discontinuities are of small enough magnitude or can be regarded as uniform, the simplifications of irrotational flow theory apply outside boundary layers and wakes.

9.4 Some Simple Relations for the Steady Adiabatic Flow of a Perfect Gas. Choking[†]

From equations (9.2,14) and (9.3,14) it follows that along any streamline in the steady, adiabatic flow of a perfect gas with constant specific heats

$$h_E = i + \frac{V^2}{2} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{V^2}{2} = c_pT + \frac{V^2}{2} = \frac{a^2}{\gamma - 1} + \frac{V^2}{2}$$
$$= c_pT_0 = \frac{{a_0}^2}{\gamma - 1} = \text{const.} \qquad (9.4,1)$$

Suffix 0 refers to the gas at rest. We have further that, if the flow is isentropic, then along the streamline

$$p/\rho^\gamma = p_0/{\rho_0}^\gamma = \text{const.} \qquad (9.4,2)$$

In certain applications we require to consider a flow expanding from rest, e.g. the flow in some types of wind tunnels or in rocket nozzles. In such cases it is convenient to refer properties of the gas to their values when the gas is at rest. In other applications we consider a body in an otherwise uniform stream, e.g. aircraft in flight, and then we require to refer the gas properties to their values in the undisturbed stream. The following lists a number of formulae which will be found useful in both these cases and which the reader should have no difficulty in proving.

Let $t = T/T_0$, then it readily follows from (9.3,7) and (9.4,1) that

$$t = \left[1 + \frac{\gamma - 1}{2}M^2\right]^{-1} = \left[1 + \frac{M^2}{5}\right]^{-1}, \quad \text{for } \gamma = 1{\cdot}4, \qquad (9.4,3)$$

$$V/a_0 = \sqrt{\left[\frac{2}{\gamma - 1}(1 - t)\right]} = \sqrt{[5(1 - t)]}, \quad \text{for } \gamma = 1{\cdot}4, \qquad (9.4,4)$$

$$M = V/a = \sqrt{\left[\frac{2}{\gamma - 1}\frac{1 - t}{t}\right]} = \sqrt{\left[\frac{5(1 - t)}{t}\right]}, \quad \text{for } \gamma = 1{\cdot}4. \qquad (9.4,5)$$

If the flow is isentropic, then from (9.4,1) and (9.4,2)

$$\left.\begin{aligned} p/p_0 &= \left[1 + \left(\frac{\gamma - 1}{2}\right)M^2\right]^{-[\gamma/(\gamma-1)]} = t^{\gamma/(\gamma-1)} \\ &= \left[1 + \frac{M^2}{5}\right]^{-7/2} = t^{7/2}, \quad \text{for } \gamma = 1{\cdot}4, \end{aligned}\right\} \qquad (9.4,6)$$

and

$$\left.\begin{aligned} \rho/\rho_0 &= \left[1 + \left(\frac{\gamma - 1}{2}\right)M^2\right]^{-[1/(\gamma-1)]} = t^{1/(\gamma-1)} \\ &= \left[1 + \frac{M^2}{5}\right]^{-5/2} = t^{5/2}, \quad \text{for } \gamma = 1{\cdot}4. \end{aligned}\right\} \qquad (9.4,7)$$

[†] In this section body forces will be neglected.

From (9.4,1) we see that the velocity is a maximum when $a^2 = \gamma p/\rho = 0$, and $M = V/a = \infty$, and this maximum velocity is given by

$$V_{\max} = \sqrt{\left(\frac{2}{\gamma - 1}\right)}\, a_0 = 2{\cdot}24 a_0, \quad \text{for } \gamma = 1{\cdot}4. \tag{9.4,8}$$

When the velocity is equal to the local speed of sound (i.e. M is unity) it is called the critical velocity, written V^*, and from (9.4,1) it is given by

$$\begin{aligned} V^{*2} &= \frac{2a_0^2}{\gamma + 1} = \frac{5}{6} a_0^2, \quad \text{for } \gamma = 1{\cdot}4, \\ &= \left(\frac{\gamma - 1}{\gamma + 1}\right) V_{\max}^2 = \frac{V_{\max}^2}{6}. \end{aligned} \tag{9.4,9}$$

The corresponding temperature T^* is given by

$$t^* = T^*/T_0 = 2/(\gamma + 1) = 0{\cdot}833, \quad \text{for } \gamma = 1{\cdot}4. \tag{9.4,10}$$

If the flow is isentropic, then the corresponding values of the pressure and density follow from (9.4,6) and (9.4,7), with $M = 1$,

$$\left.\begin{aligned} p^*/p_0 &= [2/(\gamma + 1)]^{\gamma/(\gamma-1)} = 0{\cdot}528, \quad \text{for } \gamma = 1{\cdot}4, \\ \text{and} \qquad \rho^*/\rho_0 &= [2/(\gamma + 1)]^{1/(\gamma-1)} = 0{\cdot}636, \quad \text{for } \gamma = 1{\cdot}4. \end{aligned}\right\} \tag{9.4,11}$$

With uniform flow conditions at infinity and with constant entropy everywhere, the following formulae apply. Here suffix 1 refers to undisturbed stream quantities.

The energy and Bernoulli equations are now equivalent, since the entropy is everywhere constant, so that we have

$$\begin{aligned} \frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{V_1^2}{2} &= \frac{a_1^2}{\gamma - 1} + \frac{V_1^2}{2} = c_p T_1 + \frac{V_1^2}{2} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{V^2}{2} \\ &= \frac{a^2}{\gamma - 1} + \frac{V^2}{2} = c_p T + \frac{V^2}{2} = \text{const.} \end{aligned} \tag{9.4,12}$$

and

$$p_1/\rho_1^{\gamma} = p/\rho^{\gamma} = \text{const.} \tag{9.4,13}$$

It then follows that

$$\frac{T}{T_1} = \left[\frac{1 + \frac{\gamma - 1}{2} M_1^2}{1 + \frac{\gamma - 1}{2} M^2}\right] = \left[\frac{5 + M_1^2}{5 + M^2}\right], \quad \text{for } \gamma = 1{\cdot}4, \tag{9.4,14}$$

$$p/p_1 = (T/T_1)^{\gamma/(\gamma-1)} = \left[\frac{5 + M_1^2}{5 + M^2}\right]^{7/2}, \quad \text{for } \gamma = 1{\cdot}4, \tag{9.4,15}$$

$$\rho/\rho_1 = (T/T_1)^{1/(\gamma-1)} = \left[\frac{5 + M_1^2}{5 + M^2}\right]^{5/2}, \quad \text{for } \gamma = 1{\cdot}4, \tag{9.4,16}$$

$$V/V_1 = (M/M_1)(T/T_1)^{1/2} = \frac{M}{M_1}\left[\frac{5 + M_1^2}{5 + M^2}\right]^{1/2}, \quad \text{for } \gamma = 1{\cdot}4. \tag{9.4,17}$$

When $M = 1$ (critical conditions) we have

$$T^*/T_1 = \frac{2}{\gamma+1}\left[1 + \frac{\gamma-1}{2} M_1^2\right] = \left(\frac{5 + M_1^2}{6}\right), \quad \text{for } \gamma = 1{\cdot}4, \tag{9.4,18}$$

$$V^*/V_1 = \frac{1}{M_1}\left(\frac{T^*}{T_1}\right)^{1/2} = \frac{1}{M_1}\left[\frac{5 + M_1^2}{6}\right]^{1/2}, \quad \text{for } \gamma = 1{\cdot}4, \tag{9.4,19}$$

$$p^*/p_1 = (T^*/T_1)^{\gamma/(\gamma-1)} = \left(\frac{5 + M_1^2}{6}\right)^{7/2}, \quad \text{for } \gamma = 1{\cdot}4, \tag{9.4,20}$$

and $$\rho^*/\rho_1 = (T^*/T_1)^{1/(\gamma-1)} = \left(\frac{5 + M_1^2}{6}\right)^{5/2}, \quad \text{for } \gamma = 1{\cdot}4. \tag{9.4,21}$$

The local pressure coefficient is defined by

$$C_p = (p - p_1)/\tfrac{1}{2}\rho_1 V_1^2 = \left(\frac{p}{p_1} - 1\right)\Big/\frac{\gamma}{2} M_1^2 \tag{9.4,22}$$

and hence the value of C_p when $M = 1{\cdot}0$ is

$$\left.\begin{aligned} C_p^* &= \frac{2}{\gamma M_1^2}\left\{\left[\frac{2}{\gamma+1} + \frac{\gamma-1}{\gamma+1} M_1^2\right]^{\gamma/(\gamma-1)} - 1\right\} \\ &= \frac{10}{7M_1^2}\left\{\left[\frac{5 + M_1^2}{6}\right]^{7/2} - 1\right\} \quad \text{for } \gamma = 1{\cdot}4. \end{aligned}\right\} \tag{9.4,23}$$

We also have from equations (9.4,13)–(9.4,20) that

$$\frac{\rho V}{\rho^* V^*} = M\left[\frac{\gamma+1}{2 + (\gamma-1)M^2}\right]^{(\gamma+1)/[2(\gamma-1)]} = M\left[\frac{6}{5 + M^2}\right]^3, \quad \text{for } \gamma = 1{\cdot}4. \tag{9.4,24}$$

It readily follows that $d/dM(\rho V/\rho^* V^*) = 0$, when $M = 1{\cdot}0$, and that then $\rho V/\rho^* V^*$ reaches its maximum value of unity. Fig. 9.4,1 shows $(\rho V/\rho^* V^*)$ as a function of M and of p/p_0.

We see that to each value of $\rho V/\rho^* V^*$ less than unity there are two possible values of M, one subsonic and the other supersonic. The significance of the quantity ρV lies in the fact that for steady flow along a stream tube of small cross-section $\rho V A$ = constant, where A is the cross-sectional area. Hence if A^* is the throat area

$$A^*/A = \rho V/\rho^* V^*. \tag{9.4,25}$$

It follows that A must be a minimum when $M = 1{\cdot}0$, and hence, as we have already seen in § 9.1, if the velocity in the tube is to increase continuously with distance along the tube from subsonic to supersonic speeds the tube must first contract to a throat where $M = 1{\cdot}0$ and then expand.

Consider the flow along a convergent divergent tube or nozzle with constant values of p_0, ρ_0 and T_0 at entry such as would occur if the flow entered the tube from a large reservoir, and suppose the exit pressure (p_e) can be varied. Then as p_e is reduced from p_0 the velocity and mass flow rate (ρVA) will increase until p_e reaches a value p_{e1}, say, for which at the minimum section of the tube (the throat) the local value of $M = 1{\cdot}0$ and $\rho V = \rho^* V^*$. The mass flow rate is then $\rho^* V^* A^*$, where A^* is the cross-sectional area of the throat. It follows from the foregoing that with the given entry conditions no greater mass flow rate along the tube can be obtained. The influence of any further reduction of p_e cannot penetrate upstream of the throat, since the speed of sound is attained there, and hence the mass flow rate as well as the flow upstream of the throat remain unchanged. The flow is then referred to as *choked*. When the flow is just choked the flow downstream of the throat is subsonic.

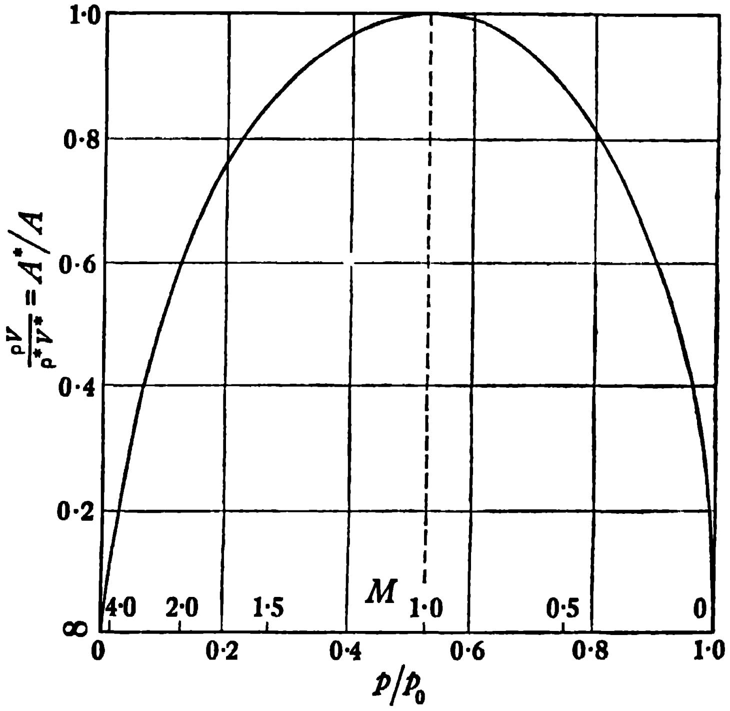

Fig. 9.4,1. One-dimensional isentropic flow. Variation of $\rho V/\rho^* V^* (=A^*/A)$ with pressure ratio and Mach number.

The following sequence of flow patterns then occur as p_e is reduced below p_{e1}.

(1) The flow becomes supersonic and expands downstream of the throat for a distance and this distance increases with reduction of p_e; this supersonic region of flow ends with a sudden compression through a normal shock wave (see § 9.8) after which it expands subsonically to attain the applied value of p_e at the exit.

(2) With reduction of p_e to p_{e2}, say, the shock wave moves downstream until it reaches the exit, at which stage the flow downstream of the throat

is completely supersonic but with an exit pressure just inside the tube p_{e3} say, less than p_{e2}, the pressure rise across the shock being equal to $p_{e2} - p_{e3}$.

(3) With further reduction of p_e below p_{e2}, the flow and pressure distribution inside the tube remain unchanged but adjustment of the pressure to p_e is made in the emergent jet downstream of the exit. This compressive adjustment is made by a train of successive shocks and expansion fans (see § 9.7) so that the pressure in the jet varies about p_e with distance downstream in an oscillatory manner. However, when p_e falls to the value p_{e3}, the pressures at the exit inside and outside the tube are equal, and the train of shocks and expansion fans in the jet disappear.

We see that for this single exit pressure the complete flow in nozzle and jet is once more isentropic and the flow in the nozzle is referred to as *fully expanded.*

(4) For values of p_e less than p_{e3} the flow inside the tube remains unaffected but now the required expansive adjustment of pressure from the exit pressure just inside the nozzle to p_e occurs by a sequence of expansion fans and shocks.

The above description of the flow in the nozzle and jet is idealised in the sense that the effects of the boundary layer in the nozzle and of turbulent mixing of the jet and external fluid have been ignored. The effect of the boundary layer can be significant in effectively modifying the shape of the nozzle (see § 6.21.1), and the pressure rise across the shock can readily induce boundary layer separation with resulting large-scale effects on the flow. Mixing of the jet and external fluid causes the trains of shocks and expansion fans to weaken with distance downstream of the exit so that the oscillations of pressure in the jet damp out with distance from the exit. However, in a general qualitative sense the above description remains valid.

9.5 Variation of Entropy with Local Reservoir Pressure in Adiabatic Flow

We define *reservoir conditions* as those which would be obtained if the flow at any point were brought to rest isentropically. We have seen (equation 9.2,21) that for a perfect gas with constant specific heats the entropy per unit mass can be written

$$s = c_v \log [p/\rho^\gamma] + \text{const.}$$

Hence s increases as p/ρ^γ increases.

If we denote reservoir conditions by suffix 0, it follows from our definition that at any point

$$p/\rho^\gamma = p_0/\rho_0{}^\gamma,$$

where p_0 and ρ_0 will vary along a streamline unless the flow is isentropic. But, since the flow is adiabatic, the reservoir temperature is constant along a streamline, and therefore from the equation of state

$$p_0/\rho_0 = \text{const.}$$

which may be written

$$p_0^{\gamma-1}[p_0/\rho_0^{\gamma}] = \text{const.}$$

It follows that in adiabatic flow p_0 decreases as p_0/ρ_0^{γ} increases and vice versa. Therefore s increases as p_0 decreases and vice versa. However, we have learnt that in all natural processes s cannot decrease. It follows that in all such processes p_0 cannot increase.

9.6 Potential and Stream Functions. The Potential Function Equation

If the flow is irrotational, then the components of vorticity are zero, i.e.

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}.$$

As in incompressible flow these are necessary and sufficient conditions for the existence of a velocity potential function ϕ such that

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z}, \tag{9.6,1}$$

$$(\text{or} \quad \mathbf{V} = -\text{grad}\ \phi).$$

The equation of continuity for steady flow is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{9.6,2}$$

In two dimensions (x, z) this becomes

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial z}(\rho w) = 0$$

and hence we can postulate a stream function ψ such that

$$\rho u = -\rho_0\frac{\partial \psi}{\partial z}, \quad \rho w = \rho_0\frac{\partial \psi}{\partial x}, \tag{9.6,3}$$

where ρ_0 is some standard density.

Reverting to three dimensions, we can convert equation (9.6,2), when the flow is isentropic, into an equation for ϕ, the so-called potential function equation, as follows. Equation (9.6,2) can be written

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \left[\frac{u}{\rho}\frac{\partial \rho}{\partial x} + \frac{v}{\rho}\frac{\partial \rho}{\partial y} + \frac{w}{\rho}\frac{\partial \rho}{\partial z}\right] = 0.$$

From (9.6,1)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\nabla^2\phi,$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Since the flow is isentropic we can write

$$\frac{u}{\rho}\frac{\partial \rho}{\partial x} = \frac{u}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s \frac{\partial p}{\partial x} = \frac{u}{\rho a^2}\frac{\partial p}{\partial x},$$

where a is the speed of sound.

But from equation (9.3,2), if we neglect the body forces, we have

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial}{\partial x}\frac{V^2}{2} = -\frac{\partial}{\partial x}\left(\frac{\phi_x^2 + \phi_y^2 + \phi_z^2}{2}\right)$$

where the suffices x, y, z denote in the usual way the partial derivatives with respect to x, y, z, respectively.

Hence

$$\frac{u}{\rho a^2}\frac{\partial p}{\partial x} = \frac{\phi_x}{a^2}\frac{\partial}{\partial x}\left(\frac{\phi_x^2}{2} + \frac{\phi_y^2}{2} + \frac{\phi_z^2}{2}\right)$$

and similar expressions follow for $\dfrac{v}{\rho a^2}\dfrac{\partial p}{\partial y}$ and $\dfrac{w}{\rho a^2}\dfrac{\partial p}{\partial z}$.

Equation (9.6,2) therefore can be written

$$\nabla^2\phi - \frac{1}{a^2}\left[\phi_x\frac{\partial}{\partial x} + \phi_y\frac{\partial}{\partial y} + \phi_z\frac{\partial}{\partial z}\right]\left[\frac{\phi_x^2}{2} + \frac{\phi_y^2}{2} + \frac{\phi_z^2}{2}\right] = 0$$

$$\text{or}\quad \phi_{xx}\left[1 - \frac{\phi_x^2}{a^2}\right] + \phi_{yy}\left[1 - \frac{\phi_y^2}{a^2}\right] + \phi_{zz}\left[1 - \frac{\phi_z^2}{a^2}\right]$$
$$- \frac{2}{a^2}\phi_{yz}\phi_y\phi_z - \frac{2}{a^2}\phi_{zx}\phi_z\phi_x - \frac{2}{a^2}\phi_{xy}\phi_x\phi_y = 0. \quad (9.6,4)$$

We note further that for a perfect gas with constant specific heats a^2 is related to ϕ by the equation

$$\frac{a^2}{\gamma - 1} + \frac{\phi_x^2 + \phi_y^2 + \phi_z^2}{2} = \text{const.} \quad (9.6,5)$$

This follows from equation (9.3,8).

For incompressible flow ($a = \infty$) equation (9.6,4) reduces to the Laplace equation

$$\nabla^2\phi \equiv \phi_{xx} + \phi_{yy} + \phi_{zz} = 0.$$

It will be clear that the non-linear equation (9.6,4) is too complex to be solved in general, and any attempt to solve it must depend for its success on further simplifying assumptions.

9.7 Mach Waves. Simple Wave Flow

We have seen in § 9.1 that a *small* disturbance in two dimensional supersonic flow is propagated along two wave fronts or Mach waves which extend downstream of it and which everywhere make the local Mach angle with the stream direction. It was emphasised that, since the disturbance was assumed small, the Mach waves were 'sound' waves, that is, the component normal to them of the relative velocity of the stream was equal to the speed of sound and flow changes produced by Mach waves are isentropic.

Now consider a supersonic flow with a lower bounding surface made up of two straight surfaces at a very *small* angle to each other, as in Fig. 9.7,1. We will suppose the flow approaching the corner to be uniform and subject only to the influence of the lower boundary. Then, since the corner angle is taken to be small, it represents a small disturbance and will generate a Mach wave of sufficient strength to turn the flow through the required infinitesimal angle. In the case illustrated in Fig. 9.7,1(a) the corner is an interior one and the wave required is, as we shall see, one of compression (denoted by a full line), whilst in the case illustrated in Fig. 9.7,1(b) the corner is an exterior one and the required wave is one of expansion (denoted by a dotted line). It must be emphasised, however, that whilst for purposes of illustration the angles of turn are shown as finite the effect of a single Mach wave is strictly infinitesimal, and it requires a family of Mach waves to produce a finite change of flow direction. Since the flow approaching the Mach wave is uniform, the Mach angle is constant along it and hence it is straight. Therefore, there can be no difference between flow conditions at any two points along the wave just aft of it and hence the downstream flow must be uniform.

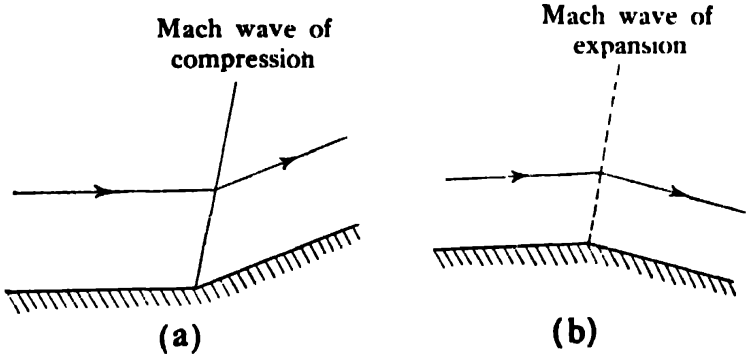

Fig. 9.7,1.

If we now imagine a succession of such corners as is illustrated in Fig. 9.7,2(a) and (b), we see that the flow is made to conform to the boundary by a succession of straight Mach waves of strength and sign depending on the angles of turn required, and between consecutive Mach waves the flow conditions are constant.

If we now make the elements of straight surfaces and the angles smaller and smaller, then in the limit we have a curved surface and we see, as is illustrated in Fig. 9.7,2(c) and (d), that the surface generates an infinite family of Mach waves, of such a strength as to induce the required curvature of the flow to conform to the boundary. Where the surface is concave the waves are compressive and where the surface is convex the waves are expansive. Again, we see that the Mach waves are straight and along each the flow conditions are constant.

Precisely the same arguments would apply had we considered the flow subject solely to the influence of an upper boundary. In general, therefore, in a supersonic flow subject to the influence of both upper and lower boundaries we find two families of Mach waves, one generated from one

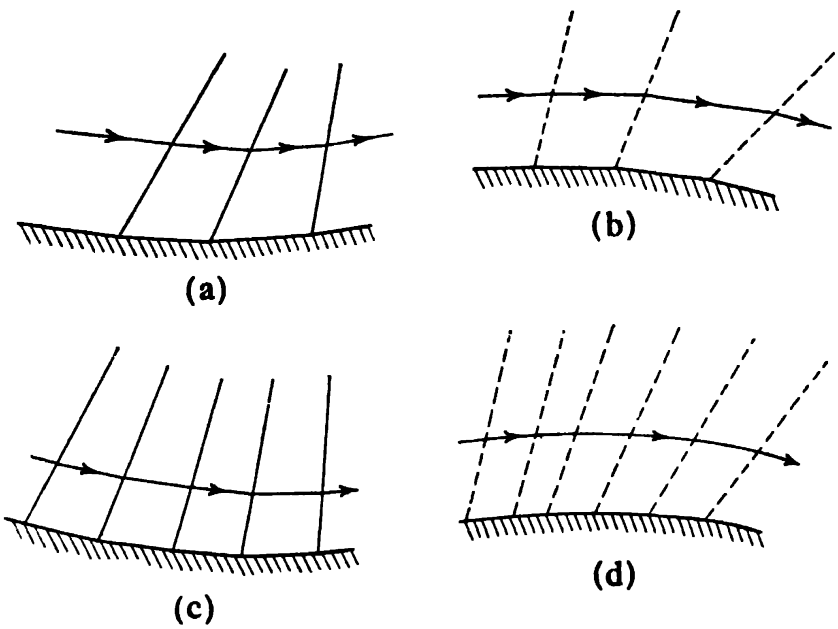

Fig. 9.7,2. Families of Mach waves generated by a lower bounding surface.

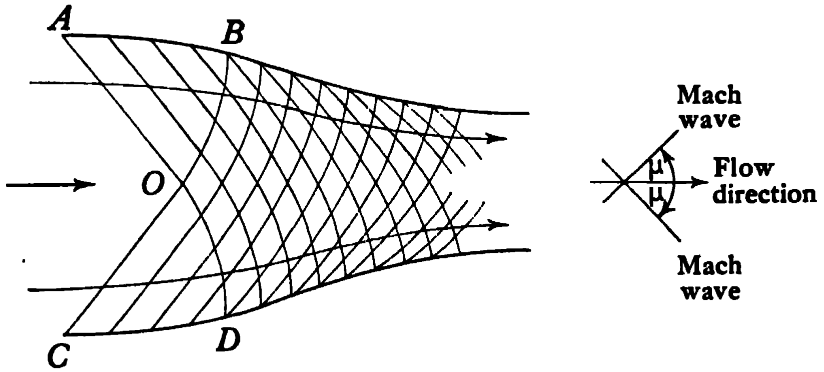

Fig. 9.7,3. Families of Mach waves generated by upper and lower boundaries.

boundary and one from the other. At any point in the flow the stream direction bisects the angle $\left(2\mu = 2\sin^{-1}\dfrac{1}{M}\right)$ between the two Mach waves passing through the point. This is illustrated in Fig. 9.7,3. In this case, however, the flow conditions along a Mach wave of one family change because of the effect of the other family, and so where the two families of waves interact the Mach waves are no longer straight but the flow remains isentropic. However, there are always regions lying between the initial region of uniform flow and the region of influence of both families where the effect of only one family is felt, as for example regions *AOB* and *COD* of Fig. 9.7,3. All such regions of flow where an initially uniform flow is subject to the influence of one boundary only are referred to as *simple wave flows*. As we have seen, a characteristic of such flows is that the Mach waves are straight and flow conditions along a Mach wave are constant.

In a compressive flow the Mach number is decreasing and the Mach angle increasing, so that succeeding Mach waves of a family make increasingly larger angles with the stream direction. Conversely in an expanding flow the succeeding Mach waves of a family make smaller and smaller angles with the stream direction. We find that in consequence there is a tendency for compressive waves of a family to run together and form an envelope (see Fig. 9.7,4). Such an envelope can never occur with expansion waves.

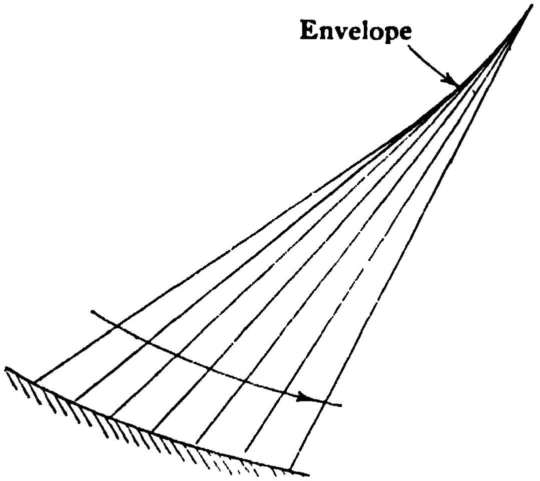

Fig. 9.7,4. Compression waves forming an envelope.

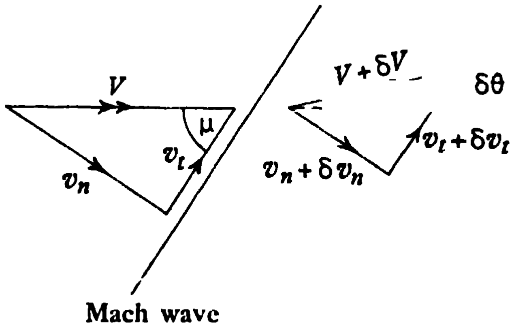

Fig. 9.7,5.

In the region of such an envelope we have the concentration or focusing of the effects of a large number of compressive waves and these build up to form a compression of significant amplitude. Now it is known that a disturbance of finite amplitude moves normal to itself relative to the fluid somewhat faster than a small disturbance, i.e. faster than the speed of sound. Further, it is known that such disturbances tend to change shape until they present a front of very rapid, almost discontinuous compression to the oncoming fluid. Such a front is called a *shock wave*, and we find that in general whenever a supersonic flow is required to compress it does so at least in part by means of a shock wave. We shall have more to say about shock waves later, but we may here note that shock waves and Mach waves provide the total mechanism whereby the boundaries of a supersonic flow impose their influence on the flow.

Let us revert to the simple wave flow. Since flow conditions along any wave are constant and since the flow is isentropic, we can infer that for given initial conditions there must be a unique relation between θ, the flow direction relative to some datum and any one of V, M or p. We now seek that between θ and M. Consider (see Fig. 9.7,5) part of a Mach wave from a lower boundary in a simple wave flow and suppose that the velocity of the approaching flow is V, with components v_n normal to the wave and v_t parallel to it. The Mach wave is taken as stationary relative to the frame of reference. We suppose the flow suffers an infinitesimal change of direction $\delta\theta$ across the wave, and the velocity and its components change to $V + \delta V$, $v_n + \delta v_n$, $v_t + \delta v_t$. Likewise the density before and after the wave is ρ and $\rho + \delta\rho$ respectively. Then continuity of mass flow across the wave requires that

$$\rho v_n = (\rho + \delta\rho)(v_n + \delta v_n). \tag{9.7,1}$$

Conservation of momentum parallel to the wave requires that

$$\rho v_n v_t = (\rho + \delta\rho)(v_n + \delta v_n)(v_t + \delta v_t). \tag{9.7,2}$$

It follows from (9.7,1) and (9.7,2) that

$$\delta v_t = 0,$$

i.e. the velocity component parallel to the wave must remain unchanged. Hence

$$V \cos\mu = (V + \delta V)\cos(\mu - \delta\theta)$$

or

$$\delta V \cot\mu = -V\,\delta\theta,$$

and therefore

$$\delta\theta = -\frac{\delta V}{V}(M^2 - 1)^{\frac{1}{2}}. \tag{9.7,3}$$

Here we are measuring θ as positive if anti-clockwise from the datum direction; with the same convention we would obtain the same relation for a wave from an upper boundary, but with a plus sign on the right hand side. Hence, we can write

$$\delta\theta = \mp\frac{\delta V}{V}(M^2 - 1)^{\frac{1}{2}}, \tag{9.7,4}$$

where the upper sign refers to a Mach wave in a simple wave flow generated from a lower boundary and the lower sign refers to a wave in a simple wave flow generated from an upper boundary. Since, from Bernoulli's law,

$$\delta p = -\rho V\,\delta V$$

we can also write

$$\left.\begin{aligned}\delta\theta &= \pm\frac{\delta p}{\rho V^2}(M^2 - 1)^{\frac{1}{2}}\\ &= \pm\frac{\delta p}{\gamma p M^2}(M^2 - 1)^{\frac{1}{2}}.\end{aligned}\right\} \tag{9.7,5}$$

We now see that compression waves imply a reduction of the component v_n, and hence a refraction of the stream direction away from the normal to a wave, and are therefore associated with a concave curvature of the boundary. Likewise expansion waves cause an increase of the component v_n and are associated with a convex curvature of the boundary.

Further, we have the energy equation in the form

$$\frac{a^2}{\gamma - 1} + \frac{V^2}{2} = \text{const.},$$

and hence

$$\delta V = -\frac{2}{(\gamma - 1)M}\,\delta a.$$

But $M = V/a$, and therefore

$$\delta M = \frac{\delta V}{a} - \frac{V}{a^2}\,\delta a$$

$$= \frac{\delta V}{a}\left[1 + \left(\frac{\gamma - 1}{2}\right)M^2\right].$$

Hence, equation (9.7,4) becomes

$$\delta\theta = \mp\frac{\delta M}{M\left[1 + \left(\frac{\gamma - 1}{2}\right)M^2\right]}(M^2 - 1)^{\frac{1}{2}}. \qquad (9.7,6)$$

This equation can be integrated to give

$$\theta = \mp\vartheta + \text{const.}$$

where ϑ is a function of Mach number given by

$$\left.\begin{aligned} \vartheta &= \left(\frac{\gamma + 1}{\gamma - 1}\right)^{\frac{1}{2}}\tan^{-1}\left\{\left(\frac{\gamma - 1}{\gamma + 1}\right)^{\frac{1}{2}}\cot\mu\right\} + \mu - \frac{\pi}{2} \\ &= \omega + \mu - \frac{\pi}{2}, \\ \text{where}\quad \omega &= \left(\frac{\gamma + 1}{\gamma - 1}\right)^{\frac{1}{2}}\tan^{-1}\left\{\left(\frac{\gamma - 1}{\gamma + 1}\right)^{\frac{1}{2}}\cot\mu\right\}. \end{aligned}\right\} \qquad (9.7,7)$$

It is readily apparent that $\vartheta = 0$ when $M = 1{\cdot}0$ and $\mu = \pi/2$, so that the constant in equation (9.7,7) can be identified as the value of θ where $M = 1{\cdot}0$; we shall denote this by $\theta(1)$. Equation (9.7,7) then tells us that apart from the datum value of $\theta(1)$ the values of θ and M in *all* simple wave flows are uniquely related. Thus, once we have tabulated or plotted ϑ as a function of M, we can readily determine with the aid of the relevant relations given in § 9.4 the flow conditions in any simple wave flow given the boundary shape and initial conditions.

To illustrate the use of equation (9.7,7) consider the following example. An otherwise uniform and unbounded flow is expanded past a smooth cylindrical surface from a Mach number of 1·3 to a Mach number of 1·5. Find the change in flow direction and the ratio of the final static pressure to the initial static pressure.

The answer is as follows. Let suffix 1 and 2 refer to the beginning and end of the expansion, then from equation (9.7,7) the deflection is

$$\theta_2 - \theta_1 = \left(\frac{\gamma + 1}{\gamma - 1}\right)^{\frac{1}{2}}\left\{\tan^{-1}\left[\left(\frac{\gamma - 1}{\gamma + 1}\right)^{\frac{1}{2}}\cot\mu_2\right] - \tan^{-1}\left[\left(\frac{\gamma - 1}{\gamma + 1}\right)^{\frac{1}{2}}\cot\mu_1\right]\right\} + \mu_2 - \mu_1.$$

Now, with $\gamma = 1{\cdot}40$, $\left(\dfrac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}} = 6^{\frac{1}{2}} = 2{\cdot}450$, and with the data given

$$\cot \mu_1 = (1{\cdot}3^2 - 1)^{\frac{1}{2}} = 0{\cdot}831, \quad \text{and hence } \mu_1 = 50{\cdot}27°,$$

$$\cot \mu_2 = (1{\cdot}5^2 - 1)^{\frac{1}{2}} = 1{\cdot}118, \quad \text{and hence } \mu_2 = 41{\cdot}80°.$$

Therefore

$$\tan^{-1}\left[\left(\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{2}} \cot \mu_2\right] = \tan^{-1}(0{\cdot}457) = 24{\cdot}57°$$

and

$$\tan^{-1}\left[\left(\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{2}} \cot \mu_1\right] = \tan^{-1}(0{\cdot}339) = 18{\cdot}75°.$$

Therefore

$$\theta_2 - \theta_1 = 2{\cdot}45 \times 5{\cdot}82° - 8{\cdot}47° = \underline{5{\cdot}78°}.$$

For isentropic flow we have (equation 9.4,14)

$$\frac{p_2}{p_1} = \left[\frac{1 + 0{\cdot}2M_1^2}{1 + 0{\cdot}2M_2^2}\right]^{7/2} = \left(\frac{1{\cdot}338}{1{\cdot}45}\right)^{7/2} = \underline{0{\cdot}754}.$$

It may be noted that the angle ω in equation (9.7,7) is not without physical significance. The student should not have any difficulty in showing that it is the angle between the Mach wave at Mach number M and that at Mach number unity.

Some related values of ϑ, ω, μ and M are given in the following Table:

TABLE 9.7,1.

$\vartheta°$	$\omega°$	$\mu°$	M	$\vartheta°$	$\omega°$	$\mu°$	M
0	0	90	1·000	35·0	99·57	25·43	2·329
1·0	23·43	67·57	1·082	40·0	106·72	23·21	2·538
2·0	30·00	62·00	1·133	45·0	113·79	21·21	2·765
3·0	34·82	58·18	1·177	50·0	120·61	19·39	3·013
4·0	38·80	55·20	1·218	55·0	127·29	17·71	3·287
5·0	42·26	52·74	1·257	60·0	133·84	16·16	3·594
6·0	45·38	50·62	1·294	65·0	140·30	14·70	3·941
7·0	48·25	48·75	1·330	70·0	146·68	13·32	4·339
8·0	50·92	47·08	1·366	75·0	152·98	12·02	4·801
9·0	53·43	45·57	1·401	80·0	159·22	10·78	5·348
10·0	55·82	44·18	1·435	85·0	164·42	9·58	6·006
12·0	60·30	41·70	1·503	90·0	177·68	7·32	7·751
14·0	64·46	39·54	1·571	100·0	183·77	6·23	9·210
16·0	68·39	37·61	1·639	110·0	195·87	4·13	13·875
18·0	72·13	35·87	1·707	120·0	207·90	2·10	27·34
20·0	75·71	34·29	1·775	130·0	219·91	0·09	630·25
25·0	84·15	30·85	1·950	130·45	220·45	0	∞
30·0	92·05	27·95	2·134				

It will be seen from equation (9.7,7) that ϑ attains a maximum when $M = \infty$ and $\mu = 0$, and this maximum is

$$\frac{\pi}{2}\left(\frac{\gamma+1}{\gamma-1}\right)^{\frac{1}{2}} - \frac{\pi}{2} = 130{\cdot}45°.$$

This therefore represents the maximum possible deflection of a stream line in a simple wave flow from its direction when $M = 1{\cdot}0$.

A particular case of a simple wave flow is the famous Prandtl-Meyer solution[1] for the expansion of a supersonic flow round an external corner.

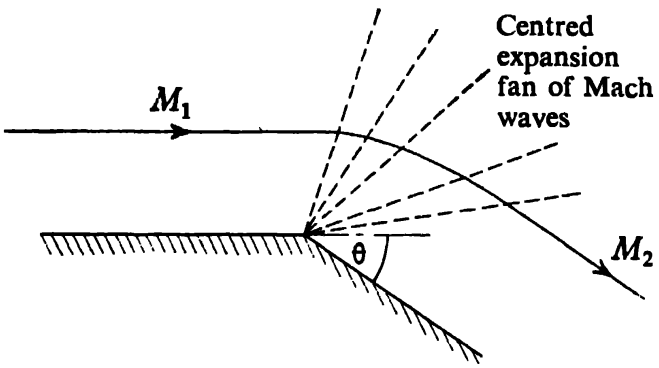

Fig. 9.7,6. Centred expansion round a corner (Prandtl-Meyer flow).

In this case all the Mach waves are centred on the corner forming an expansion fan. As in the general case, they are straight and along each the flow conditions are constant; this can be independently inferred from the fact that the flow has no characteristic length to control its scale and so along each ray through the corner conditions must be constant.† At all points equation (9.7,7) applies.

The analytic simplicity of simple wave flows is especially welcome since they are of considerable practical importance. As we shall see, the flow near either the upper or lower surface of a wing in two dimensional supersonic flow can be assumed in most cases to be influenced solely by that particular surface and in consequence can be regarded as a simple wave flow. This enables us to develop a rapid method of acceptable accuracy for most applications for determining the pressure distribution on wing sections in supersonic flow. This will be discussed more fully in § 9.9.

9.8 Shock Waves

We have already noted that in regions of compression the Mach waves tend to coalesce to form a wave of finite amplitude and such a wave then develops a steep, almost discontinuous front. In crossing this front the flow experiences a rapid increase of pressure and in consequence a rapid rise in density and temperature and a rapid fall in velocity. We call such a front a *shock wave.*

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[1] L. Prandtl, *Phys. Zeits.*, **8**, 23 (1907); Th. Meyer, *Mitt. Forschungsarbeiten*, 62 (1908).

† The Prandtl-Meyer solution was in fact developed long before the general concept of a simple wave flow.

The process of change across a shock wave is usually so rapid and the thickness of the wave is so small that we may for analytic purposes treat it as a discontinuity in the flow. We must note that the shock wave is an irreversible process, involving a rapid conversion of mechanical into heat energy, and hence there is a change of entropy across it. However, the shock involves no addition of mass or total energy and no impulsive force, and hence the principles of conservation of mass, momentum and total energy apply across it. This enables us to determine the flow conditions aft of the shock in terms of conditions in front of it.

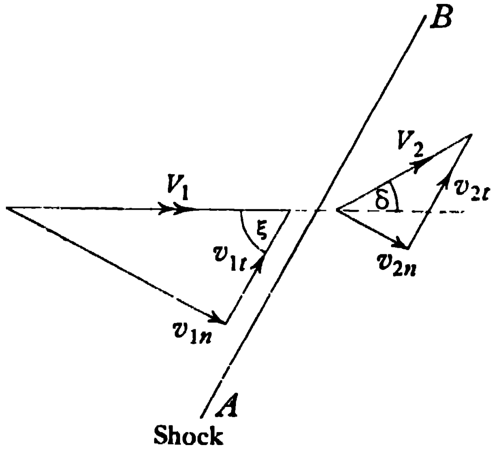


Fig. 9.8,1.

In Fig. 9.8,1 AB represents a portion of shock front taken as stationary relative to the frame of reference and V_1 represents the incident velocity and V_2 the velocity after the shock. V_1 has velocity components v_{1n} and v_{1t}, normal and parallel to the shock front, respectively, and the corresponding components of V_2 are v_{2n} and v_{2t}. The angle between V_1 and the shock front is ξ and the deflection of the flow by the shock is δ. In general we will use suffix 1 to denote quantities in front of the shock and suffix 2 to denote quantities aft of the shock.

Conservation of mass flow across the shock gives

$$\rho_1 v_{1n} = \rho_2 v_{2n}. \qquad (9.8,1)$$

Conservation of momentum normal and parallel to the shock front yields

$$p_1 + \rho_1 v_{1n}^2 = p_2 + \rho_2 v_{2n}^2 \qquad (9.8,2)$$

and

$$\rho_1 v_{1n} v_{1t} = \rho_2 v_{2n} v_{2t}. \qquad (9.8,3)$$

Conservation of total energy across the shock gives

$$\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \tfrac{1}{2}(v_{1n}^2 + v_{1t}^2) = \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \tfrac{1}{2}(v_{2n}^2 + v_{2t}^2)$$

$$= c_p T_0 = \frac{(\gamma+1)}{2(\gamma-1)} V^{*2}, \qquad (9.8,4)$$

where T_0 is the reservoir temperature and V^* is the critical velocity. We note that both T_0 and V^* are constant across the shock. From equations (9.8,1) and (9.8,3) we see that

$$v_{1t} = v_{2t} = v_t, \text{ say,} \qquad (9.8,5)$$

and hence any velocity change introduced by the shock is solely in the component normal to it; the transverse component is unchanged. Equation (9.8,4) therefore becomes

$$\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1}+\tfrac{1}{2}v_{1n}^2=\frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2}+\tfrac{1}{2}v_{2n}^2=\frac{(\gamma+1)}{2(\gamma-1)}V^{*2}-\frac{v_t^2}{2}=\frac{\gamma+1}{2(\gamma-1)}v_n{}^{*2}, \tag{9.8,6}$$

where we may interpret $v_n{}^*$ as the normal component of velocity when this component is just equal to the local speed of sound.

Divide equation (9.8,2) by (9.8,1) and we get

$$\frac{p_1}{\rho_1 v_{1n}}+v_{1n}=\frac{p_2}{\rho_2 v_{2n}}+v_{2n}=B\text{, say,} \tag{9.8,7}$$

and substituting for p_1/ρ_1 and p_2/ρ_2 in (9.8,6) we can reduce this equation to

$$B(v_{1n}-v_{2n})=\frac{\gamma+1}{2\gamma}(v_{1n}^2-v_{2n}^2).$$

Therefore either $v_{1n}=v_{2n}$, and there is no discontinuity, or

$$B=\frac{\gamma+1}{2\gamma}(v_{1n}+v_{2n}). \tag{9.8,8}$$

Equations (9.8,6), (9.8,7) and (9.8,8) then yield

$$v_n{}^{*2}=v_{1n}v_{2n}\,. \tag{9.8,9}$$

We can readily deduce from this important relation that if the *normal component* of the Mach number is greater than unity on one side of the shock, then it must be less than unity on the other. We may expect that the strength of the shock is related to the deviation of the normal component of Mach number from unity.

From equations (9.8,1) and (9.8,9) it follows that

$$\rho_2/\rho_1=v_{1n}^2/v_n{}^{*2}, \tag{9.8,10}$$

and from equation (9.8,2) we have

$$p_2=p_1+\rho_1[v_{1n}^2-v_n{}^{*2}]. \tag{9.8,11}$$

These last two relations yield

$$(p_2-p_1)/(\rho_2-\rho_1)=v_n{}^{*2} \tag{9.8,12}$$

which can be regarded as a generalised form of the small disturbance formula $dp/d\rho=a^2$ (see § 10.3).

From equations (9.8,6), (9.8,10) and (9.8,11) we can determine the values of V_2, p_2 and ρ_2 aft of the shock as well as the flow deflection angle δ, given the flow conditions in front of the shock and the shock wave inclination angle ξ. How we express the resulting relations between quantities in front of and behind the shock is a matter of choice and convenience.

The following formulae summarise the relations in terms of M_1 and ξ; the student should have no difficulty in deriving the relations for himself.

$$\frac{p_2}{p_1} = \frac{1}{(\gamma + 1)}[2\gamma M_1^2 \sin^2 \xi - (\gamma - 1)], \tag{9.8,13}$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_1^2 \sin^2 \xi}{(\gamma - 1)M_1^2 \sin^2 \xi + 2}, \tag{9.8,14}$$

$$\tan \delta = \frac{2[M_1^2 \sin^2 \xi - 1]}{\tan \xi[2 + M_1^2(\gamma + \cos 2\xi)]}, \tag{9.8,15}$$

$$M_2^2 = \frac{1 + \frac{\gamma - 1}{2} M_1^2}{\gamma M_1^2 \sin^2 \xi - \frac{\gamma - 1}{2}} + \frac{M_1^2 \cos^2 \xi}{1 + \frac{\gamma - 1}{2} M_1^2 \sin^2 \xi}, \tag{9.8,16}$$

$$\frac{V_2}{V_1} = \left\{\cos^2 \xi + \sin^2 \xi \left[\frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1)M_1^2 \sin^2 \xi}\right]^2\right\}^{\frac{1}{2}}. \tag{9.8,17}$$

$M_1 \sin \xi$ is the normal component of Mach number in front of the shock and therefore we anticipate that $(M_1 \sin \xi - 1)$ is related to the strength of the shock. Indeed it follows from (9.8,13) and (9.8,14) that

$$\frac{p_2 - p_1}{p_1} = \frac{2\gamma}{(\gamma + 1)}[M_1^2 \sin^2 \xi - 1]$$

and

$$\frac{\rho_2 - \rho_1}{\rho_1} = \frac{2(M_1^2 \sin^2 \xi - 1)}{(\gamma - 1)M_1^2 \sin^2 \xi + 2}.$$

Hence, we may define $(M_1^2 \sin^2 \xi - 1)$ as the strength of the shock.

If we now consider the ratio of reservoir pressures p_{01}/p_{02}, we find that it increases steadily with $M_1 \sin \xi$ and for small enough values of $(M_1^2 \sin^2 \xi - 1)$ it can be expanded in the form

$$\frac{p_{01}}{p_{02}} = 1 + \frac{2\gamma}{(\gamma + 1)^2} \frac{(M_1^2 \sin^2 \xi - 1)^3}{3} + \dots. \tag{9.8,18}$$

We have seen in § 9.5 that in adiabatic flow the entropy increases if the reservoir pressure falls and consequently in all natural, irreversible processes the reservoir pressure must fall. This implies that for the discontinuity to be possible

$$M_1 \sin \xi > 1$$

and therefore from equation (9.8,9) $v_{1n} > v_n^*$ and $v_{2n} < v_n^*$. It follows that the component of Mach number normal to the shock must be greater than unity in front of the shock and less than unity after it; they are both equal to unity only in the case of an infinitesimally weak shock, i.e. a compression Mach wave.

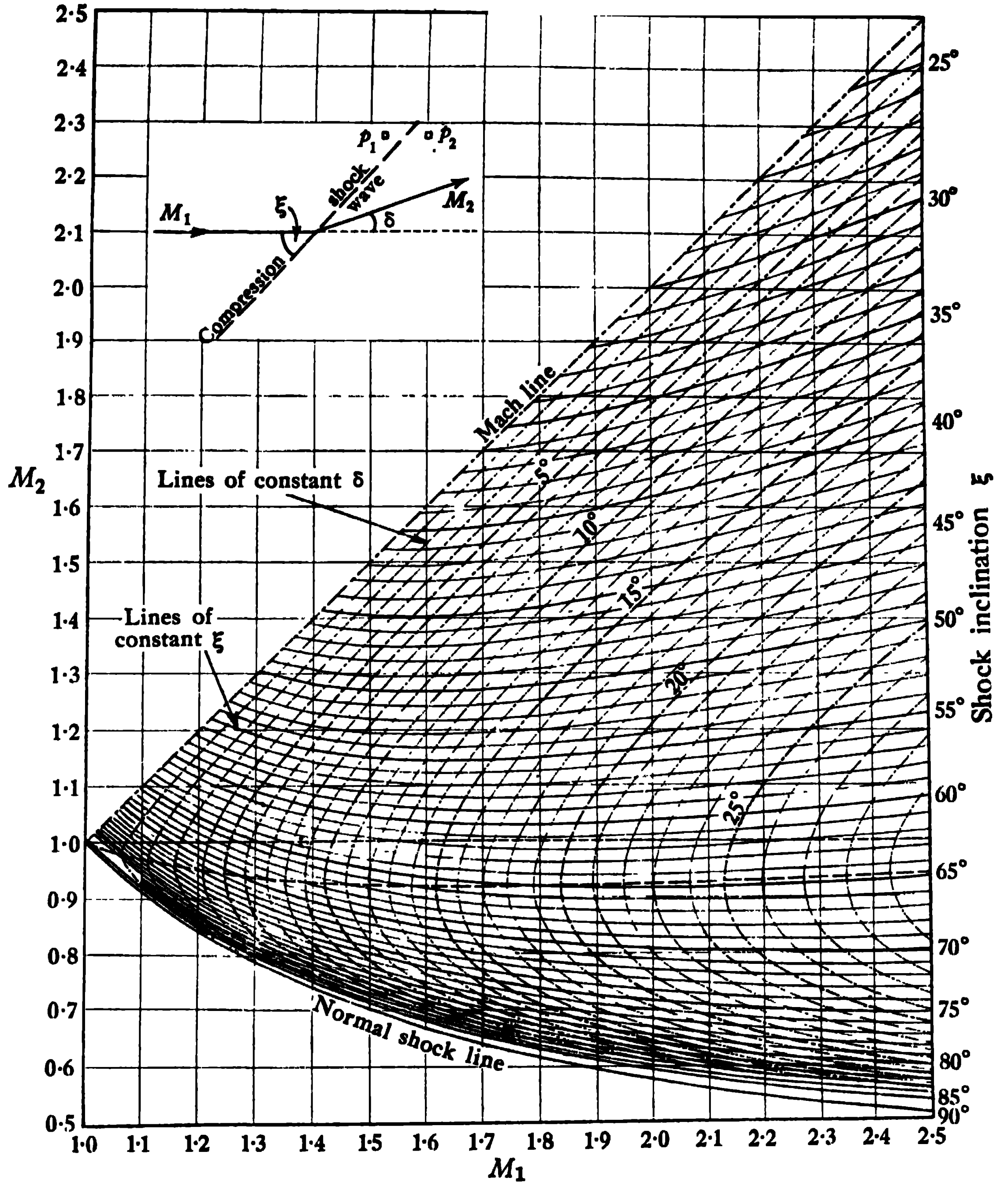


Fig. 9.8,2. M_2 as a function of M_1 for various values of δ and ξ.

We may note that the entropy change across the shock may likewise be expanded in terms of $(M_1{}^2 \sin^2 \xi - 1)$ to give

$$\Delta s = 2c_p \frac{(\gamma - 1)}{(\gamma + 1)^2} \frac{(M_1{}^2 \sin^2 \xi - 1)^3}{3} + \ldots .$$

Thus, if the shock strength is such that we may ignore terms of order $(M_1{}^2 \sin^2 \xi - 1)^3$ or $\left(\frac{p_2 - p_1}{p_1}\right)^3$, then the flow across the shock can be regarded as isentropic.

Figs. 9.8,2 and 9.8,3 present in graphical form M_2 and p_2/p_1 as functions

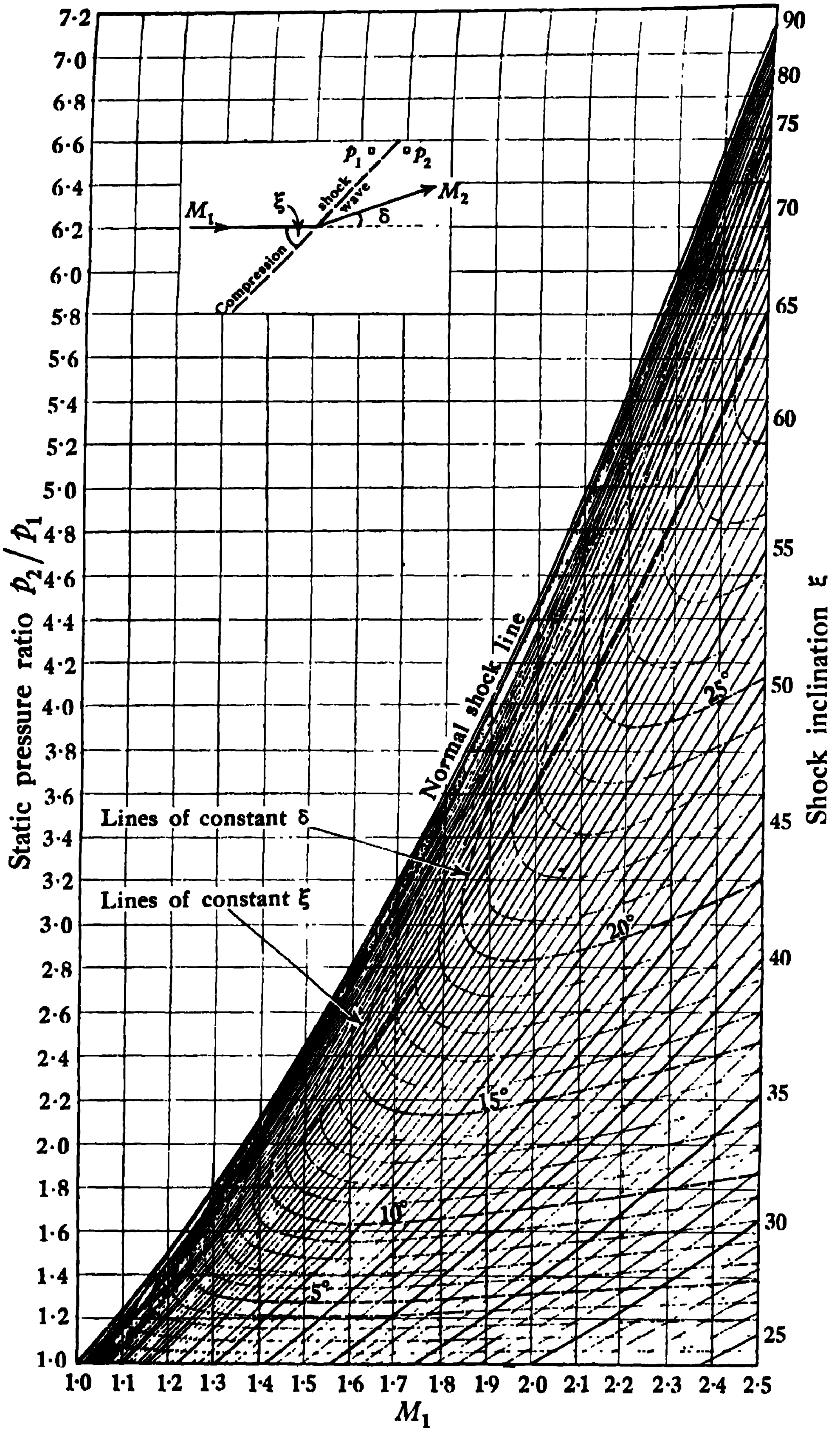


Fig. 9.8,3. Static pressure ratio p_2/p_1 as a function of M_1 for various values of δ and ξ.

of M_1 for various values of ξ and δ. It will be apparent that for any value of M_1 and a given deflection δ less than a certain critical value depending on M_1 there are two possible values of M_2, one corresponding to a weak shock with M_2 usually greater than unity and one corresponding to a strong shock with M_2 less than unity. If we consider the flow past a wedge of semi-angle δ at zero incidence in an incident stream of Mach number M_1, then we find that in general the flow is turned parallel to the sides of the wedge by means of shock waves springing from the wedge apex, as illustrated in Fig. 9.8,4.

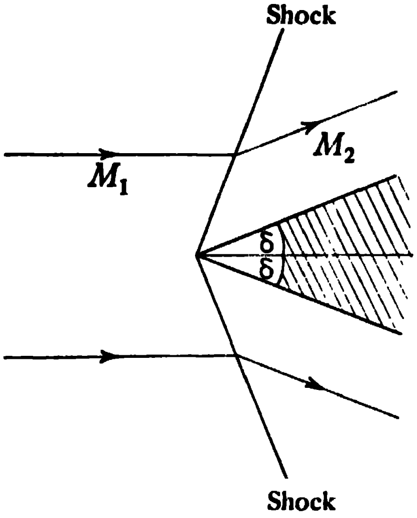
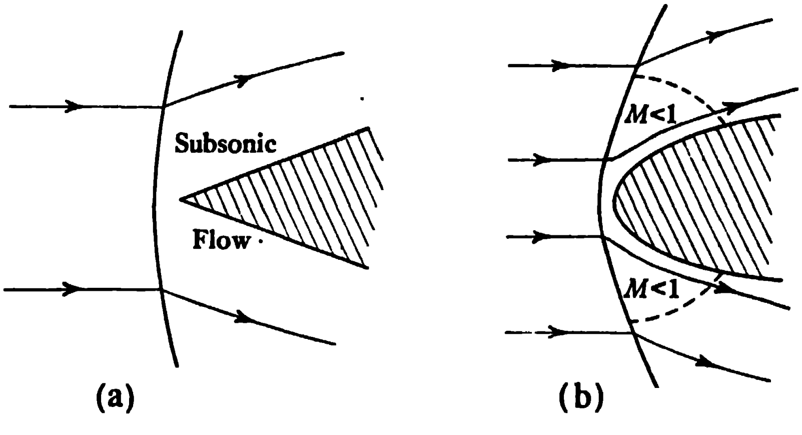


Fig. 9.8,4. Attached shocks at nose of wedge.



(a) (b)

Fig. 9.8,5. Bow shocks in front of (a) wedge, (b) round-nosed section.

In such a case, however, of the two possible shocks that give the required deflection δ the shock that occurs is the weak one. If δ is larger than the critical value for the incident Mach number M_1, then a *bow shock* forms ahead of the wedge nose as illustrated in Fig. 9.8,5(a). Behind the bow shock, over the central portion, the flow will be subsonic.

A plot of the critical value of δ as a function of M_1 is shown in Fig. 9.8,6. The greatest possible deflection occurs for $M_1 = \infty$ and then $\delta_{\max} = 46°$.

Hence any section demanding a greater deflection angle than this cannot have an attached shock at any Mach number. In particular, a round-nosed section must always have a detached or bow shock ahead of it at supersonic speeds, as illustrated in Fig. 9.8,5(b).

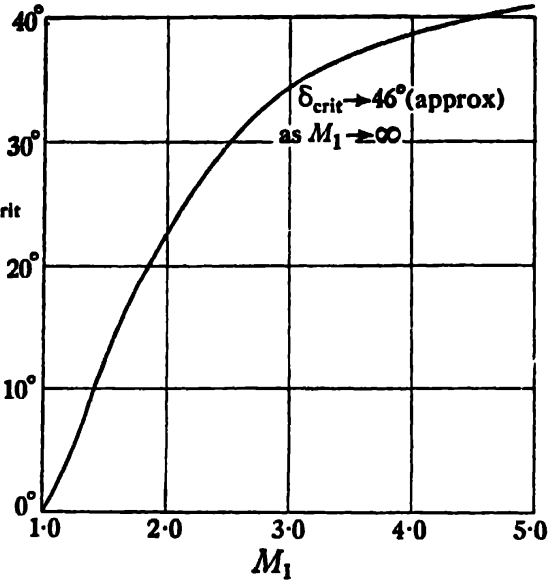


Fig. 9.8,6. Maximum deflection caused by shock as function of Mach number ahead of shock.

9.9 The Shock-Expansion Method

If one considers the typical flow past a sharp-nosed wing section at supersonic speeds it is as illustrated in Fig. 9.8,7. At the nose or leading edge there are normally attached shocks of strength and inclination to turn the incident flow parallel to the upper and lower surface there. This is only true if the required deflection for each surface is less than the critical for the incident Mach number, otherwise the shocks become detached. On the other hand, if the section incidence is large enough for the deflection of the incident stream over the upper surface to require an expansion round the leading edge, then it occurs there by means of a centred expansion (see § 9.7 and inset to Fig. 9.8,7). For attached shocks or centred expansions we can readily determine the flow conditions immediately aft of the leading edge on both upper and lower surfaces, given the incident flow conditions and the required deflections. These can be most easily obtained from tables and graphs for plane shock waves and simple wave flows. Aft of the leading edge each surface generates a family of Mach waves; these are expansion waves if the surfaces are convex. At the trailing edge there are normally two more shock waves whereby the flow is brought back to a pressure and direction approximating to undisturbed stream values. At a high enough incidence, however, a centred expansion may occur on the lower surface there intead of a shock to provide the necessary flow deflection.

The Mach waves generated from the surfaces will eventually meet and interact with the leading and trailing edge shocks, causing the latter to weaken and curve; far from the wing the shocks tend to become Mach waves and take up the Mach angle to the incident stream. The family of Mach waves from a wing surface will reflect from the leading edge shock with which it interacts to produce a reflected family of Mach waves which will return to the surface. But this reflected family is very weak and its effect near the surface can for most practical purposes be disregarded. It follows that between the front and rear shocks the flow near either surface of the wing is subject solely to the influence of the family of Mach waves from that surface, and hence the flow can be treated as a simple wave flow. Since we can determine the flow conditions at the leading edge just aft of the leading edge shock (or expansion) we can readily determine them at all points along the surface back to the trailing edge from simple wave flow tables, given the wing geometry. From the pressure distribution thus obtained the aerodynamic characteristics of the wing (lift, drag, pitching moment, etc.) readily follow.

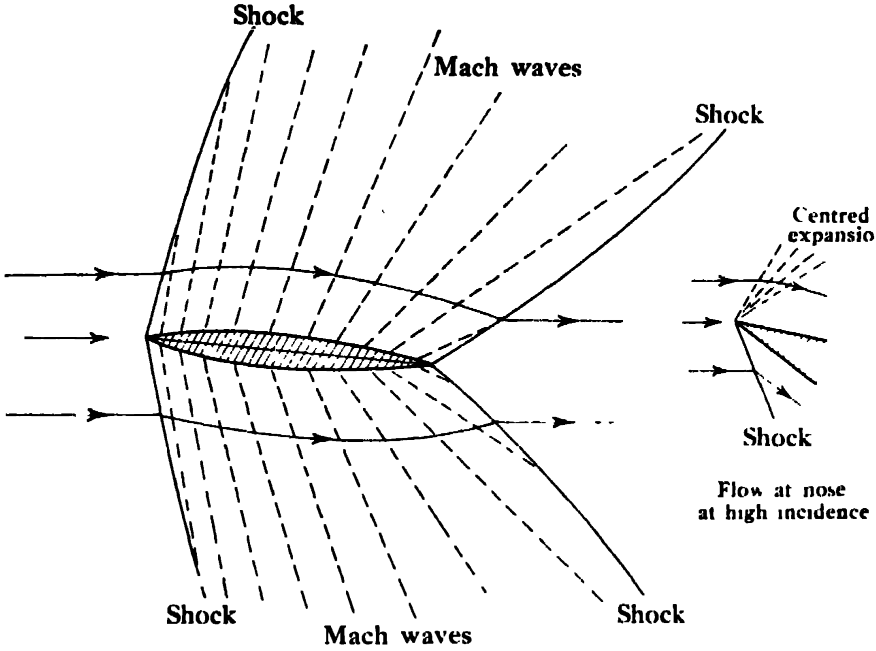


Fig. 9.8,7. Typical pattern of shocks and Mach waves for supersonic flow past a sharp-nosed section.

Two points should be noted. For the purely supersonic flow considered, the flow over the upper surface of the section is independent of the shape of the lower surface and vice versa. This is sometimes referred to as the Principle of Independence. Secondly, the method assumes inviscid flow and, as described above, neglects the effect of boundary layers in modifying the flow pattern. Usually these effects are small and can be readily estimated, but shock wave and boundary layer interaction effects can sometimes be of considerable importance particularly if they result in flow separation.

The following is a simple example illustrating the application of the method.

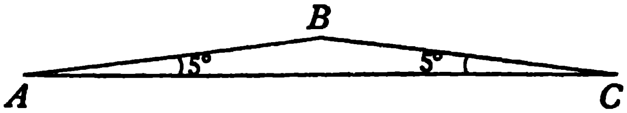


Fig. 9.8,8.

In the sketch ABC represents a wing section. Find the pressure distribution on the section and its lift coefficient at a Mach number of 2·0 and an incidence of 7°, where AC is taken as the datum chord line. At $M_1 = 2{\cdot}0$ the following values of deflection angle (δ) and pressure ratio (p_2/p_1) apply across oblique shocks:—

δ	p_2/p_1	δ	p_2/p_1
0	1·000	7°48′	1·524
1°15′	1·072	10°36′	1·762
4°40′	1·293		

From Fig. 9.8,9 we see that at A we have a shock on the lower surface turning the flow through 7°, and on the upper surface there we have a centred expansion turning the flow through 2°. At B we have a second centred expansion turning the flow through a further 10°.

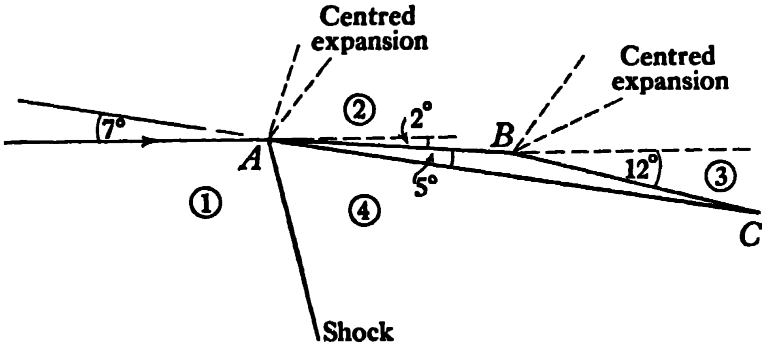


Fig. 9.8,9.

Let suffices 1, 2, 3 and 4 refer to the regions as shown and U_1 be the undisturbed stream velocity. From the shock data given we find by graphical interpolation that for a 7° deflection by a shock with $M_1 = 2{\cdot}0$

$$p_4/p_1 = 1{\cdot}476.$$

Hence, on the lower surface

$$C_{p_4} = \frac{p_4 - p_1}{\frac{1}{2}\rho_1 U_1^2} = \frac{0{\cdot}476 p_1}{\frac{1}{2}\rho_1 U_1^2} = \frac{0{\cdot}476 \times 2}{\gamma M_1^2} = \underline{0{\cdot}170}.$$

Also from the simple wave flow table we find that, for a 2° expansive deflection from $M_1 = 2{\cdot}0$, $M_2 = 2{\cdot}076$, and a further 10° expansive deflection leads to $M_2 = 2{\cdot}473$.

But in an isentropic expansion with $\gamma = 1{\cdot}40$ we have

$$p_1/p = [(5 + M^2)/(5 + M_1^2)]^{7/2},$$

and hence

$$p_1/p_2 = 1{\cdot}121, \quad \text{or} \quad p_2/p_1 = 0{\cdot}892.$$

Therefore, $$C_{p_2} = -0{\cdot}108 \times \frac{2}{\gamma M_1^2} = \underline{-0{\cdot}039}.$$

Similarly, $$p_1/p_3 = 2{\cdot}018, \quad \text{or} \quad p_3/p_1 = 0{\cdot}495,$$

and hence $$C_{p_3} = -0{\cdot}505 \times \frac{2}{\gamma M_1^2} = \underline{-0{\cdot}180}.$$

Therefore $$\begin{aligned} C_L &= 0{\cdot}170 \cos 7° + \tfrac{1}{2}[0{\cdot}039 \cos 2° + 0{\cdot}180 \cos 12°] \\ &= 0{\cdot}169 + 0{\cdot}020 + 0{\cdot}088 = \underline{0{\cdot}278}. \end{aligned}$$

9.10 The Total Pressure Registered by a Pitot Tube in Compressible Flow

In subsonic flow the total pressure registered by a pitot tube aligned with the stream is the same as the local reservoir pressure. Thus, if H denotes the pitot pressure, p_0 the reservoir pressure, p_1 the static pressure and U_1 the undisturbed stream velocity, then from § 9.4 we find

$$\frac{H}{p_1} = \frac{p_0}{p_1} = \left[1 + \left(\frac{\gamma - 1}{2}\right)M_1^2\right]^{\gamma/(\gamma-1)} = \left[1 + \frac{M_1^2}{5}\right]^{7/2}, \quad \text{for } \gamma = 1{\cdot}4. \tag{9.10,1}$$

The non-dimensional 'dynamic' pressure is then given by

$$q \equiv \frac{H - p_1}{\frac{1}{2}\rho_1 U_1^2} = \frac{2}{\gamma M_1^2}\left\{\left[1 + \left(\frac{\gamma - 1}{2}\right)M_1^2\right]^{\gamma/(\gamma-1)} - 1\right\}. \tag{9.10,2}$$

For values of M_1 less than $\left[\dfrac{2}{(\gamma - 1)}\right]^{\frac{1}{2}}$ we can expand the R.H.S. thus:

$$\left.\begin{aligned} q &= 1 + \tfrac{1}{4}M_1^2 + \left(\frac{2 - \gamma}{24}\right)M_1^4 + \frac{(2 - \gamma)(3 - 2\gamma)}{142} M_1^6 + \dots \\ &= 1 + \tfrac{1}{4}M_1^2 + \frac{1}{40} M_1^4 + \frac{1}{1600} M_1^6 + \dots \quad \text{for } \gamma = 1{\cdot}4. \end{aligned}\right\} \tag{9.10,3}$$

Hence in subsonic flow the pressure difference $H - p_1$ is greater than $\frac{1}{2}\rho_1 U_1^2$ by the factor $1 + \frac{1}{4}M_1^2 + \frac{1}{40} M_1^4 + \dots$

In supersonic flow there is a shock wave ahead of the pitot tube and it is assumed that immediately in line with the tube the shock wave is normal to the flow. Hence H is equal to the reservoir pressure behind the shock (p_{02}).

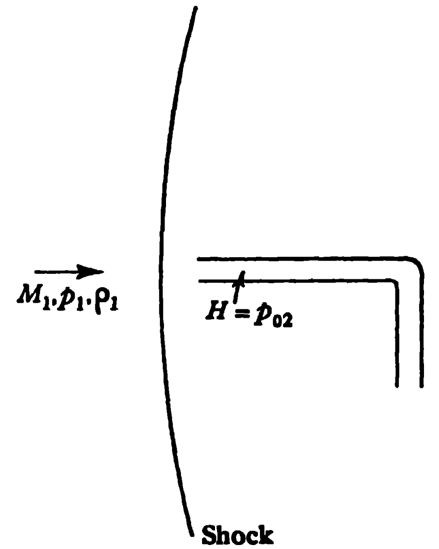


Fig. 9.10,1.

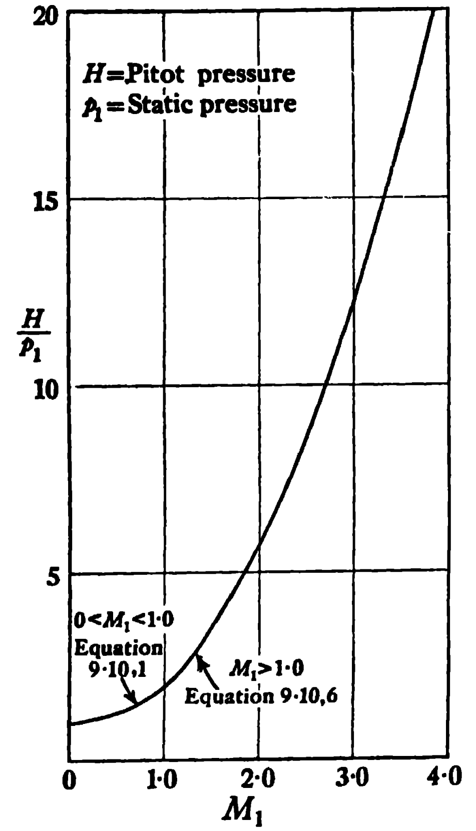


Fig. 9.10,2. Variation of pitot pressure with Mach number

From the shock wave relations of § 9.8 we find that the ratio of reservoir pressures across a normal shock wave is

$$\frac{p_{01}}{p_{02}} = \left[\frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1}\right]^{1/(\gamma-1)}\left[\frac{(\gamma-1)M_1^2+2}{(\gamma+1)M_1^2}\right]^{\gamma/(\gamma-1)}. \qquad (9.10,4)$$

But if p_1 is the static pressure just ahead of the shock

$$\frac{p_1}{p_{01}} = \left[1 + M_1^2\left(\frac{\gamma-1}{2}\right)\right]^{-[\gamma/(\gamma-1)]}. \qquad (9.10,5)$$

Hence

$$\left.\begin{aligned}\frac{p_{02}}{p_1} = \frac{H}{p_1} &= \left[\frac{\gamma+1}{2}M_1^2\right]^{\gamma/(\gamma-1)} \Bigg/ \left[\frac{2\gamma}{(\gamma+1)}M_1^2 - \frac{(\gamma-1)}{(\gamma+1)}\right]^{1/(\gamma-1)} \\ &= 167M_1^7/(7M_1^2-1)^{5/2}, \quad \text{when } \gamma = 1{\cdot}4.\end{aligned}\right\} \qquad (9.10,6)$$

This may be compared with the corresponding relation for subsonic flow given in equation (9.10,1). We can plot, as in Fig. 9.10,2, H/p_1 as a function of M_1, using equation (9.10,1) for the subsonic range and (9.10,6) for the supersonic range. We find that the two functions and their first and second derivatives with respect to M_1^2 are continuous at $M_1 = 1{\cdot}0$.

9.11 Linearised Perturbation Potential Function Theory*

9.11.1 General Relations

In § 9.6 we derived the general equation for the potential function in irrotational isentropic flow. We remarked that the equation is too complex to permit of general solution, and further approximations and simplifications are necessary before its solution can be attempted. Since in many practical instances we are concerned with the steady flow past a thin wing or body at small incidence an acceptable assumption is that the disturbances introduced by the body can be regarded as small. More specifically we assume that the velocity components in the field of the body can be written $(U_1 + u, v, w)$ where U_1 is the undisturbed stream velocity with the x axis taken in the direction of U_1, and u/U_1, v/U_1, w/U_1 are all small quantities such that their squares and products and higher order terms can be neglected compared with first order terms. Thus, we say that $\phi_x + U_1$, ϕ_y, ϕ_z, are all small compared with U_1, as are the derivatives of these quantities with respect to x, y and z compared with U_1/c, where c is a typical linear dimension of the body.

Equation (9.6,4) then reduces to

$$\phi_{xx}\left[1 - \frac{\phi_x^2}{a^2}\right] + \phi_{yy} + \phi_{zz} = 0,$$

and from equation (9.6,5) we have

$$a^2 = a_1^2 - U_1 u(\gamma - 1)$$

so that the equation for ϕ becomes

$$\phi_{xx}\left\{1 - \frac{M_1^2\left(1 + 2\dfrac{u}{U_1}\right)}{\left[1 - M_1^2\dfrac{u}{U_1}(\gamma - 1)\right]}\right\} + \phi_{yy} + \phi_{zz} = 0.$$

We see that, provided (a), $\dfrac{u}{U_1}$ is small compared with $(1 - M_1^2)$ and (b), $M_1^2\dfrac{u}{U_1}$ is small compared with unity, the potential function equation can be written

$$(1 - M_1^2)\phi_{xx} + \phi_{yy} + \phi_{zz} = 0. \qquad (9.11,1)$$

Proviso (a) implies that equation (9.11,1) becomes invalid at Mach numbers near unity, i.e. transonic Mach numbers, whilst proviso (b) implies that it ceases to apply at high Mach numbers, i.e. hypersonic speeds, the limits of validity depending on the average magnitude of the disturbance velocity ratio u/U_1 and therefore on the geometry of the body.†

It will be seen that equation 9.11,1 has the welcome simplification of being linear and it is referred to as the linearised potential function equation. From now on we shall treat ϕ as the *perturbation potential function,* i.e.

$$-\frac{\partial\phi}{\partial x}=u, \quad -\frac{\partial\phi}{\partial y}=v, \quad -\frac{\partial\phi}{\partial z}=w. \qquad (9.11,2)$$

To the order of approximation specified we readily find that

$$M^2=M_1^2\left[1+\frac{2u}{U_1}\left(1+\frac{\gamma-1}{2}M_1^2\right)\right] \qquad (9.11,3)$$

and hence from equations (9.4,14)–(9.4,16) we have

$$T/T_1=1-M_1^2(\gamma-1)u/U_1, \qquad (9.11,4)$$

$$p/p_1=1-\gamma M_1^2u/U_1, \qquad (9.11,5)$$

and

$$\rho/\rho_1=1-M_1^2u/U_1. \qquad (9.11,6)$$

It follows that

$$C_p=(p-p_1)/\tfrac{1}{2}\rho_1U_1^2=2\left(\frac{p}{p_1}-1\right)\Big/\gamma M_1^2$$

$$=-2u/U_1. \qquad (9.11,7)$$

9.11.2 Subsonic Flow

For subsonic flow we have $M_1 < 1{\cdot}0$ and we write equation (9.11,1) in the form

$$\left.\begin{aligned}&\phi_{xx}+\phi_{yy}+\phi_{zz}=0\\ &\quad\text{where}\quad \beta^2=(1-M_1^2).\end{aligned}\right\} \qquad (9.11,8)$$

The close similarity of this equation to the Laplace equation for incompressible flow ($\beta = 1{\cdot}0$) suggests that a simple transformation should be possible from a problem in compressible flow to a related problem in incompressible flow for which the solution can be presumed known.[1]

To illustrate this consider the equation in two dimensions (x, z),

$$\beta^2\phi_{xx}+\phi_{zz}=0 \qquad (9.11,9)$$

and the related incompressible flow equation

$$\phi_{xx}+\phi_{zz}=0. \qquad (9.11,10)$$

Then we see that if

$$\phi_I=-f(x,z) \qquad (9.11,11)$$

satisfies equation (9.11,10)

$$\phi_C=-f(x,\beta z)/\beta^n \qquad (9.11,12)$$

satisfies (9.11,9), where n is any number.

---

[1] S. Goldstein and A. D. Young, *R. & M.* No. 1909 (1943).

Let us suppose that ϕ_I is the perturbation potential function for some wing section I in incompressible flow with the undisturbed stream velocity U_1. Then the boundary condition on I requires that

$$(dz/dx)_{SI} = w/(U_1 + u) = w/U_1, \text{ to the order of accuracy of the theory,}$$
$$= f_z(x, z)_{SI}/U_1.$$

Here suffix S denotes values at the surface of the section.

But $$f_z(x, z)_S = f_z(x, \pm 0) + z_S f_{zz}(x, \pm 0) + \dots$$

where the + sign refers to the upper surface and the − sign to the lower surface. Since f and its derivatives are all small quantities as is z_S, being the section ordinate, it follows that to the order of accuracy of the theory we can write

$$f_z(x, z)_S = f_z(x, \pm 0).$$

Hence we have

$$(dz/dx)_{SI} = f_z(x, \pm 0)/U_1. \tag{9.11,13}$$

It follows similarly that if ϕ_C, as defined in equation (9.11,12), is the perturbation potential function for some wing section C in compressible flow with the undisturbed stream velocity U then

$$(dz/dx)_{SC} = f_z(x, \pm 0)/(U_1\beta^{n-1})$$
$$= \frac{(dz/dx)_{SI}}{\beta^{n-1}}. \tag{9.11,14}$$

Hence we infer that at corresponding values of x the ordinates of wing C are $1/\beta^{n-1}$ those of wing I. Thus, to solve for the flow past a given section C at Mach number M_1, multiply its ordinates (and incidence) in the ratio $\beta^{n-1} : 1$ to obtain wing I. Determine the perturbation potential function for wing I in the form $-f(x, z)$, then that for wing C is $-f(x, \beta z)/\beta^n$.

The pressure coefficient on wing C is from equation (9.11,7)

$$C_{pC} = -(2u/U_1)_S = \frac{2f_x(x, 0)/U_1}{\beta^n} = \frac{C_{pI}}{\beta^n}, \tag{9.11,15}$$

where the two pressure coefficients are at the same value of x.

If we take $n = 1$, then we see that wings I and C are identical, and it follows from equation (9.11,15) that

$$C_{pC} = C_{pi}/\beta \tag{9.11,16}$$

where C_{pi} is the pressure coefficient on wing C in incompressible flow. This important relation is known as the *Prandtl-Glauert Law*. The same relation follows, as indeed it must, if other values of n are chosen, since linearised theory applied to incompressible flow yields the result that the pressure coefficient on a wing section is proportional to its thickness.

From 9.11,16 we deduce that at a given incidence the lift coefficient and pitching moment coefficient of a wing section in compressible flow are similarly related to their values for the same wing section in incompressible flow by

$$C_{LC} = C_{Li}/\beta, \tag{9.11,17}$$

and

$$C_{MC} = C_{Mi}/\beta. \tag{9.11.18}$$

These relations are found to be in reasonable agreement with experiment for thin wings and small incidences, i.e., where the basic assumptions of the theory are not contravened. We note that, since $\beta \to 0$ as $M_1 \to 1{\cdot}0$, the results must clearly become invalid for values of M_1 close to unity.

We can deduce nothing about the variation of the boundary layer drag from the Prandtl-Glauert relation since this relation does not apply to viscous flow effects. Experiment and calculations based on boundary layer theory[1] both agree in demonstrating small but not always negligible effects of Mach number on boundary layer drag coefficient provided the flow is everywhere subsonic. For higher Mach numbers in the transonic range large effects can be evident (see § 9.13).

The general analysis for three dimensional wings and bodies follows similar lines but will not be reproduced here.[2] The results can, however, be quoted. The perturbation velocity potential ϕ_C for any wing, fuselage or wing-fuselage combination C to which linearised theory is applicable is found by first reducing its lateral (y and z) ordinates in the ratio $\beta : 1$ to obtain body I. If

$$\phi_I = -f(x, y, z) \tag{9.11,19}$$

is the perturbation velocity for body I in incompressible flow, then that for body C in compressible flow is

$$\phi_C = -\frac{1}{\beta^2} f(x, \beta y, \beta z). \tag{9.11,20}$$

At corresponding points on bodies C and I the pressure coefficients are related by

$$C_{pC} = C_{pI}/\beta^2. \tag{9.11,21}$$

We see that in this case only one transformation is valid. However, for a thin wing with span large compared with the thickness, the analysis permits of some relaxation of the boundary conditions leading again to an infinite number of possible transformations as in the case of two dimensions. The most useful of these transformations is that for $n = 1$ and yields the result that we obtain wing I from wing C by simply reducing the spanwise (y) ordinates in the ratio $\beta : 1$, leaving the other ordinates unchanged. Then, if $\phi_I = -f(x, y, z)$ is the perturbation potential function for wing I in incompressible flow,

$$\phi_C = -\frac{1}{\beta} f(x, \beta y, \beta z) \tag{9.11,22}$$

and

$$C_{pC} = C_{pI}/\beta. \tag{9.11,23}$$

9.11.3 Supersonic Flow

For supersonic flow equation (9.11,1) can be written

$$\left.\begin{aligned} B^2\phi_{xx} - \phi_{yy} - \phi_{zz} &= 0 \\ \text{where} \qquad B^2 &= M_1{}^2 - 1. \end{aligned}\right\} \tag{9.11,24}$$

In two dimensions (x, z) this equation becomes

$$B^2\phi_{xx} - \phi_{zz} = 0. \tag{9.11,25}$$

This is a simple form of wave equation and its general solution is readily verified to be

$$\phi = f_1(x - zB) + f_2(x + zB) \tag{9.11,26}$$

where f_1 and f_2 can be any functions of $x - zB$ and $x + zB$ respectively, their form being dependent on the boundary conditions. The lines $x - zB = \text{const.}$ and $x + zB = \text{const}$, are in fact the lines of action of the two families of Mach waves, since within the accuracy of linearised theory all Mach waves are straight and parallel to the corresponding Mach wave directions in the undisturbed flow. Since disturbances are always propagated rearwards from a boundary the function $f_1(x - zB)$ must represent the potential function for the family of waves generated from a lower boundary, whilst $f_2(x + zB)$ represents that for the family of waves generated by an upper boundary.

If we now consider again the case of a thin wing section in an otherwise unbounded and initially uniform flow linearised theory presents the picture shown in Fig. 9.11,1.

Here only the two systems of parallel Mach waves operate on the flow, linearised theory does not include shock-waves within its scope since these are essentially non-linear and of finite magnitude. As will be apparent from the figure the principle of independence applies, the flow over the upper surface being influenced only by the Mach wave family generated by that surface, whilst that over the lower surface is likewise influenced only by the family generated by that surface. Hence we can write

$$\phi_U = f_1(x - zB)$$

for the flow over the upper surface, and $\phi_L = f_2(x + zB)$ for the flow over the lower surface.

The boundary condition on the upper surface requires that

$$(dz/dx)_{SU} = w/U_1 = \left[-\frac{\partial f_1/\partial z}{U_1}\right]_S$$

$$= B(f_1')/U_1 \qquad (9.11,27)$$

where suffix S denotes conditions at the surface, the dash denotes differentiation of f_1 with respect to its argument $(x - zB)$, and we have introduced approximations as in the previous section consistent with the assumptions of linearised theory. But the pressure coefficient at the surface (equation 9.11,7) is

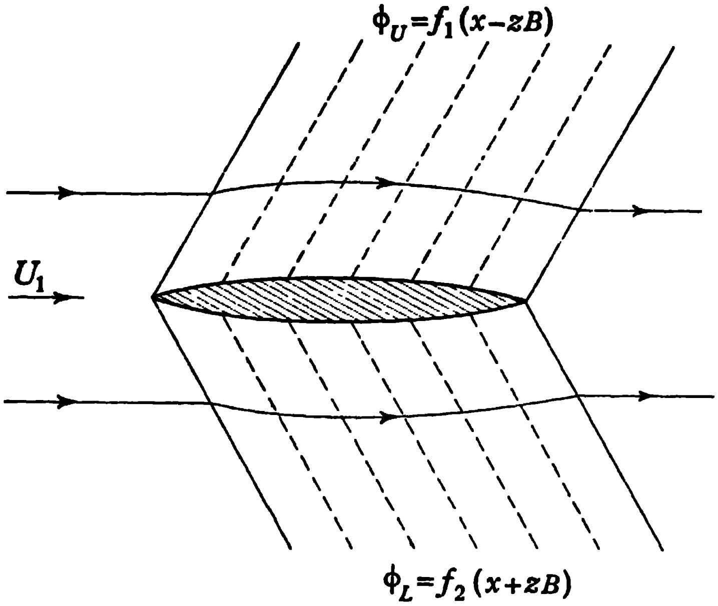


Fig. 9.11,1. Supersonic flow past a wing section, according to linearised theory.

$$C_{pU} = -2u/U_1 = 2f_1'/U_1. \qquad (9.11,28)$$

It follows from 9.11,27 and 9.11,28 that

$$C_{pU} = \frac{2}{B}\left(\frac{dz}{dx}\right)_{SU}. \qquad (9.11,29)$$

Similarly for the lower surface we have

$$C_{pL} = -\frac{2}{B}\left(\frac{dz}{dx}\right)_{SL}. \qquad (9.11,30)$$

We thus obtain the remarkably simple result that the pressure coefficient at a point is $\pm 2/B$ times the deflection of the flow at that point from the undisturbed stream direction (θ), the + sign refers to the upper surface and the — sign refers to the lower surface, and θ is positive if anti-clockwise.

This result was first discovered by Ackeret[1] and it is known as *Ackeret's Law*. Hence, subject to the assumptions of linearised theory, the pressure coefficient at a point on a wing in supersonic flow depends only on the local slope of the surface there and is independent of the rest of the wing section.

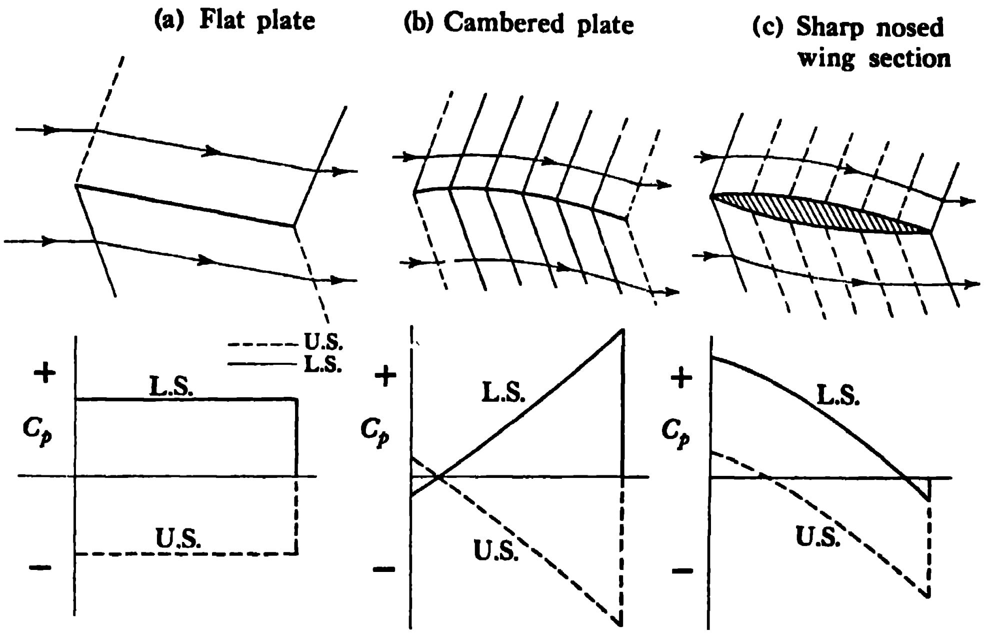


Fig. 9.11,2. Illustration of pressure distribution according to linearised supersonic flow theory on (a) a flat plate, (b) a cambered plate and (c) a sharp-nosed wing section.

This is in sharp contrast to what we find in subsonic flow where the pressure coefficient at a point depends on the complete wing shape and incidence.

It follows that the lift coefficient of a sharp nosed aerofoil of chord c is

$$\begin{aligned} C_L &= -\frac{2}{Bc}\left[\int_0^c \left(\frac{dz}{dx}\right)_{SU} dx + \int_0^c \left(\frac{dz}{dx}\right)_{SL} dx\right] \\ &= \frac{4}{Bc}[z_{\mathrm{L.E.}} - z_{\mathrm{T.E.}}] \qquad (9.11,31) \\ &= 4\alpha/B, \end{aligned}$$

where the incidence α is defined as the angle of the line joining the leading edge (L.E.) and trailing edge (T.E.) relative to the undisturbed stream direction. We here note that wing camber has no effect on the lift developed, in contrast to the effect of camber in subsonic flow.

By way of illustration Fig. 9.11,2 shows the pressure distributions derived by this theory for a flat plate, cambered plate and a biconvex wing at incidence.

The pressure distribution will have a resultant in the drag direction, the rate of work done by this drag provides the energy propagated away from the wing in the wave system. This *wave drag*, as it is called, is from equations 9.11,29 and 9.11,30 given by

$$C_{DW} = \frac{2}{Bc}\left[\int_0^c \left(\frac{dz}{dx}\right)^2_{SU} dx + \int_0^c \left(\frac{dz}{dx}\right)^2_{SL} dx\right]$$

$$= \frac{2}{B}(2\alpha^2 + \overline{\varepsilon_U{}^2} + \overline{\varepsilon_L{}^2}) \qquad (9.11,32)$$

where $\overline{\varepsilon_U{}^2}$ and $\overline{\varepsilon_L{}^2}$ are the mean square slopes of the upper and lower surface relative to the chord line, respectively. The term $4\alpha^2/B$ may be regarded as

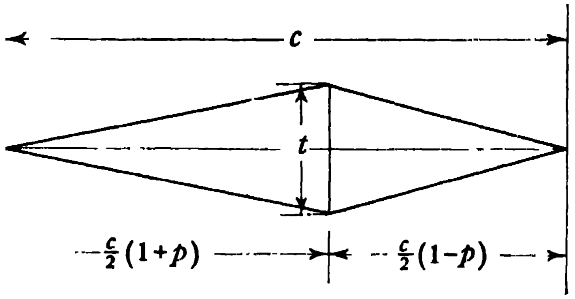


Fig. 9.11,3.

the wave drag coefficient due to lift; from equation 9.11,31 it will be seen to be $C_L\alpha$, the remaining terms, namely, $2(\overline{\varepsilon_U{}^2} + \overline{\varepsilon_L{}^2})/B$ may be regarded as the wave drag coefficient due to shape (or thickness).

Likewise the pitching moment coefficient about the leading edge is given by

$$C_M = -\frac{2\alpha}{B} + \frac{2}{Bc^2}\int_0^c (\varepsilon_U + \varepsilon_L)x\, dx. \qquad (9.11,33)$$

For a symmetrical aerofoil $\varepsilon_U = -\varepsilon_L$ and then

$$C_M = -2\alpha/B. \qquad (9.11,34)$$

As examples of equation (9.11,32) we may note that for a circular arc symmetrical aerofoil

$$C_{DW} = \left[4\alpha^2 + \frac{16}{3}\left(\frac{t}{c}\right)^2\right]\Big/B.$$

For a symmetrical diamond shaped aerofoil, as illustrated in Fig. 9.11,3, we have

$$C_{DW} = \left[4\alpha^2 + \frac{4}{(1-p^2)}\frac{t^2}{c^2}\right]\Big/B$$

and hence C_{DW} is a minimum when $p = 0$ and then

$$C_{DW} = \left[4\alpha^2 + 4\frac{t^2}{c^2}\right]\Big/B.$$

We note that the contribution to wave drag due to thickness is proportional to $(t/c)^2$, hence it is important at supersonic speeds to keep the ratio of thickness to chord as small as possible.

We may again remark that since all aerodynamic coefficients vary as $1/B$ with Mach number according to this theory and since $B \rightarrow 0$ as $M_1 \rightarrow 1{\cdot}0$ the theory and its results become invalid for Mach numbers close to unity. A typical comparison of lift and drag coefficients of a wing section as functions of Mach number as predicted by linearised theory and as measured experimentally is shown in Fig. 9.11,4.

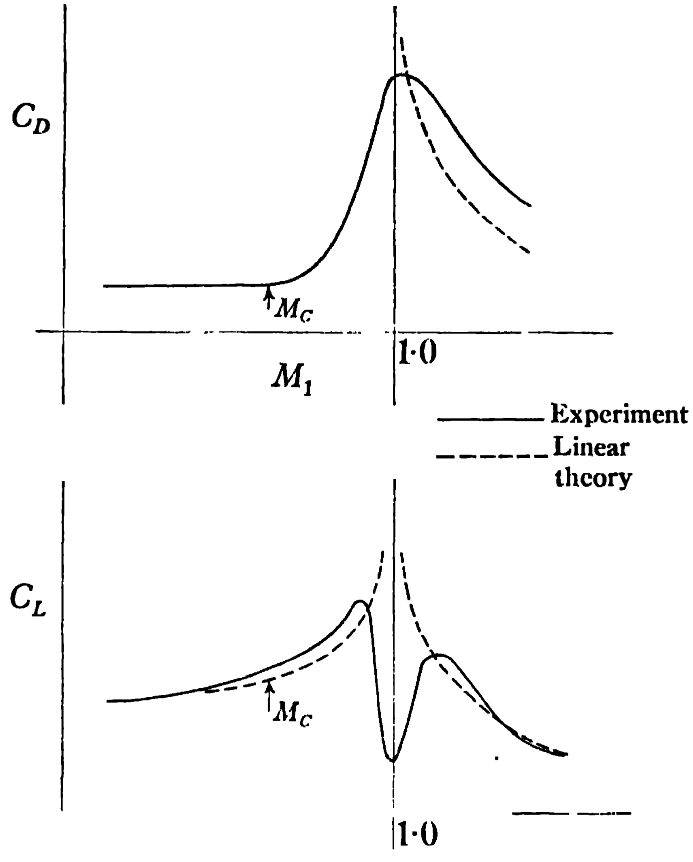


Fig. 9.11,4. Sketch illustrating measured and theoretical variations of C_D and C_L with Mach number for a typical aerofoil section.

It may be noted that the experimental results for drag would include profile or boundary layer drag (see Chapter 6) which of course does not figure in the linearised inviscid flow theory.

The application of linearised theory to three dimensional supersonic flow is a subject that has received very considerable attention for many years and is too vast to be presented even in summary form here. For further details the reader is referred to G. N. Ward[1] and W. R. Sears.[2]

9.12 The Method of Characteristics

The method of characteristics is one of very wide application and scope, and in its most general form it can be used in principle to determine the flow past any shaped body in an inviscid fluid, provided the flow everywhere is supersonic, and provided adequate computing facilities are available. We shall confine ourselves to discussing briefly its application to steady two-dimensional, supersonic, irrotational flow, where the method is particularly simple to apply.

The term 'characteristics' is used in connection with the theory of a certain class of differential equations, called hyperbolic, to which the equations of motion of a supersonic flow happen to belong. For such equations with two independent variables, say, it is found that, in general, if the values of the dependent variables are specified along any line in the plane of the independent variables, then the equations provide information about the gradients of the dependent variables normal to that line, so that the values of the dependent variables can ultimately be determined within a region containing the line. There are, however, certain families of lines, n if there are n dependent variables, such that the basic equations yield only relations for the dependent variables along the lines, and give no information as to their derivatives normal to the lines. These lines are called *characteristics* and the relations along them are called *characteristic relations*. Their usefulness lies in the fact that the relations that hold along them can be treated as ordinary differential equations to which the main system of equations can be transformed. We can then make use of the network of characteristic lines and their corresponding relations to solve the equations by a step by step process working outwards into a region beginning with given boundary conditions on a line which is not itself a characteristic. This technique will be made clearer in what follows; for the present we note that we can take the hint provided by the theory of hyperbolic differential equations and seek for the characteristics of supersonic flow and the relations that hold along them.

For two dimensional, irrotational flow of uniform entropy there are two dependent variables and hence there are two families of characteristics. It needs little thought to realise that these families are in fact the two families of Mach waves or lines. Since the equations are such that a knowledge of the values of the dependent variables along a characteristic does not lead to any information about their gradients normal to the characteristic, it follows that valid solutions can exist with discontinuities in these gradients. But we have seen that Mach waves are wave fronts normal to which small disturbances generated at the boundaries are propagated. Disturbances taking the form of sudden changes in flow direction can only be compressive in nature and they take the form of shock waves, but disturbances of a smaller order taking the form of discontinuities in rate of change of flow direction or higher derivatives of rate of change of flow direction take the form of Mach waves. Hence we can identify the Mach waves, or more strictly the lines (called Mach lines) defining the Mach wave fronts, as the characteristics of the flow. It remains to determine the characteristic relations that hold along them.

Consider the variation of flow conditions between two neighbouring Mach waves of one family from an upper boundary as influenced by the Mach waves of the other family, $\beta_1 \ldots \beta_n$ from the lower boundary, as illustrated in Fig. 9.12,1.

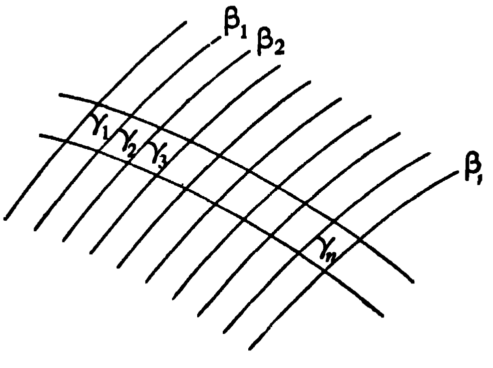


Fig. 9.12,1.

The change from cell γ_1 to cell γ_2 is that produced by Mach wave β_1 and hence must be governed by the differential equation (9.7,3), and the same relation must similarly apply to the change from cell γ_2 to cell γ_3 and so on. Hence we deduce that this relation must apply all along a Mach wave from an upper boundary. But we have seen that in conjunction with the energy equation this relation can be integrated to yield

$$\theta = -\vartheta + \text{const.}$$

where ϑ is given in equation (9.7,7), and hence this last relation must hold along the Mach wave. Similarly we can deduce that along a Mach wave from a lower boundary the relation

$$\theta = \vartheta + \text{const.}$$

must apply. In each case the constant can be regarded as a number characterising the Mach wave in question. Thus we can write

$$\left.\begin{array}{ll} & \theta = -\vartheta + \alpha, \text{ say,} \\ \text{along a member of the family of Mach waves from above} & \\ \text{for which} & \\ & \dfrac{dy}{dx} = \tan(\theta - \mu), \\ \text{and} & \theta = \vartheta + \beta, \text{ say,} \\ \text{along a member of the family of Mach waves from below} & \\ \text{for which} & \\ & \dfrac{dy}{dx} = \tan(\theta + \mu). \end{array}\right\} \qquad (9.12,1)$$

The value of α (or β) for any particular Mach wave is determined by the conditions at the boundary from which it springs.

Any point in the flow can then be characterised by the numbers α, β of the two waves through it, and we see from equation (9.12,1) that the flow conditions at the point are then given by

$$\left.\begin{array}{ll} & \theta = (\alpha + \beta)/2 \\ \text{and} & \vartheta = (\alpha - \beta)/2. \end{array}\right\} \qquad (9.12,2)$$

Thus, *if we know the values of α and β at a point we can immediately determine the flow conditions there, and conversely if we know the flow conditions there we can determine the values of α and β.*

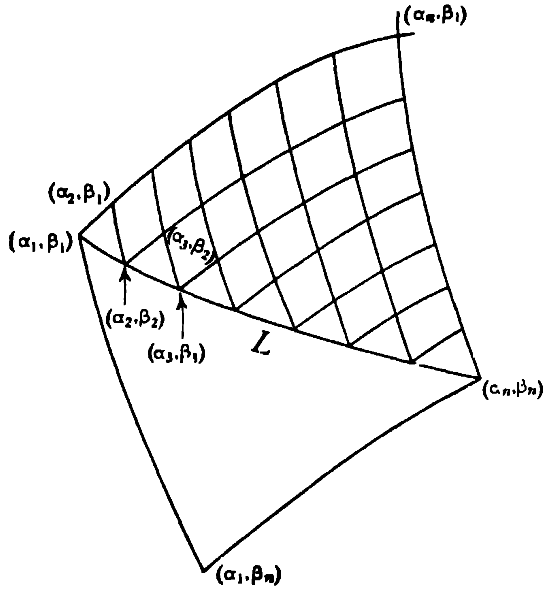


Fig. 9.12,2.

This last statement is the key to the method.

Thus, suppose the flow conditions are given along a line L, as in Fig. 9.12,2, and suppose L is not a characteristic. Then at a series of points 1, 2, etc. along L we can determine the characteristic numbers (α_1, β_1), (α_2, β_2) etc. These points can be made as close together as permissible computing time and the accuracy required will demand. Knowing the flow conditions at points 1 and 2 we can determine the direction of the Mach waves through them, and if we assume these are straight over the small distances involved we can draw the β wave through 1 and the α wave through 2 to meet at the point (α_2, β_1). Similarly we can determine the point (α_3, β_2) and so on until we arrive at the point (α_n, β_{n-1}). These points now define a new line on which we know the characteristic numbers and therefore the flow characteristics, and we can therefore proceed as before to determine points (α_3, β_1), (α_4, β_2), . . . , (α_n, β_{n-2}) and so on. In this way we can determine the flow conditions everywhere in the 'triangle' defined by L and the characteristics β_1 and α_n through its end points. Similarly we can determine the flow on the other side of L within the 'triangle' defined by L and the characteristics α_1 and β_n through its end points. We conclude, therefore, that given the flow conditions on L we can determine them throughout the 'quadrilateral' defined by the Mach lines through its end points. An improvement to the accuracy of the method is readily obtained by taking the Mach line between two neighbouring points on it in a direction which is a mean for the two points.

With the aid of suitable Tables† the method has application to a wide range of problems with given boundary conditions. The particular simplicity of the method for problems of two-dimensional, irrotational flow of constant entropy derives from the fact that the characteristic relations can be integrated outright in a form independent of the problem considered (equations 9.12,1 above). More generally, these relations are in the form of differential equations to which approximations in the form of finite difference relations are used in the step by step process of solution. For further details reference should be made to Temple,[1] Howarth,[2] Ferri,[3] Sears,[4] and Courant and Friedrichs.[5]

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† As, for example, *Tables for Compressible Airflow*, ed. L. Rosenhead (O.U.P.).

9.13 Transonic Flow*

There is no precise definition of transonic flow but the term is generally associated with flows in which there are appreciable regions of supersonic and subsonic flow co-existent outside boundary layers and wakes. Roughly speaking, for aircraft of modern design it will extend from about $M_1 = 0{\cdot}8$ to $M_1 = 1{\cdot}2$, the lower limit being defined by the *critical Mach number*, i.e. the main stream Mach number at which a local Mach number of unity is first attained on the aircraft, and the upper limit is determined by the main stream Mach number at which the nose shock waves first become attached if the nose is sharp or at which their distance of detachment becomes very small if the nose is round.

It will be clear from our previous discussion that subsonic and supersonic flows differ profoundly, particularly in the way the influence of the boundaries is propagated into the flow. It is not surprising to find, therefore, that the problems of transonic flow are in many respects more difficult than the problems of purely subsonic or purely supersonic flow. It will be apparent that before we can begin to understand transonic flows we must have a firm grasp of the principles of subsonic and supersonic flows; it is for this reason that this subject has not been dealt with earlier in this chapter.

To illustrate the main points, we will consider how the pressure distribution on a typical aerofoil section changes with main stream Mach number as the latter is increased from a subsonic value to a value greater than unity.†

Fig. 9.13,1 shows the type of pressure distribution obtained at a small positive incidence at low and moderate main stream Mach numbers for which the variation of pressure with Mach number is given with fair accuracy by the Prandtl-Glauert Law (equation 9.11,16). We see that there are regions of fairly high suction particularly on the upper surface, and these correspond to regions of locally higher velocity and Mach number than the main stream velocity and Mach number. We find, therefore, that with increase of main stream Mach number it reaches a value, called the *critical Mach number* and denoted by M_c, at which the local Mach number on the wing reaches unity. In Fig. 9.13,1 this value of main stream Mach number has been slightly exceeded. The corresponding lift coefficient is indicated in the small left-hand diagram.

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* The reader new to the subject may find it helpful to read §§ 11.1 to 11.4 before reading this section.

† The authors are indebted to Dr. D. W. Holder, Aerodynamics Division, N.P.L., for Fig. 9.13,1–5.

With further increase of main stream Mach number a limited region in which the flow is supersonic develops from the surface and grows, expanding outwards, forwards and backwards. The local flow in passing through the supersonic region is speeded up and its pressure is reduced, but towards the rear of that region it must slow up again and compress back to subsonic speeds. This compression occurs very largely through the mechanism of a shock wave at the rear of the supersonic region. This is illustrated in Fig. 9.13,2 in which we see the characteristic region of high suction upstream of the shock followed by a sudden compression at the shock. The appearance of the shock on the upper surface does not by itself cause any marked change in the behaviour of the lift, as is apparent from the left-hand diagram.

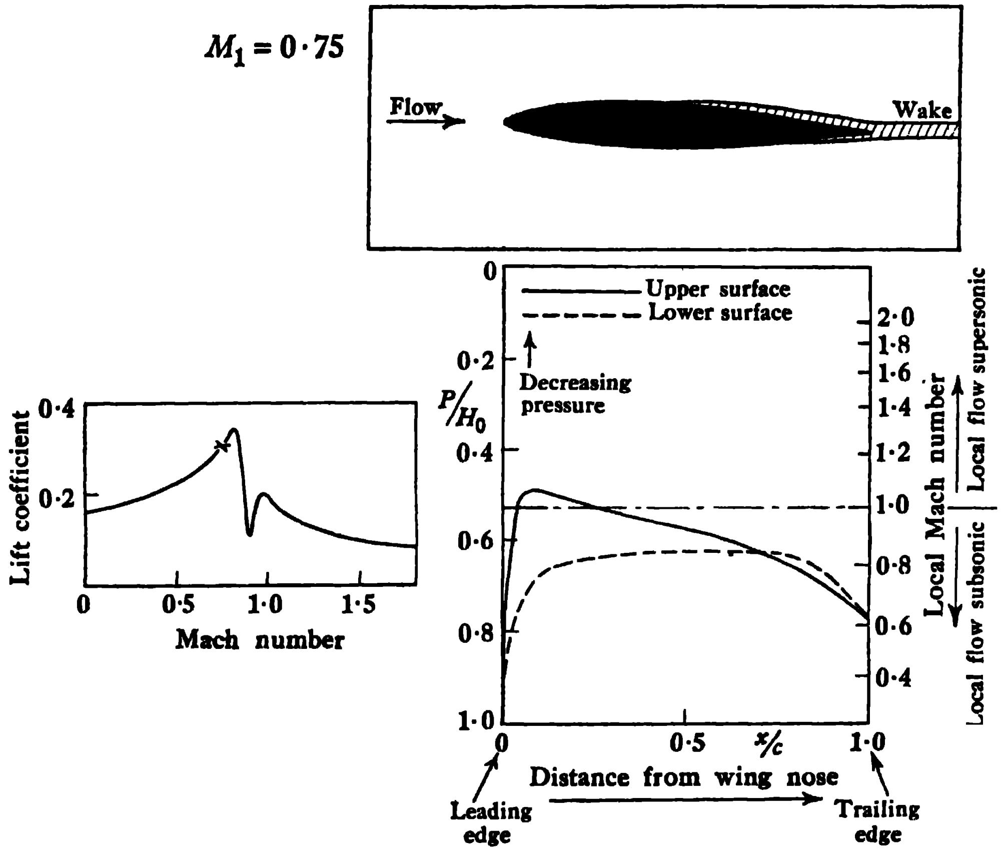

Fig. 9.13,1. Typical pressure distribution on a wing at low and moderate main stream Mach numbers and a small angle of attack (lower right-hand diagram). The scale of p/H_0 represents decreasing pressure (and increasing velocity) upwards. The appropriate lift on the wing is indicated by the cross in the bottom left-hand diagram.

However, we note that in Fig. 9.13,2 the local Mach number on the lower surface has just attained unity, and with further increase in main stream Mach number a supersonic region rapidly develops over the lower surface and is also terminated by a shock. It is at this stage that the more serious aerodynamic effects of transonic flow become apparent. It will be clear that the net lift on the wing at any particular main stream Mach number will then be largely determined by the relative positions of the upper and lower surface shocks, since the associated pressure distributions will have opposing effects. It frequently happens that the growth of the supersonic region on the lower surface and the accompanying rearward shock movement with increase of main stream Mach number is very rapid, so much so that the lower surface shock moves aft of the upper surface shock for a range of Mach numbers, as illustrated in Fig. 9.13,3. At that stage the lift falls rapidly and may even become negative. This in itself is undesirable, but there may also be large and dangerous changes of pitching moment with small changes of Mach number. Further we recall that a shock wave involves the irreversible dissipation of mechanical into heat energy, and this is manifest as an increase of drag. This drag increase (*wave drag*) will be directly related to the size and intensity of the shock waves on the wing, which in turn are determined by the extent of the supersonic region ahead of the shock. At first, for main stream Mach numbers only a little greater than the critical, M_c, this effect on drag is small but later it increases rapidly, being roughly proportional to $(M_1 - M_c)^4$, and it can become several times larger than the boundary layer drag (see Fig. 9.11,4).

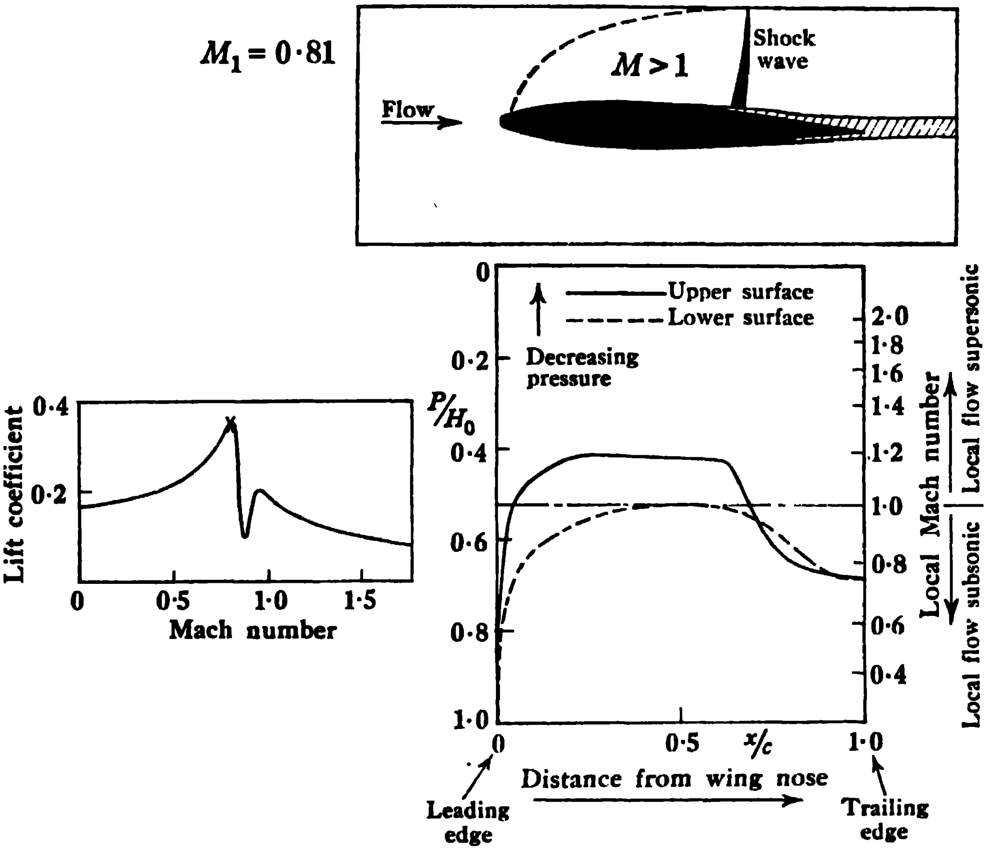

Fig. 9.13,2. Typical pressure distribution on a wing with upper surface shock.

The sequence of events is made more complicated by the effects of interaction between the shock waves and the boundary layers. The boundary layer in passing through the pressure rise at the shock foot will thicken and may separate, and in consequence the pressure distribution and drag may be considerably modified. It is known that separation of the boundary layer from the upper surface due to the shock there reduces the pressure aft of the shock and therefore tends to suck the lower surface shock back and so reduces the lift and changes the pitching moment.

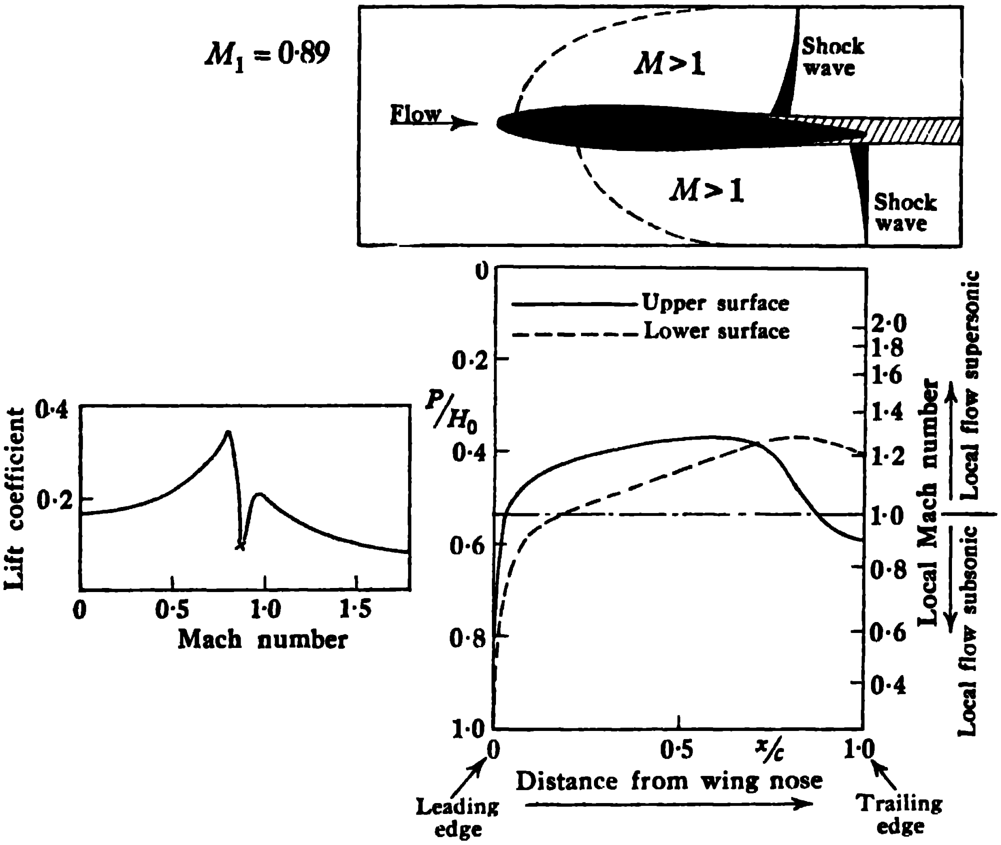

Fig. 9.13,3. Typical pressure distribution on a wing with upper and lower surface shocks, the lower surface shock having moved aft of the upper surface shock.

A further point to note is that the suction coefficient just ahead of a shock does not increase indefinitely with increase of main stream Mach number but attains a maximum and then falls. This is bound up with a general observation that on wing sections the maximum local Mach number ahead of a shock attained with subsonic main stream Mach numbers is of the order of 1·4–1·5 and once attained it remains roughly constant with further increase of M_1. Reasons for this cannot be given here but its effect is to enhance the lift reduction when the lower surface shock appears and moves rapidly backward, since at that stage the suction on the upper surface is waning with increase of Mach number.

Eventually, however, at a main stream Mach number close to unity both shocks move back to the trailing edge and the wing recovers some of its lift. At that stage the form of the pressure distribution is more characteristic of supersonic than subsonic flow (cf. Fig. 9.11,2*c*). With further increase of Mach number through unity a weak bow shock appears far ahead of the wing and moves rapidly towards it, increasing in strength as the distance between bow shock and wing nose diminishes. As we have already learnt,

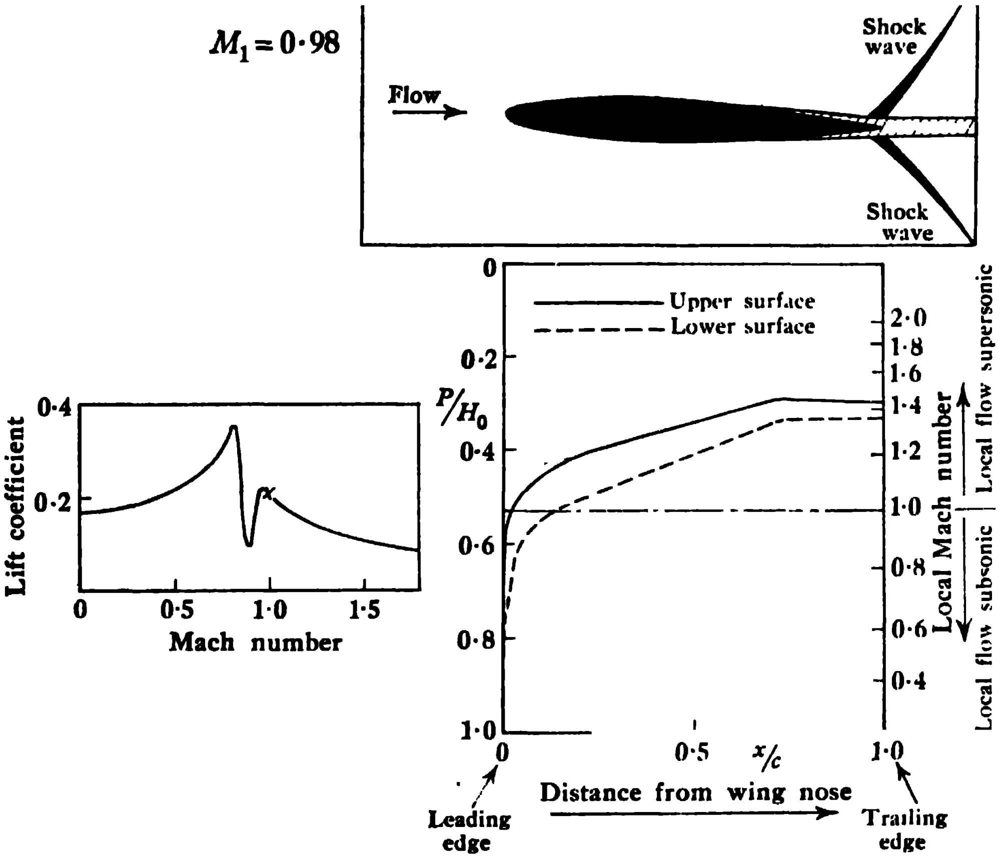

Fig. 9.13,4. Typical pressure distribution on a wing at a main stream Mach number close to unity, with both upper and lower shocks close to the trailing edge.

if the wing nose is round this shock remains detached from the nose but at a small distance from it however high the main stream Mach number, but if the wing nose is sharp then at some Mach number depending on the nose angle and incidence the shock becomes attached in the form of two shocks springing from the nose. These points are illustrated in Figs. 9.13,5 and 9.13,6.

The general sequence of shock movements on a section with increase of Mach number is also illustrated in Plate 9.1.

Curves illustrating typical variations of the drag and lift coefficients of a wing section with Mach number through the transonic range are given in Fig. 9.11,4. A possible variation of pitching moment coefficient is similarly illustrated in Fig. 9.13,7.

Certain salient features of these variations of the aerodynamic characteristics of an aerofoil with Mach number are worth emphasising. We note that the drag coefficient rapidly increases with Mach number due to the development of shock waves, but it reaches a maximum for a Mach number close to unity and then falls with further increase of Mach number, tending approximately at the higher supersonic speeds to the relation predicted by

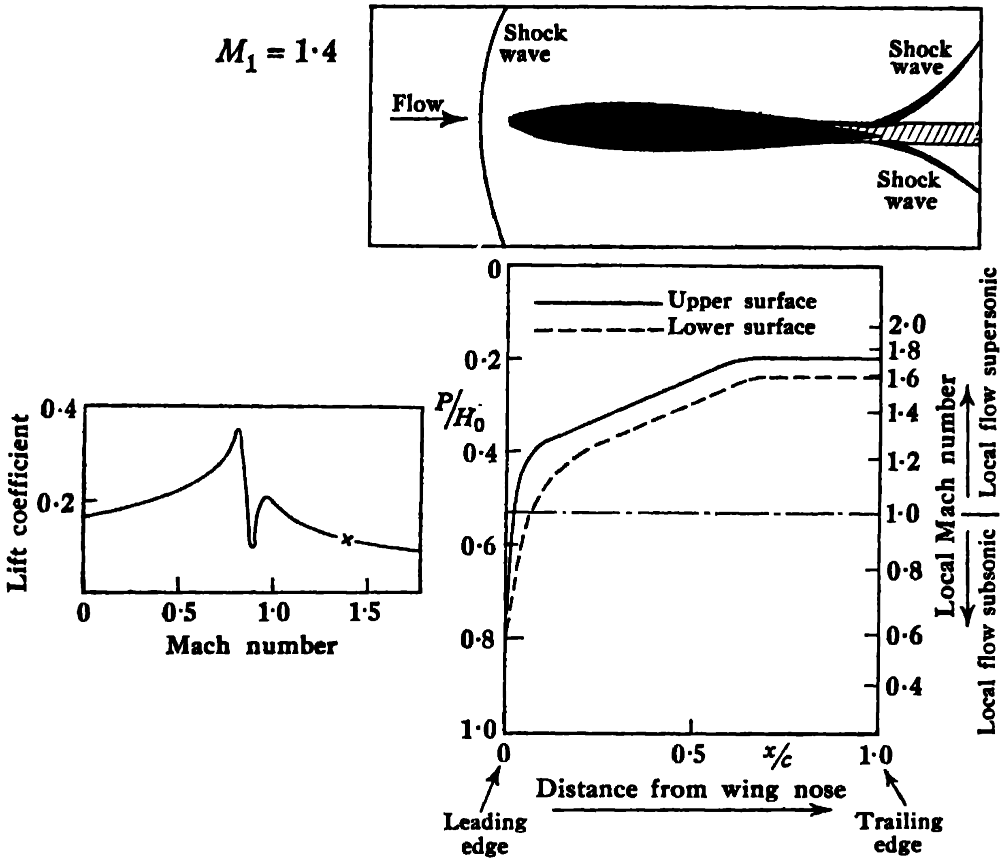

Fig. 9.13,5. Typical pressure distribution at low supersonic speeds, with detached bow shock wave ahead of the wing. Note the similarity of the pressure distribution to those illustrated in Figs. 9.13,4 and 9.13,6.

linearised theory after allowances are made for the boundary layer drag. The maximum value of C_D is given roughly by $3(t/c)^{2/3}$ based on frontal area, where t/c is the thickness-chord rate of the section. We also note the sharp fall in lift coefficient to a minimum, the recovery of lift coefficient with further increase of Mach number, and the behaviour of the lift at the higher supersonic Mach numbers when it is once more in general conformity with the predictions of linearised theory. Finally, we note the erratic variations of the pitching moment coefficient with Mach number in the transonic range as the pressure distribution due to lift changes from a typical subsonic one, with its centre of pressure somewhere near the point a quarter of the chord aft of the leading edge, to a typical supersonic one, with its centre of pressure near the mid-chord point. It will be clear

PLATE 9.1

SCHLIEREN PICTURES OF SHOCK FORMATIONS ON A WING SECTION

(*a*)–(*l*) 10% thick section at 2° incidence for Mach numbers from 0·7 to 1·18.

(*m*) $12\frac{1}{2}$% double wedge section at zero incidence and $M_1 = 1{\cdot}60$.

Note. For pictures (*a*)–(*l*) the boundary layers aft of about 0·15*c* back from the leading edge were made turbulent by means of air jets. The small disturbances due to these jets are clearly visible.

(These pictures were taken by the Staff of the Aerodynamics Division, National Physical Laboratory, and are published by kind permission of the Director, N.P.L.)

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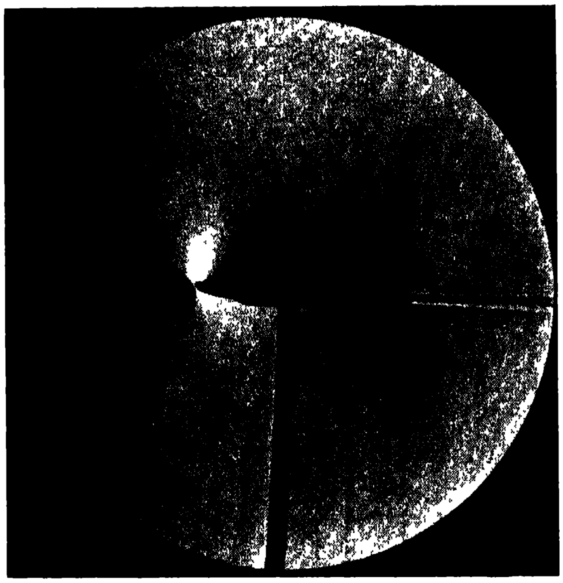
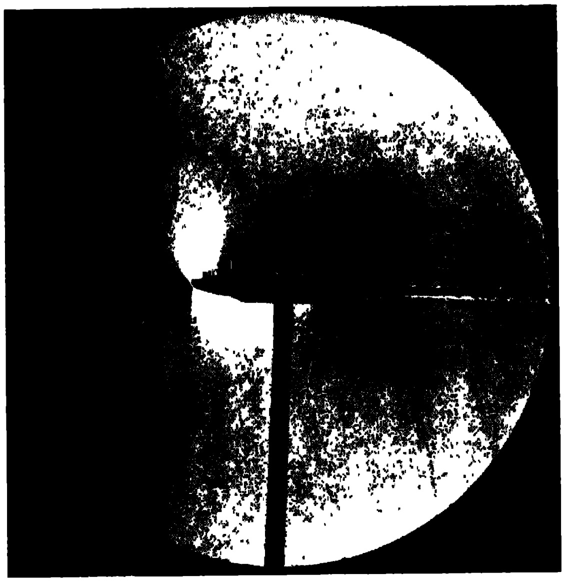
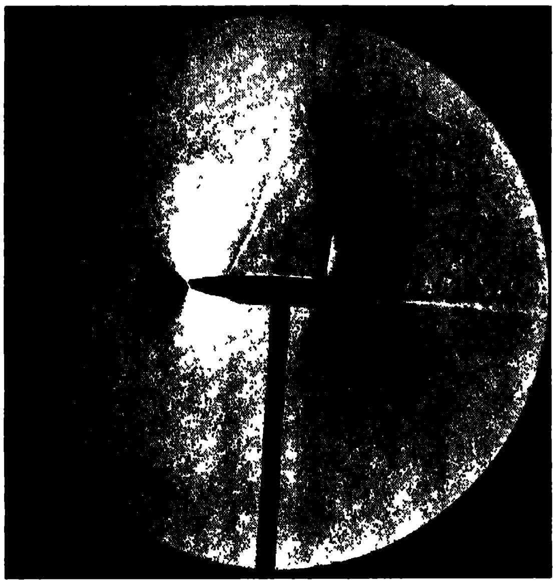


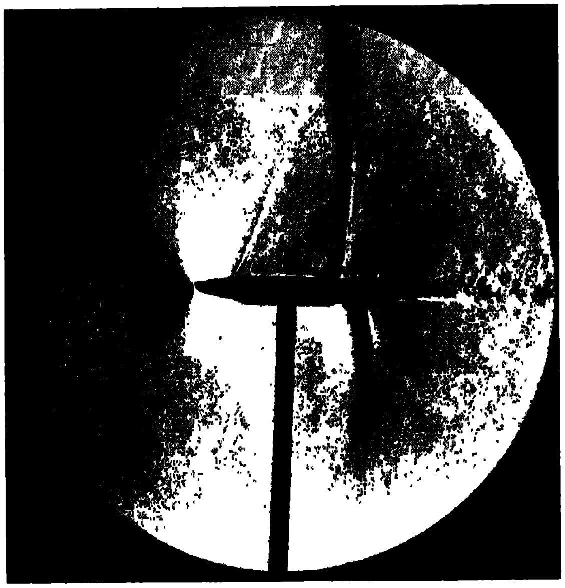
(*a*) $M_1 = 0{\cdot}7$



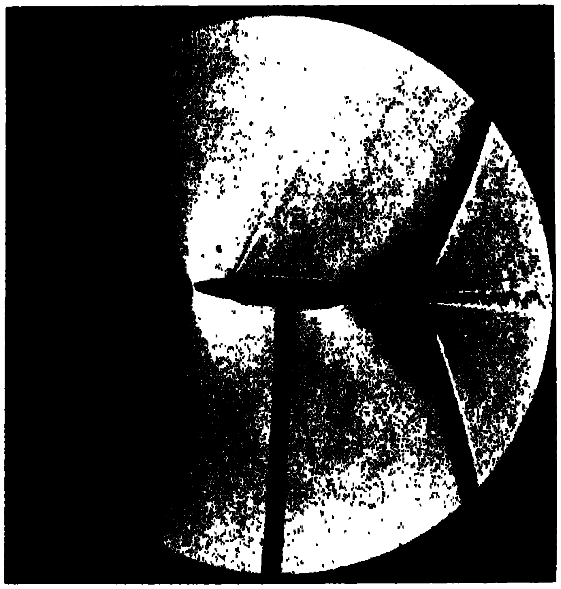
(*b*) $M_1 = 0{\cdot}75$



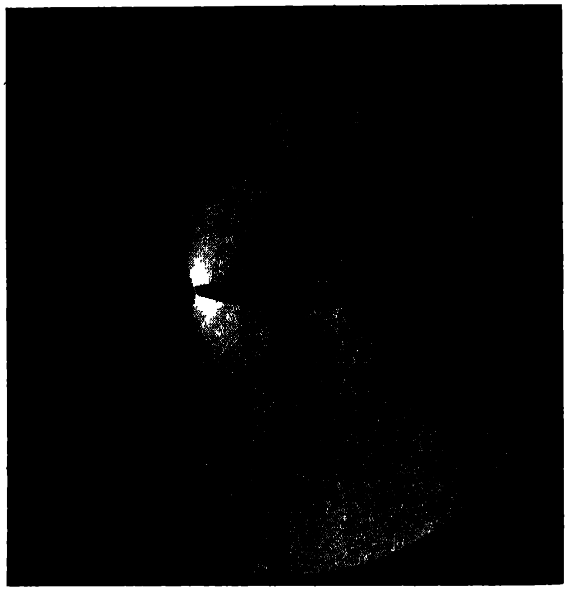
(*e*) $M_1 = 0{\cdot}84$



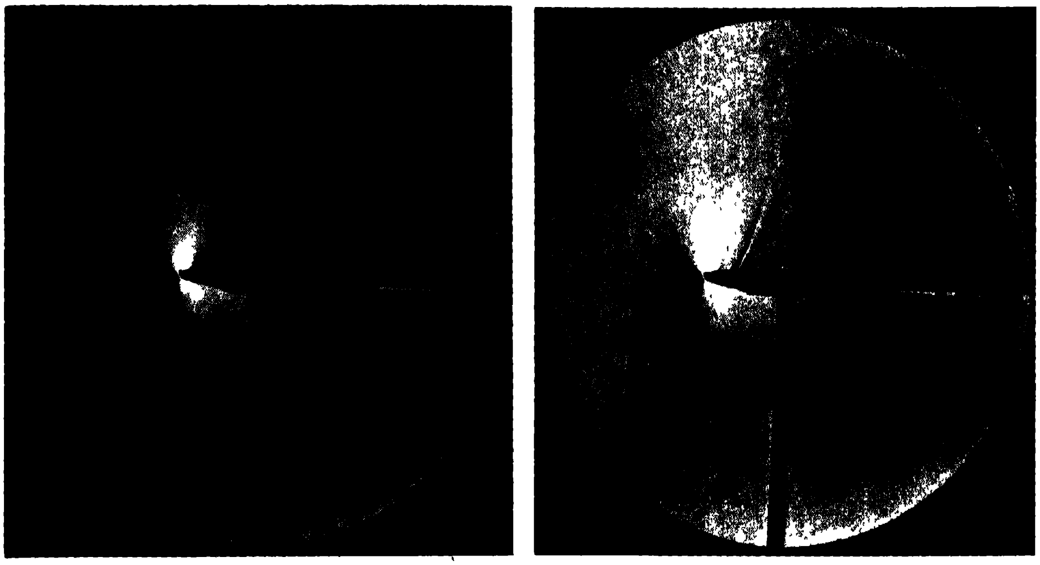
(*f*) $M_1 = 0{\cdot}88$



(*i*) $M = 0{\cdot}97$

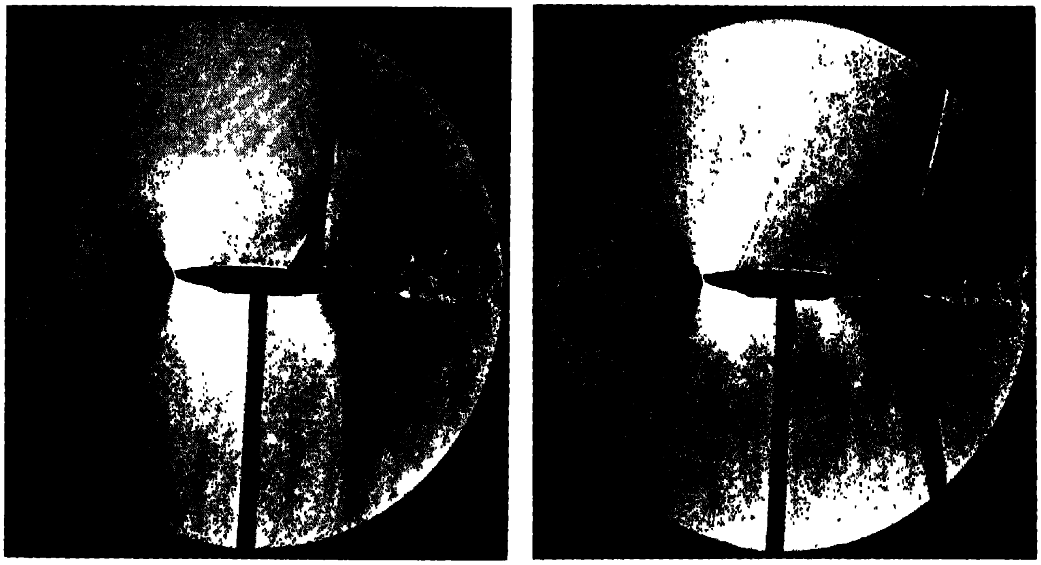


(*j*) $M = 0{\cdot}98$



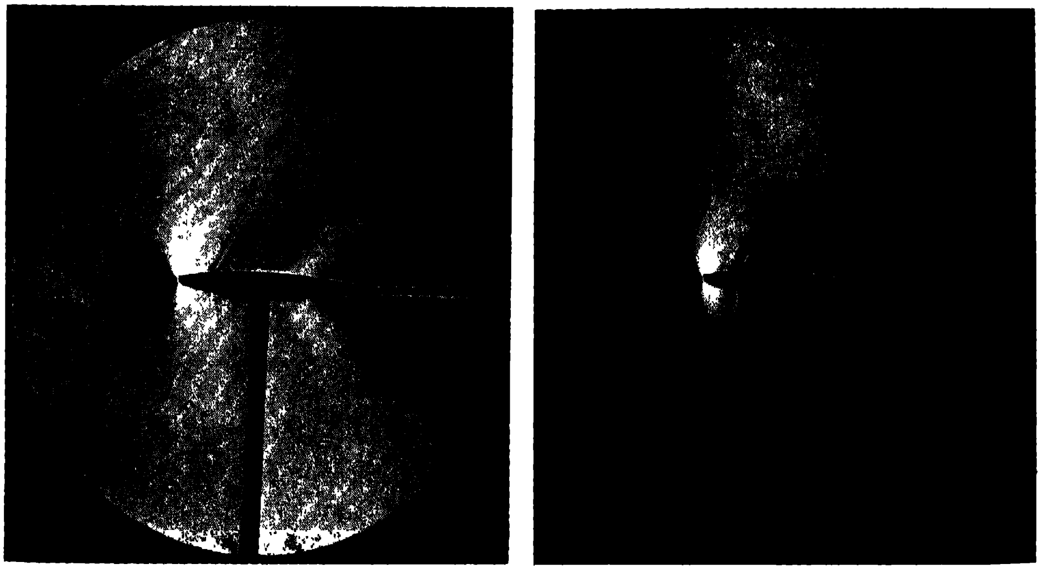
(*c*) $M_1 = 0{\cdot}775$

(*d*) $M_1 = 0{\cdot}82$



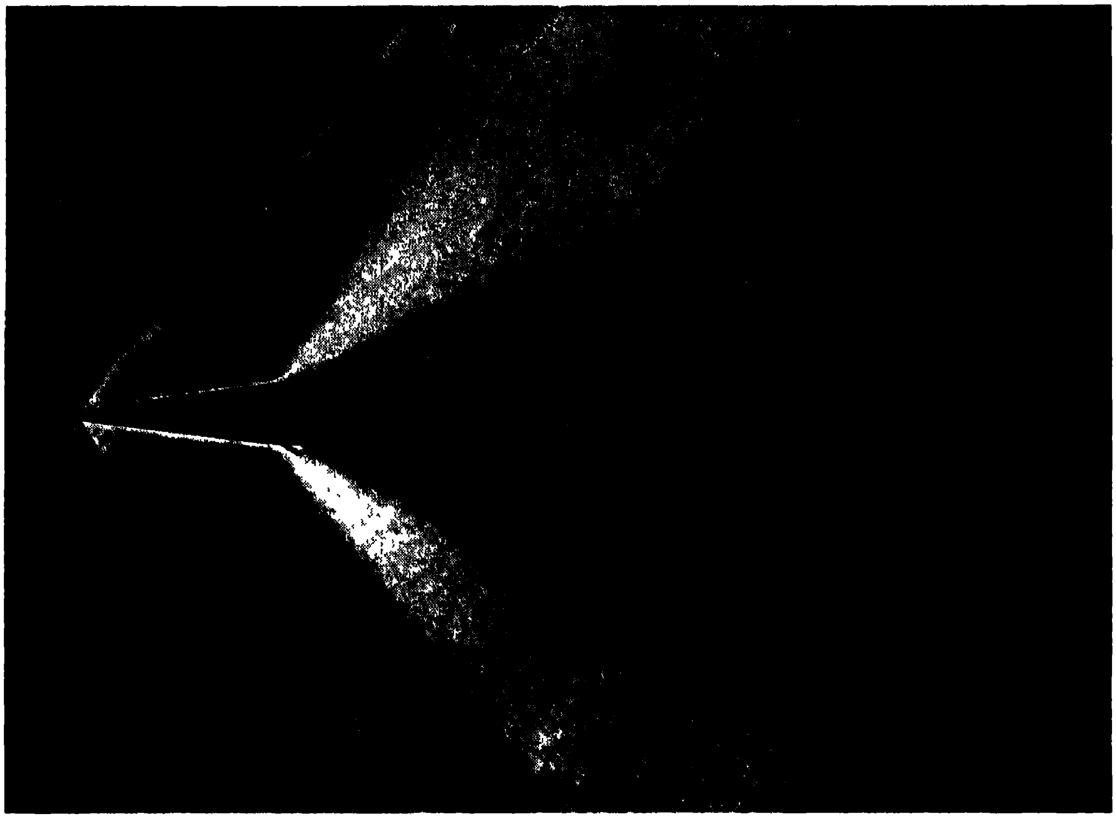
(*g*) $M_1 = 0{\cdot}90$

(*h*) $M_1 = 0{\cdot}95$



(*k*) M 1·10

(*l*) M = 1·18



(*m*) $M_1 = 1{\cdot}60$

that to the designer of an aeroplane destined to fly at transonic speeds these rapid changes of aerodynamic characteristics with small changes of Mach number present extremely serious problems. The problems are not made

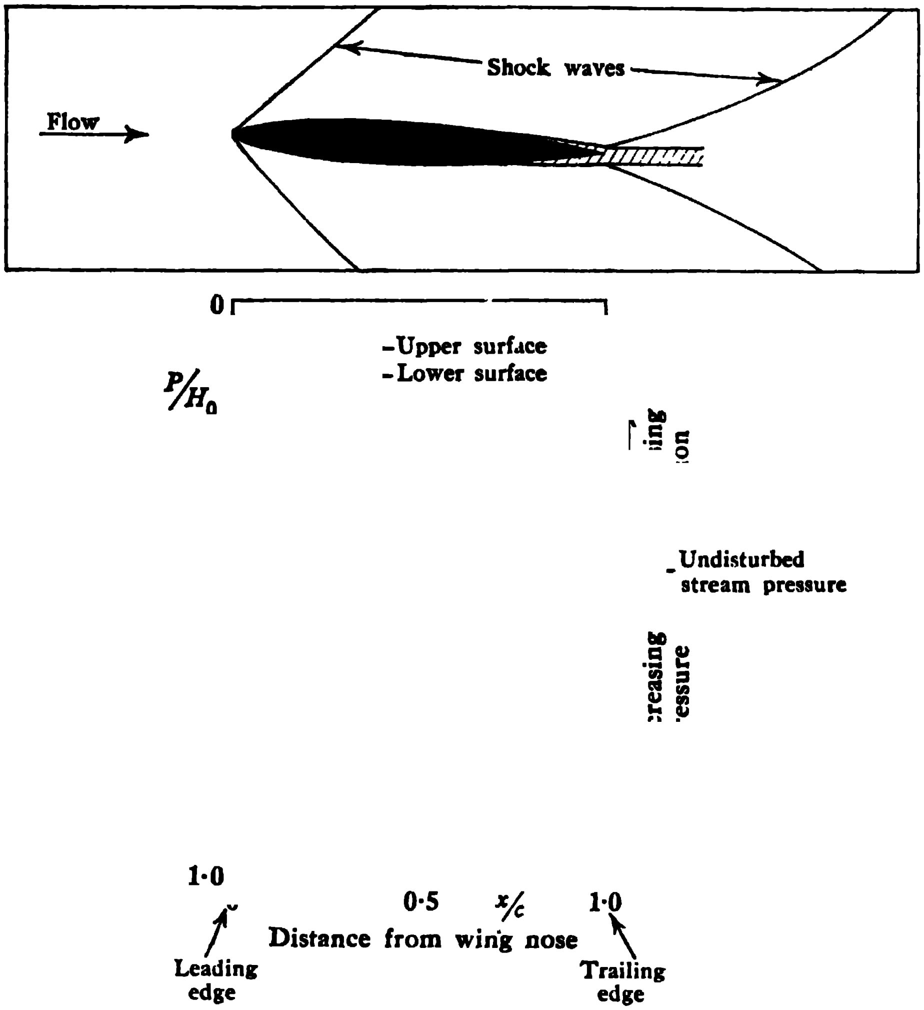


Fig. 9.13,6. Typical pressure distribution at supersonic speeds on a sharp-nosed aerofoil.

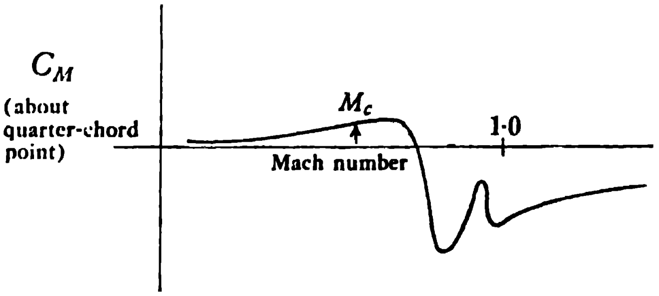


Fig. 9.13,7. Sketch illustrating possible variation of C_M of wing section with Mach number in transonic range.

any easier by possible boundary layer separation induced by the shock waves and the associated buffeting and vibration. In so far as these rapid changes are the direct result of the development of the shock waves the phenomenon has been loosely, if graphically, described as the *shock stall*, and the Mach number at its onset is called the *shock stall Mach number*.

It must be emphasised that important changes are inevitable as the main stream Mach number increases from subsonic to supersonic, but it is clearly desirable to make these changes as smooth and continuous as possible. But we have seen that the changes are likely to be particularly erratic and violent if with small changes of Mach number the relative positions of the two shocks change rapidly. It is, therefore, most important to keep the shock waves as far as possible moving rearwards in step with increase of Mach number. This design aim has yet to be completely fulfilled but much is already known to help achieve it and to alleviate the worst effects of the shock stall.

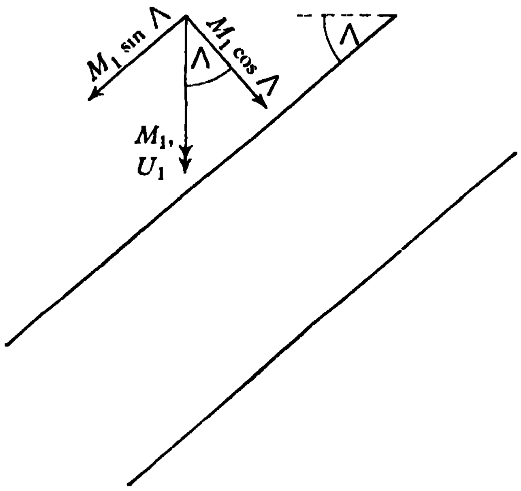


Fig. 9.13,8.

Thus, a study of available experimental data makes it abundantly clear that the adverse effects of the shock stall increase rapidly as well as appear at progressively earlier Mach numbers with increase of wing thickness-chord ratio and with increase of fuselage fineness ratio (defined by the ratio of the maximum diameter to the length). Conversely, by making these ratios small, the effects of the shock stall can be rendered comparatively mild. Hence, reduction of these ratios must be given prime consideration in the design of an aircraft for high speed flight. In addition wing section, planform, aspect ratio and trailing edge angle all play an important part and must be carefully considered in any design.

Perhaps the most important single design device for alleviating and delaying the worst effects of the shock stall is the use of sweepback of the wings and tail unit of an aeroplane. A study of Fig. 9.13,8 illustrates in a simple manner the effect of sweepback.

The figure shows a wing of infinite span with a sweep angle Λ in a stream of Mach number M_1, and velocity U_1. We can resolve the stream into one of Mach number $M_1 \cos \Lambda$ and velocity $U_1 \cos \Lambda$ normal to the wing span and one of Mach number $M_1 \sin \Lambda$ and velocity $U_1 \sin \Lambda$ parallel to the wing span. Apart from secondary effects on the boundary layer of the wing, the latter stream component has no effect on the pressure distribution on the wing. To appreciate why this is so we can regard the wing as equivalent to an unswept wing in a stream of velocity $U_1 \cos \Lambda$ and Mach number $M_1 \cos \Lambda$ moved parallel to itself with velocity $U_1 \sin \Lambda$, and this latter motion can have no direct aerodynamic effect on the infinite wing.

Alternatively, we can regard the swept wing as the above unswept wing viewed relative to a frame of reference moving parallel to the wing with constant velocity $U_1 \sin \Lambda$. Again, such a movement of the frame of reference can have no effect on the wing pressure distribution. We conclude, therefore, that the effective Mach number and velocity of the wing relative to the fluid are both reduced by sweep by the factor $\cos \Lambda$, and hence the adverse effects of the shock stall are delayed to a Mach number greater than for the unswept wing (of the same wing section as that in a plane normal to the span) by a factor $\sec \Lambda$. For $\Lambda = 45°$ this represents an increase in shock stalling Mach number of over 40 per cent.*

A real aircraft of finite aspect ratio and with swept wings obviously cannot show the full increase of the shock stall Mach number for a given angle of a sweep as the wing of infinite span, since the flow over its centre section and tips must differ markedly from the flow over the infinite wing. Nevertheless, very significant increases can result in practical cases and sweepback is an essential feature in the design of all modern high speed aircraft. By careful design it is possible to have an aircraft flying at low supersonic speeds with what is effectively subsonic flow over almost all of its surface.†‡

9.14 Hypersonic Flow. Rarefied Gas Flow

The flow past a body at sufficiently high Mach numbers is characterised by the fact that the leading edge shock waves lie close to the body surface. When this happens we refer to the flow as *hypersonic flow*. Strictly, the Mach number at which the flow can first be regarded as hypersonic depends on the body thickness or fineness ratio; the larger the latter the lower is this Mach number. But for most practical purposes we may define the hypersonic range as extending upwards from $M_1 = 5{\cdot}0$.

Hypersonic flow is characterised by the fact that the flow pattern changes very little with change of Mach number; the major changes occur in the temperature distribution. Linearised theory loses validity at these speeds and the analysis is further complicated by the possibility that if the temperature rise across the nose shock is large enough then the air will dissociate and even ionize. These problems cannot be discussed here, but in one limited sense hypersonic flow offers a measure of simplicity.

Newton, when considering the motion of a rare medium such as a gas past a body, suggested a hypothetical model in which the motion of the fluid particles was unaffected by the body until they hit it, after which the particles were deflected along its surface without change of tangential momentum. The pressure on the surface, therefore, was equal to the rate of loss of normal momentum of the fluid on deflection by the surface.

This argument readily leads to the result that if the surface inclination to the undisturbed stream direction is θ then the surface pressure differs from the undisturbed stream pressure by

$$\rho_1 U_1^2 \sin^2\theta$$

where U_1 is the undisturbed stream velocity, and hence the pressure coefficient is

$$C_p = 2\sin^2\theta. \tag{9.14,1}$$

This picture of fluid flow does not begin to approximate to reality as long as we are considering subsonic and moderate supersonic speeds, the major defect being the lack of any allowance for the way in which the influence of a boundary is propagated into the fluid. However, when we come to consider hypersonic flow, then we find that Newton's hypothesis has a measure of applicability. Thus the approaching stream is unaffected by the body until it crosses the nose shock waves, after which it is deflected in a direction more or less parallel to the body with no change of momentum parallel to the shocks, and in so far as the nose shocks lie close to the body this deflection can only occur within the narrow region between shocks and body surface. Of course, Newton's hypothesis takes no account of the thermodynamic changes that occur across the shock waves and subsequently, and we cannot therefore expect it to predict absolute values of the pressure distribution. However, we find that the simple relation

$$C_p = C_{p\max}\sin^2\theta, \tag{9.14,2}$$

which may be regarded as a generalisation of equation (9.14,1), gives surprisingly good agreement with experimental data. Here $C_{p\max}$ is the pressure coefficient at the stagnation point on the nose of the wing or body, which is to be assumed blunt, and hence $C_{p\max}$ is appropriate to stagnation conditions behind a normal shock. It is in calculating $C_{p\max}$ that a detailed assessment of the flow conditions must be made, since at high Mach numbers and high stagnation temperatures the flow conditions behind a shock will be influenced by departures of the gas behaviour from that of a perfect gas and by possible dissociation, ionization and radiation effects.

Apart from these effects, we see that Newton's long discarded particle theory of fluid flow is now found to have some relevance to flow at very high Mach numbers.

A related topic is that of the flow of a *rarefied gas*. When for any particular flow problem we treat a gas as a continuum we imply that its density is sufficiently high for the mean free path of the molecules (λ) to be very small (i.e. less than about 0·01) in terms of a typical linear dimension (L) of the body past which the gas is flowing. In that case the molecular structure of the gas can be ignored when considering the effects of the flow on the body, although the physical properties of the fluid, e.g. viscosity and conductivity, are determined by its molecular structure. For small values of the density, however, such that the ratio λ/L (called the *Knudsen number*) is of the order of 0·1, say, the molecular structure of the gas and the molecular reflection processes at the surface become of direct significance, and in particular the condition of no-slip at the surface (see Para. 6.1) ceases to hold. The flow is then referred to as *slip flow*. At still lower densities, for which the Knudsen number is of the order of unity or higher, the nature of the collision processes between the gas molecules and the surface and, in particular, the degree to which the molecules share their various forms of energy and their momenta with the surface become of prime importance. Collisions between the molecules away from the surface are then of little significance. Such a flow is called a *free molecule flow*.

Now it can be demonstrated that λ/L is of the same order of magnitude as M/R where M is the Mach number and R is the Reynolds number based on L. It follows that an increase of Knudsen number corresponds to an increase of Mach number or a reduction of Reynolds number, and rarefied gas flows can only be expected in practice at very low Reynolds numbers and moderate to large Mach numbers. The re-entry of a satellite, for example, will present problems of rarefied gas flow when travelling in the very rare regions of the atmosphere at great altitudes. However, it must be emphasised that the free stream values of the Knudsen number for a particular body may be very different from the local values behind the nose shocks or in the body boundary layers and each such region of flow needs separate consideration.

The theory of free molecule flow about a body generally involves a statistical description of the molecular velocities in the stream (taken as Maxwellian), and the assumption that incident and re-emitted molecules can be treated separately, and it requires a description of the transfer processes of momentum and energy from the molecules to the surface. Since these processes are far from understood at present the theory is incomplete, and more experimental information is required before a complete theory can be developed.

For further details of this and other topics related to this section, reference can be made to Hayes and Probstein.†

EXERCISES. CHAPTER 9.

1. An insulated vessel is divided into two compartments of equal volume by a diaphragm. One compartment is completely evacuated and unit mass of a gas is contained in the other. The diaphragm is then burst so that the gas fills both compartments. If the gas is perfect show that the final temperature of the gas is the same as its original temperature and that its entropy increases by an amount $C_v(\gamma - 1) \ln 2$.

2. If the gas referred to in Question 1 is air, its temperature is 15° C, and its mass is 1 slug, find the work that must be done on it in ft-slug-sec. units to restore it to its original state by a reversible process. What is the equivalent of this work in S.I. units?

   (Answer $6{\cdot}16 \times 10^5$ ft. lbf, 835 kJ. This amount of heat must be removed from the air during the process of restoring it to its original state to balance the work done on it.)

3. From equation 9.3,33 deduce the corresponding form of Bernoulli's equation for steady incompressible inviscid flow, viz.:—

$$\frac{p}{\rho} + \frac{V^2}{2} + \chi = \text{const. along a streamline.}$$

4. In the working section of a wind tunnel of cross-sectional area $0{\cdot}6\ \text{m}^2$ the Mach number is 0·86. In the settling chamber of cross-sectional area $4{\cdot}0\ \text{m}^2$ the pressure, density and temperature are $1{\cdot}014 \times 10^5\ \text{N/m}^2$, $1{\cdot}144\ \text{kg/m}^3$ and 35° C, respectively. Find the pressure, density and temperature in the working section, neglecting the effects of viscosity and treating the flow as one-dimensional.

   (Answer $6{\cdot}31 \times 10^4\ \text{N/m}^2$, $0{\cdot}813\ \text{kg/m}^3$ 268° K(−5° C))

5. A large air reservoir in which the pressure is kept constant at a value of 1.44×10^4 lbf./ft.^{2} and the temperature is constant at 15° C is connected via a convergent-divergent nozzle to another large vessel in which the pressure can be varied. The throat area and exit area of the nozzle are 2 sq.in. and 8 sq.in., respectively. Find
   (a) the maximum rate of mass flow of air through the nozzle,
   (b) the two values of the Mach number at the end of the nozzle for which isentropic flow throughout the nozzle occurs with this maximum rate of mass flow,
   (c) the pressure at the end of the nozzle corresponding to these Mach numbers.

   Treat the flow as one-dimensional and inviscid.

   (Answers (a) 0·145 slug/s (b) 0·147, 2·94, (c) $1{\cdot}42 \times 10^4$ lbf/ft^2, 429 lbf/ft^2.)

6. A rocket motor is designed to give 1000 lbf. thrust at 40,000 ft., and at that height the flow in the motor nozzle and subsequent jet is isentropic. The combustion chamber pressure and temperature are $4{\cdot}32 \times 10^4$ lbf/ft^2 and 3000° K, respectively, and the value of γ for the gases of combustion is 1·3. Find the exit Mach number and the cross-sectional areas of the exit and throat of the nozzle.

   Treat the flow as one-dimensional and inviscid.

   (The pressure of the atmosphere at 40,000 ft. is 391 lbf/ft^2

   (Answer 3·62, 0·15 ft^2 and 0·0145 ft^2.)

7. Show that the temperature rise in °C at the stagnation point of an aircraft is very nearly $\left(\frac{\text{speed in m.p.h.}}{100}\right)^2$.

8. A simple wave flow with an initial Mach number of 1·46 is deflected expansively through an angle of 5°. Find the Mach number after deflection, and the pressure coefficient $c_p = (p - p_1)/\tfrac{1}{2}\rho_1 U_1^2$, where suffix 1 refers to initial conditions. Compare this pressure coefficient with that predicted by the linearised (Ackeret) theory.

   (Answer 1·63, $c_p = -0{\cdot}149$, linearised theory gives $c_p = -0{\cdot}154$)

9. Show that, for a small flow deflection θ from given initial conditions, c_p (as defined in equation 8) for both a simple wave flow and across a shock wave is given to the second order in θ by

$$c_p = \frac{2}{(M_1^2 - 1)^{\frac{1}{2}}}\,\theta + \frac{\gamma M_1^4 + (M_1^2 - 2)^2}{2(M_1^2 - 1)^2}\,\theta^2.$$

(Note, the relations between c_p and θ for the two cases differ only in terms of order θ^3, since the entropy change across the shock is of that order.)

10. Prove the simple wave flow relation of equation (9.7,7) for a centred expansion (Prandtl-Meyer expansion) by using polar coordinates and making use of the inference that flow conditions along any ray through the centre of the expansion are constant.

11. Show that across a shock wave in a perfect diatomic gas ($\gamma = 1{\cdot}40$) there cannot be more than a sixfold increase of density.

12. For an oblique shock you are given that $M_1 = 2{\cdot}0$ and $\xi = 45°$. Find the deflection δ. (Answer 14°44′)

13. At the nose of a wedge of semi-angle 10° at zero incidence an attached shock of angle 30° is observed. Find M_1, M_2 and the ratio p_1/p_2. (Answer 2·73, 2·26, 0·497)

14. A symmetrical double wedge section of semi-angle at nose and tail of 3·46° is at an incidence of 2° and a Mach number of 1·57. Find its lift and wave drag coefficients, using shock-expansion theory, and compare the answers with those given by Ackeret's theory. (Answer $C_L = 0{\cdot}115$, $C_{DW} = 0{\cdot}016$; linearised theory gives $C_L = 0{\cdot}116$, $C_{DW} = 0{\cdot}016$)

15. An intermittent supersonic tunnel is such that air initially in a large container at atmospheric pressure and 15°C passes through the working section into an evacuated vessel. Find the static pressure and pitot pressure in the working section if the Mach number there is 2·1. Take the atmospheric pressure as $1{\cdot}014 \times 10^5 \mathrm{N/m^2}$. (Answer $1{\cdot}10 \times 10^4\ \mathrm{N/m^2}$, $6{\cdot}80 \times 10^4\ \mathrm{N/m^2}$.)

16. The minimum pressure coefficient measured on a particular wing section at a given incidence and a main stream Mach number of 0·2 was −0·15. On the basis of subsonic flow linearised theory determine its critical Mach number. What would be the critical Mach number if the wing were sheared so that its leading edge became inclined at 45° to the main stream direction keeping the section shape and incidence unchanged in the main stream direction? (Answer 0·857, 1·17)

17. Prove the formula quoted on p. 508 for the wave drag coefficients according to linearised supersonic flow theory of (a) a circular arc symmetrical aerofoil, (b) a symmetrical diamond shaped aerofoil.

18. Show that, according to linearised supersonic flow theory, if a number of points on a wing section including its leading and trailing edge are specified then the section of least wave drag is obtained by joining successive points by straight lines.

19. The result given in Exercise 9 can be interpreted as giving a second order correction to the linearised Ackeret relation connecting c_p and deflection θ. This was pointed out by Busemann. Show that if it is applied to a wing section
    (i) $dC_L/d\alpha$ is the same as that predicted by the Ackeret relation,
    (ii) The wave drag is the same as that predicted by the Ackeret relation if the section is doubly symmetric,
    (iii) The aerodynamic centre is forward of the midpoint of the chord.

    Note that the aerodynamic centre is the point on the chord line for which $\partial C_M/\partial C_L = 0$.

20. Suppose that in steady two dimensional, irrotational inviscid flow at transonic speeds past a slender wing section we can write the velocity components as $(V^* + u, v)$ where u/V^*, and v/V^* are small compared with unity, V^* is the critical velocity. Show that

$$\frac{a^2}{V^{*2}} = 1 - (\gamma - 1)\frac{u}{V^*}$$

and

$$M^2 = 1 + (\gamma + 1)\frac{u}{V^*}.$$

If the flow is nearly parallel to the x axis and $\partial u/\partial y$ can be assumed small compared with $\partial u/\partial x$ show that the potential function equation reduces to

$$\frac{(\gamma + 1)}{V^*}\phi_x\phi_{xx} - \phi_{yy} = 0$$

where

$$\phi_x = -u, \text{ and } \phi_y = -v.$$

This is the so-called transonic flow equation and is due to von Kármán. Note that it is non-linear.

CHAPTER 10

OSCILLATIONS AND WAVES

10.1 Introduction

Oscillations and waves are periodic phenomena, i.e. phenomena in which a given configuration of the system recurs at regular intervals of time.† The distinction between an oscillation and a wave motion cannot be drawn in any precise way but the typical phenomena of the two kinds are sufficiently distinct to warrant the distinctive names. As an example of an oscillation we may take the motion of a homogeneous liquid in a fixed vertical U tube under the influence of gravity. When the system is in equilibrium the levels of the free surfaces in the two limbs of the U tube are the same but a motion may be initiated by applying an increased pressure momentarily in one of the limbs. In the absence of viscosity each free surface would then oscillate in a simple harmonic motion whose periodic time depends on gravity (g) and the length (l) of the tube between the free surfaces (see § 10.2). As an example of wave motion we may take a regular train of waves at the surface of the sea. Let z be the vertical displacement of a particle of fluid lying in the surface. Then in the simplest case we have

$$z = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \tag{10.1,1}$$

where a, T and λ are constants, t is the time and x is the horizontal coordinate of the particle in its undisturbed position, measured from a fixed origin in a direction perpendicular to the wave crests (which are here parallel straight lines). In general the fluid particle will also oscillate horizontally but we need not consider this further at present. It is evident from (10.1,1) that z will be constant when

$$\frac{t}{T} - \frac{x}{\lambda} = \varepsilon \tag{10.1,2}$$

and ε is any constant. This yields

$$x = \left(\frac{\lambda}{T}\right)t - \varepsilon\lambda \tag{10.1,3}$$

and we see that the vertical displacement of the surface will be constant at a point which advances along the surface in the direction OX with the constant speed

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† Isolated waves or pulses are not periodic. Damped and divergent oscillations are not strictly periodic.

$$v = \frac{\lambda}{T} \tag{10.1,4}$$

which is called the *phase velocity* of the waves. Thus, although each individual particle of fluid merely oscillates about a fixed mean position, the whole surface profile advances over the fluid with the constant speed v. If we keep x constant in (10.1,1) we see that a given displacement z will recur at times which differ by T, so T is the *periodic time* of the motion. Likewise if we keep t constant in (10.1,1) we see that a given displacement z recurs at distances x which differ by λ, so λ is the *wavelength*. Equation (10.1,4) shows that the phase velocity is equal to the wavelength divided by the periodic time. Lastly, the constant a in (10.1,1) is the *amplitude of the wave* and the total height of the wave from trough to crest is $2a$. Equation (10.1,1) may be written in the alternative form

$$z = a \sin(\omega t - \nu x) \tag{10.1,5}$$

where

$$\omega = \frac{2\pi}{T} \tag{10.1,6}$$

and

$$\nu = \frac{2\pi}{\lambda} \tag{10.1,7}$$

The reciprocal of λ is sometimes called the *wave number* and is equal to the number of complete waves in unit length, but this term will not be employed here.

When T and λ are positive, as usual, equation (10.1,1) represents a complete train of waves propagated with the positive phase velocity v given by equation (10.1,4). Likewise

$$z = a \sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right) \tag{10.1,8}$$

represents a complete train of waves propagated with the phase velocity $-v$. When the two wave trains are superposed we have

$$\begin{aligned} z &= a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + a \sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right) \\ &= 2a \sin\frac{2\pi t}{T} \cos\frac{2\pi x}{\lambda}. \end{aligned} \tag{10.1,9}$$

When $x = \frac{1}{4}\lambda, \frac{3}{4}\lambda, \ldots [(2n+1)/4]\lambda$, this yields $z = 0$ for all values of t. Hence (10.1,9) represents a system of *standing waves* and the *nodes* are parallel straight lines at the distance apart $\lambda/2$.

In the foregoing the fluid (sea) is at rest except for the periodic motion associated with the waves. However, waves which are stationary with respect to one frame of reference (say the earth) may arise when a wave train is propagated with speed v relative to a fluid which is itself streaming with velocity $-v$ relative to the frame. Hence we have three cases, all of practical importance:

(i) Progressive waves in stationary fluid.
(ii) Standing waves in stationary fluid.
(iii) Standing waves in a streaming fluid.

So far, in discussing waves, we have considered only the motion of particles lying in the surface of the fluid but the motion is not confined to the surface. The amplitude of the motion falls as the distance of the point considered from the surface increases and is inappreciable at distances from the surface exceeding a few wavelengths. However, strictly speaking the whole of the fluid is involved in the wave motion. It must also be emphasised that the propagation of the waves is essentially brought about by two things (a) the inertia of the fluid and (b) gravity, which tends to maintain the surface as a horizontal plane. The interaction of these gives rise to the periodic motion, much as in the simple instance of the U tube mentioned at the beginning of this section.

The phase velocity of waves of any kind may be independent of the wavelength λ but when the phase velocity depends on λ the propagation is said to be *dispersive*. Whenever the propagation is dispersive an isolated group of waves has a velocity called the *group velocity* which differs from the phase velocity (see § 10.4). This sounds paradoxical but what happens is that any individual wave, according to the particular case, either advances through or recedes through the group while its amplitude always tends to zero as it approaches either end of the group. For gravity waves at the free surface of a fluid the propagation is, in general, dispersive but it is not dispersive when the wavelength is much greater than the depth of the fluid layer and this depth is constant (see § 10.5).

Gravity waves of the kind already described occur at the surfaces of lakes and in harbours but wave motions of a very different kind are also sometimes observed. These are called *seiches*† and are usually initiated by uneven fluctuations of barometric pressure over the surface of the lake, by winds or earthquakes. In the simplest kind of seiche the surface rises at one end of the lake and simultaneously falls at the other end while there is a nodal line across the middle where the vertical displacement is zero. For seiches of higher orders there will be several nodal lines. Apart from the influence of viscosity, the motions of all fluid particles would be simple harmonic and in phase. The periodic time is far longer than for ordinary surface waves. Seiches can be imitated on the small scale by setting up regular oscillations in water contained in bowls or trays.[†] Tides are oscillations of the oceans as a whole but these are forced oscillations and the forcing is provided by the direct gravitational action of the moon and sun on the water of the oceans. The further discussion of tides and seiches is beyond the scope of this book.[‡]

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† Pronounced *saysh*.

Finally, sound is a wave motion and here the oscillatory displacement is in the direction of propagation of the waves. Sound waves in a fluid are discussed in § 10.3.

10.2 Oscillations of Liquids in Tubes

We shall first consider the oscillation under gravity of a perfect fluid in an unbranched tube which is *rigid* and *slender*; this means that the length of the part of the tube filled with liquid is much greater than the greatest linear dimension of its bore. The bore may vary along the tube in size and shape but such variations will be supposed to be very gradual. Accordingly it is legitimate to assume that the velocity is uniform over the area of the bore at any section of the tube. Since the fluid is incompressible and the tube rigid, the product of the velocity q and cross-sectional area A is constant along the tube at any instant, i.e.,

$$Aq = Q \tag{10.2,1}$$

where Q is independent of position along the tube. In any oscillation of small amplitude the motion will be simple harmonic. Let x be the linear amplitude of the oscillation at a point where the sectional area is A. Then the amplitude of the volume displacement, which is constant along the tube, is

$$v = Ax \tag{10.2,2}$$

and the instantaneous value of the volume displacement is

$$v(t) = Ax \sin \omega t. \tag{10.2,3}$$

The amplitude of the velocity is then $\omega x = q$ and therefore

$$q = \frac{\omega v}{A} \tag{10.2,4}$$

by (10.2,2).

In order to calculate ω we apply the Principle of Energy (see § 3.2). Since the fluid is frictionless, the sum of its kinetic and potential energies is constant throughout any free motion. Let us measure the potential energy V from the state of the system when in equilibrium as datum; this datum state is reached at the middle of each complete swing. Exactly at the end of a swing, the fluid is momentarily at rest and the kinetic energy is zero; hence the total energy is equal to the potential energy V. On the other hand, at mid-swing V is zero and the total energy is equal to the kinetic energy T. Hence, by the Principle of Energy,

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[†] Seiches are not observed unless the ratio of greatest depth of the lake to its length is not too small; otherwise the motion is too heavily damped to be detected. Seiches were first discovered in 1730 in Switzerland and were first studied scientifically on Lake Geneva in the nineteenth century. In Britain extensive studies of seiches have been carried out at Loch Earn, Perthshire. For further information on seiches see Chapter XI of *Dynamical Oceanography* by Proudman.

[‡] For the mathematical theory of the tides, see Lamb's *Hydrodynamics*, also *Dynamical Oceanography* by Proudman. A more popular account will be found in *Tides and Waves* by Russell and Macmillan.

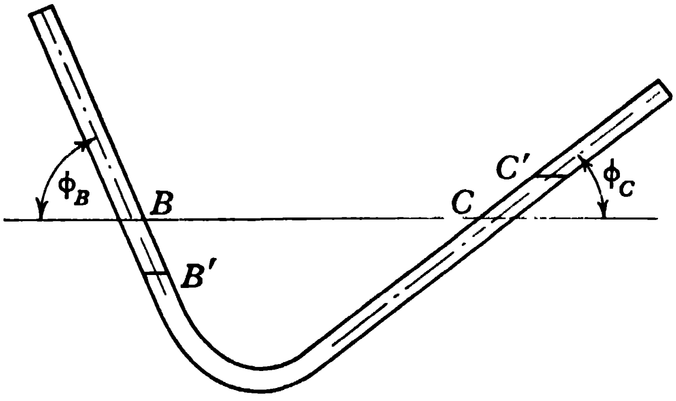

Fig. 10.2,1.

$$T = V. \tag{10.2,5}$$

Moreover, T contains ω^2 as a factor and we may write

$$T = \omega^2 T_1 \tag{10.2,6}$$

where T_1 can be calculated, as explained below. Hence (10.2,5) becomes

$$\omega^2 T_1 = V \tag{10.2,7}$$

from which ω can be found.

Let us now determine V for the U tube shown in Fig. 10.2,1. When the fluid is in equilibrium the free surfaces in the limbs are at B and C, on the same level. The sectional areas of the tube are A_B and A_C at B and C respectively while the axis of the tube is inclined to the horizontal at the angles ϕ_B and ϕ_C, as shown. At the end of a swing the free surfaces are at B' and C'. To calculate V we have merely to find the increase in gravitational potential energy brought about by transferring the fluid element BB' to CC'. This element has volume v and weight $g\rho v$ where ρ is the density of the fluid. The length of the element BB' is v/A_B and the vertical distance of its centre of gravity below B is $\frac{1}{2}(v \sin \phi_B/A_B)$. Similarly the vertical height of the centre of gravity of the element CC' above C is $\frac{1}{2}(v \sin \phi_C/A_C)$. The total rise of the centre of gravity is the sum of these and the increase of potential energy is obtained by multiplying this sum by the weight of the element. Hence

$$V = \tfrac{1}{2} g\rho v^2 \left(\frac{\sin \phi_B}{A_B} + \frac{\sin \phi_C}{A_C}\right) \tag{10.2,8}$$

We have now to calculate the kinetic energy at mid-swing. Let s be the distance measured from B along the axis of the tube to any point and let l be the total length measured from B to C. At the point corresponding to s the sectional area is $A(s)$ and the velocity is $\omega v/A(s)$ by (10.2,4). The mass of an element of length ds is $\rho\, A(s)\, ds$ and its kinetic energy is

$$dT = \tfrac{1}{2}\rho A(s)\, ds\left(\frac{\omega v}{A(s)}\right)^2 = \tfrac{1}{2}\,\frac{\omega^2 \rho v^2\, ds}{A(s)}.$$

Hence

$$T = \tfrac{1}{2}\omega^2\rho v^2\int_0^l \frac{ds}{A(s)} \tag{10.2,9}$$

and

$$T_1 = \tfrac{1}{2}\rho v^2\int_0^l \frac{ds}{A(s)}. \tag{10.2,10}$$

Accordingly (10.2,7) yields

$$\omega^2 = \frac{g\left(\dfrac{\sin\phi_B}{A_B} + \dfrac{\sin\phi_C}{A_C}\right)}{\displaystyle\int_0^l \frac{ds}{A(s)}} \tag{10.2,11}$$

on account of (10.2,8). When the tube is of uniform bore this becomes

$$\omega^2 = \frac{g(\sin\phi_B + \sin\phi_C)}{l} \tag{10.2,12}$$

and when, in addition, the limbs of the tube at B and C are both vertical we get

$$\omega^2 = \frac{2g}{l}. \tag{10.2,13}$$

The periodic time is now the same as that of a pendulum of length $l/2$.

Before leaving the unbranched tube we shall consider the two cases shown in Fig. 10.2,2 where the liquid is supposed to be held in equilibrium by a suitable *constant* pressure difference and then set in oscillation. It is easy to verify that in case (a) the expression for V is obtained from that for the U tube (equation (10.2,8)) by the substitutions

$$\phi_B = -\theta_B$$
$$\phi_C = \theta_C$$

while T is still given by (10.2,10). Hence (10.2,11) becomes

$$\omega^2 = \frac{g\left(\dfrac{\sin\theta_C}{A_C} - \dfrac{\sin\theta_B}{A_B}\right)}{\displaystyle\int_0^l \frac{ds}{A(s)}}. \tag{10.2,14}$$

In Fig. 10.2,2(b) the lower end of the tube dips into fluid contained in a large vessel and is sucked up the tube till the free surface is at C. Since we now have only one variable free surface we obtain from (10.2,14)

$$\omega^2 = \frac{g \sin \theta_C}{A_C \int_0^l \frac{ds}{A(s)}} . \tag{10.2,15}$$

Whenever the free surface at B is absent the term in $\sin \phi_B$ must be omitted from the expression (10.2,8) for the potential energy.

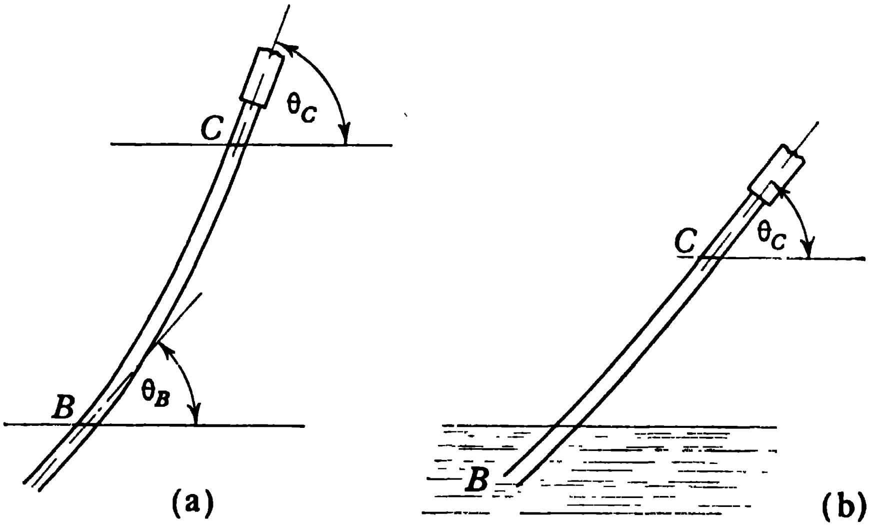

Fig. 10.2,2.

When a mass m is constrained by a spring of stiffness c and has displacement x in a straight line the equation of motion is

$$m \frac{d^2x}{dt^2} + cx = 0. \tag{10.2,16}$$

The free motion is simple harmonic and

$$\omega^2 = \frac{c}{m} \tag{10.2,17}$$

while
$$V = \tfrac{1}{2}cx^2 \tag{10.2,18}$$

and
$$T = \tfrac{1}{2}m\omega^2x^2. \tag{10.2,19}$$

For the U tube V is given by (10.2,18) and the volume displacement v takes the place of the linear displacement x. It is evident on comparison of (10.2,18) with (10.2,8) that the fluid in the tube has effectively a stiffness†

$$\sigma = g\rho\left(\frac{\sin \phi_B}{A_B} + \frac{\sin \phi_C}{A_C}\right). \tag{10.2,20}$$

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† For an inverted U tube ϕ_B and ϕ_C are negative. Hence σ is negative and the fluid is unstable.

Similarly a comparison of (10.2,19) with (10.2,9) shows that the fluid has effectively an inertia

$$\mu = \rho \int_0^l \frac{ds}{A(s)}. \tag{10.2,21}$$

Since v and x have different physical dimensions it follows that σ and C (or μ and m) have different dimensions.

We have in fact

$$\sigma \mathrel{\widehat{=}} ML^{-4}T^{-2}$$

$$\mu \mathrel{\widehat{=}} ML^{-4}.$$

For the tube shown in Fig. 10.2,2(a)

$$\sigma = g\rho\left(\frac{\sin\theta_C}{A_C} - \frac{\sin\theta_B}{A_B}\right) \tag{10.2,22}$$

while for the tube shown in Fig. 10.2,2(b)

$$\sigma = \frac{g\rho\sin\theta_C}{A_C}. \tag{10.2,23}$$

In all cases the "inertia" is given by (10.2,21). Now when the mass m is subject to the action of a simple harmonic force $F \sin \omega t$ the dynamical equation (10.2,16) becomes

$$m\frac{d^2x}{dt^2} + cx = F\sin\omega t \tag{10.2,24}$$

and there is a particular solution of the form

$$x = X\sin\omega t. \tag{10.2,25}$$

We find at once by substitution that

$$\frac{X}{F} = \frac{1}{c - \omega^2 m}. \tag{10.2,26}$$

The quantity X/F is a flexibility measured under the particular conditions of simple harmonic motion and is called a *receptance*†. By analogy, if the amplitude of the volume displacement is v when the amplitude of the pressure difference across the tube is p, the quotient

$$\beta = \frac{v}{p} \tag{10.2,27}$$

is called a receptance. Moreover, its value for a simple (i.e. unbranched) tube is given by

$$\beta = \frac{1}{\sigma - \omega^2\mu}. \tag{10.2,28}$$

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† This was at one time known as a *mechanical admittance*.

The great value of receptances is that they enable us to determine the natural frequencies of oscillation of fluids in systems of pipes with extreme ease. Suppose, for example, that n pipes have a common junction and let the receptances of the individual pipes be $\beta_1, \beta_2, \ldots, \beta_n$ which, it is to be noted, are all functions of ω. Then all the pipes are exposed to the same fluctuating pressure at the junction while the algebraic sum of the volume displacements must be zero (Principle of Continuity). Consequently we must have

$$\sum_{r=1}^{n} \beta_r = 0. \tag{10.2,29}$$

This equation determines the natural frequencies of the system. When every pipe is itself unbranched we have

$$\beta_r = \frac{1}{\sigma_r - \omega^2 \mu_r} \tag{10.2,30}$$

$$= \frac{1}{\mu_r(\omega_r^2 - \omega^2)} \tag{10.2,31}$$

where $$\omega_r^2 = \sigma_r/\mu_r. \tag{10.2,32}$$

Hence (10.2,29) becomes

$$\sum_{r=1}^{n} \frac{1}{\mu_r(\omega_r^2 - \omega^2)} = 0. \tag{10.2,33}$$

Now let the pipes be numbered so that[†] $\omega_n > \omega_{n-1} > \ldots > \omega_2 > \omega_1$. Then there is one value of ω lying in each interval $\omega_r \ldots \omega_{r-1}$. The frequency equation (10.2,33) is extremely convenient for numerical work and, as we have just seen, the $(n - 1)$ roots are already roughly located.

Receptances of systems of tubes can be found merely by solving a set of simultaneous linear equations.[‡] First, take any number n of tubes in parallel, i.e. meeting at a junction J but otherwise independent of one another. Since all the tubes are subject to the same pressure at J and the total volume displacement is the sum of those in the individual tubes, the receptance β of the system is the sum of the individual receptances. Thus

$$\beta = \sum_{r=1}^{n} \beta_r. \tag{10.2,34}$$

Next, take a pair of tubes AB (No. 1) and BC (No. 2) in series, with receptances β_1, β_2 respectively. Let the amplitude of the pressure be p_A, p_B

[†] We do not here consider cases where any of these quantities are equal.

[‡] The theory is formally the same as that of the conductance of a system of electrical conductors.

at A and B respectively, while the pressure at C is constant. Then the amplitude of the volume displacement is

$$v = \beta_1(p_A - p_B)$$

for AB and

$$v = \beta_2 p_B$$

for BC. But these must be equal, so

$$\beta_2 p_B = \beta_1(p_A - p_B) \quad \text{and} \quad p_B = \frac{p_A\beta_1}{\beta_1 + \beta_2}.$$

Hence $$v = \beta_2 p_B = \frac{p_A\beta_1\beta_2}{\beta_1 + \beta_2}$$

and the receptance of the tubes in series is

$$\beta = \frac{v}{p_A} = \frac{\beta_1\beta_2}{\beta_1 + \beta_2}. \tag{10.2,35}$$

The last equation is equivalent to

$$\frac{1}{\beta} = \frac{1}{\beta_1} + \frac{1}{\beta_2}. \tag{10.2,36}$$

This is easily generalised to the case where n tubes are in series and we obtain

$$\frac{1}{\beta} = \sum_{r=1}^{n} \frac{1}{\beta_r}. \tag{10.2,37}$$

For any simply connected system the receptance can be obtained by the combined use of (10.2,34) and (10.2,37). For example, suppose we have $AB(1)$ in series with $BC(2)$, $BD(3)$ and $BE(4)$. The receptance of BC, BD and BE in parallel is $\beta_2 + \beta_3 + \beta_4$. When this is in series with AB we have

$$\frac{1}{\beta} = \frac{1}{\beta_1} + \frac{1}{\beta_2 + \beta_3 + \beta_4}$$

or $$\beta = \frac{\beta_1(\beta_2 + \beta_3 + \beta_4)}{\beta_1 + \beta_2 + \beta_3 + \beta_4}. \tag{10.2,38}$$

The theory so far considered is valid only when the fluid is incompressible and the pipes rigid. When these conditions are not satisfied it is possible to use receptances of a more general kind. Consider the tube AB and let a periodic pressure of amplitude p_A be applied at A while the pressure at B is constant. On account of the compressibility of the fluid and lack of rigidity of the pipe walls the amplitude of the volume displacement v_B at B will not, in general, be the same as that v_A at A. We define the two receptances for pressure applied at A as follows:

$$\beta_{AA} = \frac{v_A}{p_A}, \qquad \beta_{BA} = \frac{v_B}{p_A} \tag{10.2,39}$$

where the first suffix shows the point where the volume displacement occurs and the second suffix shows where the pressure is applied. Similarly, let pressure of amplitude p_B be applied at B while the pressure at A is constant and let the amplitudes of the volume displacements at A and B be v_A, v_B respectively. Then

$$\beta_{AB} = \frac{v_A}{p_B} \quad \text{and} \quad \beta_{BB} = \frac{v_B}{p_B}. \tag{10.2,40}$$

It should be noted that v_A and v_B are positive when the displacement is from A towards B. For the particular case where the fluid is incompressible and the pipe rigid we shall have

$$\left.\begin{aligned} \beta_{AA} &= \beta_{BA} = \beta \\ \beta_{AB} &= \beta_{BB} = -\beta \end{aligned}\right\} \tag{10.2,41}$$

where β is the receptance as formerly used. In the general case, let pressures of amplitudes p_A and p_B be applied in phase at A and B respectively. Then

$$\left.\begin{aligned} v_A &= p_A\beta_{AA} + p_B\beta_{AB} \\ v_B &= p_A\beta_{BA} + p_B\beta_{BB} \end{aligned}\right\} \tag{10.2,42}$$

provided that the pressure amplitudes are not so large that the relations cease to be linear. These receptances can also be applied when the fluid is viscous but then become complex numbers.

Water hammer is considered separately in § 10.8 where the rate of damping of oscillations in U tubes is also discussed.

Example. Find the natural frequencies for a pipe system consisting of three uniform pipes JA, JB and JC.

The sectional areas of the pipes are A_A, A_B and A_C and their lengths (measured from J to the free surface) l_A, l_B and l_C. The centre lines are inclined to the horizontal at the free surfaces at the angles θ_A, θ_B and θ_C. Then since there is only one free surface in each pipe

$$\sigma_A = \frac{g\rho \sin \theta_A}{A_A}$$

and there are similar formulae for the other stiffnesses. The inertias are typified by

$$\mu_A = \frac{\rho l_A}{A_A}.$$

Equation (10.2,29) is here

$$\frac{1}{\sigma_A - \omega^2\mu_A} + \frac{1}{\sigma_B - \omega^2\mu_B} + \frac{1}{\sigma_C - \omega^2\mu_C} = 0$$

and when cleared of fractions this reduces to

$$\omega^4(\mu_B\mu_C + \mu_C\mu_A + \mu_A\mu_B)$$
$$- \omega^2(\sigma_A(\mu_B + \mu_C) + \sigma_B(\mu_C + \mu_A) + \sigma_C(\mu_A + \mu_B))$$
$$+ (\sigma_B\sigma_C + \sigma_C\sigma_A + \sigma_A\sigma_B) = 0.$$

On substitution of the values of the μ's and σ's the last equation becomes when multiplied by $A_AA_BA_C/\rho^2g^2$

$$a\left(\frac{\omega^2}{g}\right)^2 - b\left(\frac{\omega^2}{g}\right) + c = 0$$

where $a = A_Al_Bl_C + A_Bl_Cl_A + A_Cl_Al_B$

$$b = A_A(l_B \sin\theta_C + l_C \sin\theta_B) + A_B(l_C \sin\theta_A + l_A \sin\theta_C) + A_C(l_A \sin\theta_B + l_B \sin\theta_A)$$

$$c = A_A \sin\theta_B \sin\theta_C + A_B \sin\theta_C \sin\theta_A + A_C \sin\theta_A \sin\theta_B.$$

10.3 Sound Waves in a Fluid

We shall now consider the theory of plane waves in an elastic medium for the case where the oscillatory displacement of the particles is in the direction of propagation. Such waves are sometimes called *waves of compression* and the medium is subject to periodic compressions and rarefactions which are resisted by its elasticity. Periodic free motion results from the interaction of the inertia and the volume elasticity of the medium. For a solid medium other kinds of elastic waves are possible since the medium possesses elasticity of shape but only waves of compression can be propagated in a true fluid. When a tuning fork is in vibration in air a sound (note) is heard by an ear in its vicinity whereas no sound is heard when the fork vibrates in a vacuum. We are thus able to identify sound as a wave motion of the compressional type, but vibrations of any kind may give rise to sound inasmuch as they may be the means of generating compressional waves in air. We shall simplify the discussion by assuming that body forces are absent, so the pressure is uniform throughout the fluid when it is undisturbed. The temperature is also uniform.

We suppose that all the particles of fluid which lie on a plane x = constant when the fluid is undisturbed have a displacement ξ in the direction OX, where ξ is a function of the variables x and t only. The fluid is at rest, apart from the wave motion, its density is ρ and its volume elasticity (bulk modulus) is K when undisturbed. These are constant throughout the fluid and the amplitude of ξ is supposed to be so small that the variations of K during the wave motion can be neglected.† Consider the plane slab of fluid consisting of the particles whose abscissae in the undisturbed state lie between x and $x + dx$. At any instant t the displacement of the particles of abscissa x is ξ whereas the displacement of the particles of abscissa $x + dx$ is $\xi + (\partial\xi/\partial x)\,dx$, where $\partial\xi/\partial x$ is the partial derivative for t constant. Thus the distance between the faces of the slab is

---

† We are here neglecting small quantities of the second order. The equation of motion is thus *linearised*

$$\left(x + dx + \xi + \frac{\partial\xi}{\partial x}\,dx\right) - (x + \xi) = \left(1 + \frac{\partial\xi}{\partial x}\right) dx.$$

Hence the extensional strain in the direction OX, which is also the volume strain since there is no displacement in the directions OY and OZ, is $\partial\xi/\partial x$, for $(\partial\xi/\partial x)\,dx$ is the increase in length of an element of initial length dx.† Consequently the pressure is

$$p = p_0 - K\frac{\partial\xi}{\partial x} \tag{10.3,1}$$

where p_0 is the pressure in the undisturbed fluid. Now let us calculate the net accelerating force on unit area of the slab. On the left hand face the force is p whereas on the right hand face it is $-(p + (\partial p/\partial x)\,dx)$. Hence the net force is $-(\partial p/\partial x)\,dx$. The mass of the element is $\rho\,dx$ and its acceleration is $\partial^2\xi/\partial t^2$. Therefore, by the Second Law of Motion,

$$\rho\frac{\partial^2\xi}{\partial t^2} = -\frac{\partial p}{\partial x}. \tag{10.3,2}$$

But, since p_0 and K are constant, we get from (10.3,1)

$$\frac{\partial p}{\partial x} = -K\frac{\partial^2\xi}{\partial x^2}$$

and the dynamical equation becomes

$$\rho\frac{\partial^2\xi}{\partial t^2} = K\frac{\partial^2\xi}{\partial x^2}. \tag{10.3,3}$$

Now let

$$a = \sqrt{\frac{K}{\rho}} \tag{10.3,4}$$

and treat K and ρ as constants. Then (10.3,3) becomes

$$\frac{\partial^2\xi}{\partial t^2} = a^2\frac{\partial^2\xi}{\partial x^2}. \tag{10.3,5}$$

This is called the *equation of wave propagation in one dimension*.‡ The most general solution has already been obtained in equation (2.14,9) and is

$$\xi = F_1(x - at) + F_2(x + at) \tag{10.3,6}$$

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† The condition for the validity of the linearisation of the dynamical equation is that the strain $d\xi/dx$ shall be small in comparison with unity.

‡ For a general three-dimensional compressional wave the dynamical equation is $\partial^2\phi/\partial t^2 = a^2\nabla^2\phi$, and this is the general equation of wave propagation.

where F_1 and F_2 are arbitrary functions which can be differentiated twice. Let us begin by considering the solution

$$\xi = F_1(x - at). \tag{10.3,7}$$

Then ξ is constant so long as $(x - at)$ is constant or so long as

$$x = at + \varepsilon$$

where ε is any constant (see also § 10.1). Hence (10.3,7) represents a wave whose phase velocity is a. Likewise $F_2(x + at)$ represents a wave whose phase velocity is $-a$ and the most general solution of the equation (10.3,5) is obtained by the superposition of waves of phase velocities a and $-a$. Since a as given by (10.3,4) does not depend on the wavelength, the propagation is not dispersive and we may call a simply *the* velocity of propagation, i.e. *the* velocity of sound in the fluid. Equation (10.3,4) is Newton's formula for the velocity of sound.

The changes of volume and pressure in a sound wave are very small (see § 9.2 and 9.7) and there is a negligible error in assuming them to be isentropic. For a perfect gas isentropic changes obey the law (equation 9.2,22)

$$pv^\gamma = \text{constant} \tag{10.3,8}$$

where v is the volume and p the pressure of the mass of gas considered, while γ is the ratio of the specific heats c_p/c_v. Now the bulk modulus is given by (see § 1.3)

$$K = -v\frac{dp}{dv}$$

and for an isentropic change

$$p = cv^{-\gamma}.$$

Hence

$$\frac{dp}{dv} = \gamma cv^{-(\gamma+1)} = -\gamma p/v$$

and

$$K = \gamma p. \tag{10.3,9}$$

Thus Newton's formula (10.3,4) for the velocity of sound becomes†

$$a = \sqrt{\frac{\gamma p}{\rho}}. \tag{10.3,10}$$

But for a perfect gas

$$\frac{p}{\rho} = RT$$

and (10.3,10) becomes

$$a = \sqrt{\gamma RT}. \tag{10.3,11}$$

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† When Newton applied his formula to find the velocity of sound in air he used the isothermal value of K, namely p, and found that the result disagreed with experiment. Laplace explained the discrepancy and brought in the factor γ under the radical.

Hence for any perfect gas the velocity of sound is equal to the square root of the absolute temperature multiplied by a constant which depends on the nature of the gas. For air at the ordinary room temperature of 15°C (288·2°K) the velocity is 1,117 ft. per sec.

The general formula (10.3,4) for the velocity of sound can be written

$$a^2 = \frac{dp}{d\rho} \tag{10.3,12}$$

since, by (1.3,1), $K = \rho\ (dp/d\rho)$. The differential coefficient is to be evaluated for an isentropic change of density.

Example 1. *Show that the pressure satisfies the equation of wave propagation*
By (10.3,1)

$$\frac{\partial^2 p}{\partial x^2} = -K\frac{\partial^3 \xi}{\partial x^3}.$$

Also

$$\begin{aligned}\frac{\partial^2 p}{\partial t^2} &= -K\frac{\partial^3 \xi}{\partial t^2 \partial x} = -K\frac{\partial}{\partial x}\left(\frac{\partial^2 \xi}{\partial t^2}\right)\\ &= -K\frac{\partial}{\partial x}\left(a^2\frac{\partial^2 \xi}{\partial x^2}\right) = -Ka^2\frac{\partial^3 \xi}{\partial x^3}\\ &= a^2\frac{\partial^2 p}{\partial x^2}.\end{aligned}$$

Example 2. *Kinematic waves in one dimension*[1]

Suppose that the concentration k of a quantity is a function only of x and t. Let the quantity be conserved and let q be its flux. Then the equation of continuity is

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0. \tag{1}$$

Assume that q is determined by k when x is given, i.e., assume $q = q(k,x)$, and let

$$c = \left(\frac{\partial q}{\partial k}\right)_{x\ \text{const.}} = c(k,x).$$

When (1) is multiplied by c it becomes

$$\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x} = 0 \tag{2}$$

since $\dfrac{\partial k}{\partial t}\left(\dfrac{\partial q}{\partial k}\right)_{x\ \text{const.}} = \dfrac{\partial q}{\partial t}$.

Equation (2) shows that q is propagated locally with velocity c. If c is constant the general solution of (2) is

$$q = F(x - ct). \tag{3}$$

10.4 Dispersive Wave Propagation

As explained in § 10.1 the propagation of waves is called dispersive when the phase velocity v is not independent of the wavelength λ; here the velocity of a group of waves as a whole differs from the phase velocity and we now proceed to investigate the group velocity w. We begin by superposing two regular wave trains of the same amplitude but of different wavelengths. Thus the total displacement is (see equation (10.1,5))

$$z = a \sin (\omega t - \nu x) + a \sin (\omega' t - \nu' x). \tag{10.4,1}$$

In view of the trigonometric identity

$$\sin A + \sin B = 2 \cos \frac{A - B}{2} \sin \frac{A + B}{2}$$

equation (10.4,1) is equivalent to

$$z_1 = 2a \cos \left[\frac{\omega - \omega'}{2} t - \frac{\nu - \nu'}{2} x\right] \sin \left[\frac{\omega + \omega'}{2} t - \frac{\nu + \nu'}{2} x\right]. \tag{10.4,2}$$

Suppose next that ω' and ν' differ only slightly from ω and ν respectively. Then (10.4,2) shows that we have a wave motion with displacement proportional to

$$z_1 = \sin \left[\frac{\omega + \omega'}{2} t - \frac{\nu + \nu'}{2} x\right] \tag{10.4,3}$$

and of variable amplitude given by

$$a_1 = 2a \cos \left[\frac{\omega - \omega'}{2} t - \frac{\nu - \nu'}{2} x\right]. \tag{10.4,4}$$

The amplitude a_1 is a maximum when

$$\frac{\omega - \omega'}{2} t - \frac{\nu - \nu'}{2} x = 0 \quad \text{or} \quad 2n\pi,$$

where n is any integer, while a_1 is zero when

$$\frac{\omega - \omega'}{2} t - \frac{\nu - \nu'}{2} x = \frac{\pi}{2} + r\pi$$

where r is an integer. It is evident that the group of waves as a whole moves with the velocity

$$w = \frac{\omega - \omega'}{\nu - \nu'}. \tag{10.4,5}$$

For, if we increase t by an arbitrary amount Δt and simultaneously increase

x by $w\Delta t$ the amplitude a_1 remains unaltered. In the limit when $\omega' = \omega + d\omega$ and $\nu' = \nu + d\nu$ the equation (10.4,5) yields for the group velocity

$$w = \frac{d\omega}{d\nu} \tag{10.4,6}$$

whereas the phase velocity is

$$v = \frac{\omega}{\nu} \tag{10.4,7}$$

on account of equations (10.1,4), (10.1,6) and (10.1,7). By (10.4,7) equation (10.4,6) can be written

$$w = \frac{d(\nu v)}{d\nu} = v + \nu\frac{dv}{d\nu}. \tag{10,4.8}$$

But

$$\frac{dv}{d\nu} = \frac{dv}{d\lambda}\frac{d\lambda}{d\nu} = -\frac{2\pi}{\nu^2}\frac{dv}{d\lambda}$$

since

$$\nu = \frac{2\pi}{\lambda}$$

and

$$\nu\frac{dv}{d\nu} = -\frac{2\pi}{\nu}\frac{dv}{d\lambda} = -\lambda\frac{dv}{d\lambda}.$$

Hence (10.4,8) becomes

$$w = v - \lambda\frac{dv}{d\lambda}. \tag{10.4,9}$$

This is Rayleigh's formula for the group velocity. It will be seen that $w < v$ when $dv/d\lambda$ is positive and in this case the individual waves advance through the group. On the other hand $w > v$ when $dv/d\lambda$ is negative and now the individual waves recede through the group. The first case is exemplified by gravity waves in a deep sea and the second by capillary waves (see § 10.5).

Example. Group velocity for gravity waves in a sea of uniform depth h.

It is shown in § 10.5 that

$$v^2 = \frac{g\lambda}{2\pi}\tanh\left(\frac{2\pi h}{\lambda}\right).$$

Hence

$$\frac{dv}{d\lambda} = \frac{g}{4\pi v}\tanh\left(\frac{2\pi h}{\lambda}\right) - \frac{gh}{2\lambda v}\operatorname{sech}^2\left(\frac{2\pi h}{\lambda}\right)$$

and equation (10.4,9) yields

$$w = \tfrac{1}{2}v + \frac{gh}{2v}\operatorname{sech}^2\left(\frac{2\pi h}{\lambda}\right).$$

When h/λ is large (waves in deep sea) this becomes $w = \frac{1}{2}v$.

10.5 Waves in a Sea of Uniform Depth

We shall treat the sea water as a perfect fluid and begin by considering the propagation of a uniform train of waves in a uniform gravitational field while surface tension will be neglected (the influence of this is discussed later). The sea, when undisturbed, is at rest and of constant depth h while the axis OX lies on the sea bed in the direction of propagation of the waves. Then the equation to the profile of the free surface at any instant is

$$z = h + a \cos \nu (vt - x). \tag{10.5,1}$$

The conditions to be satisfied are as follows:

(a) The motion is two-dimensional in the plane OXZ and is everywhere purely periodic.
(b) The velocity normal to OX is always zero on OX, i.e. OX is a streamline.
(c) The pressure at the free surface is always equal to the constant atmospheric pressure p_0.
(d) The motion is irrotational.

We shall confine attention to waves of small amplitude. This means that both a/h and a/λ are small. The squares and products of these will be neglected so the equations are linearised.

We shall employ the stream function ψ and as a matter of convenience we shall superpose a uniform and constant velocity $-v$ on the complete system so as to bring the surface profile to rest. By the Principle of Relativity this does not affect the physical relationships in any way and we now have the simplification that the motion is *steady*, for the boundaries at the surface and sea bed are now both fixed. The stream function for the combined motion† is

$$\psi' = \psi + vz \tag{10.5,2}$$

where ψ is the stream function for the motion referred to axes fixed in the earth. Now for convenience we measure x from an origin fixed relative to the surface profile. Accordingly the surface profile is

$$z = h + a \cos \nu x. \tag{10.5,3}$$

Since the motion is steady this is a streamline and ψ must be constant on (10.5,3) and on the sea bed $z = 0$. Also the motion is irrotational, so the stream function must satisfy Laplace's equation (see § 2.12). Since the equation (10.5,3) contains $\cos \nu x$ we naturally look for a harmonic function containing $\cos \nu x$ as a factor. Now $\cos \nu x \cosh \nu z$ and $\cos \nu x \sinh \nu z$ are harmonic functions and the latter fits our problem since it vanishes with z. Accordingly we shall assume

$$\psi = c \cos \nu x \sinh \nu z \tag{10.5,4}$$

---

† See §§ 2.5 and 2.6. Note that we now use the coordinates x,z in place of x,y.

and see whether we can choose c so as to satisfy all the conditions. By (10.5,2) we now have

$$\psi' = vz + c \cos \nu x \sinh \nu z \tag{10.5,5}$$

and this is constant (zero), as required, when $z = 0$. At the free surface given by (10.5,3)

$$\psi' = vh + va \cos \nu x + c \cos \nu x \sinh \nu(h + a \cos \nu x)$$

and this is to be constant. Now we have specified that a/h is to be very small and we shall therefore have at the free surface

$$\psi' = vh + \cos \nu x[va + c \sinh \nu h]$$

and this will be constant, as required, provided that

$$c \sinh \nu h + va = 0. \tag{10.5,6}$$

The kinematic conditions are now all satisfied but we have still to satisfy condition (c) regarding the pressure at the free surface. Since the motion is steady we use Bernoulli's equation

$$\tfrac{1}{2}q^2 + \frac{p}{\rho} + gz = C_0. \tag{10.5,7}$$

The components of velocity are

$$u' = -\frac{\partial \psi'}{\partial z} = -v - \frac{\partial \psi}{\partial z}$$

$$w' = \frac{\partial \psi'}{\partial x} = \frac{\partial \psi}{\partial x}$$

where $\partial\psi/\partial x$ and $\partial\psi/\partial z$ are very small in comparison with v. Therefore

$$q^2 = u'^2 + w'^2 = v^2 + 2v\frac{\partial \psi}{\partial z} \tag{10.5,8}$$

when small quantities of the second order are neglected. Hence (10.5,7) becomes

$$v\frac{\partial \psi}{\partial z} + \frac{p}{\rho} + gz = C_0 - \tfrac{1}{2}v^2 = C.$$

But the pressure is p_0 when $z = h$ and $\partial\psi/\partial z = 0$.

Hence
$$C = \frac{p_0}{\rho} + gh$$

and we obtain

$$v\frac{\partial \psi}{\partial z} + \frac{p - p_0}{\rho} + g(z - h) = 0. \tag{10.5,9}$$

In the last equation put $z = h$. It becomes

$$v\left(\frac{\partial \psi}{\partial z}\right)_{z=h} + \frac{p(h) - p_0}{\rho} = 0. \tag{10.5,10}$$

Now $p(h) = p_0 + g\rho\, a \cos \nu x$

since by (10.5,3) the free surface is at the height $a \cos \nu x$ above $z = h$† while

$$\left(\frac{\partial \psi}{\partial z}\right)_{z=h} = c\nu \cos \nu x \cosh \nu h.$$

Hence (10.5,10) becomes

$$\cos \nu x \, [(vc\nu) \cosh \nu h + ga] = 0$$

which will always be satisfied provided that

$$vc\nu \cosh \nu h + ga = 0. \qquad (10.5,11)$$

Equations (10.5,6) and (10.5,11) both provide a value for c/a and the conditions for the equality of these is

$$v^2 = \frac{g}{\nu} \tanh \nu h.$$

In view of equation (10.1,7) the last becomes

$$v^2 = \frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi h}{\lambda}\right). \qquad (10.5,12)$$

This gives the phase velocity of gravity waves of small amplitude and the group velocity is given in the Example on § 10.4. There are two special cases of (10.5,12) that deserve special attention: (1) For a very deep sea, i.e. for h/λ very large, the equation becomes

$$v^2 = \frac{g\lambda}{2\pi} \qquad (10.5,13)$$

since $\tanh x \to 1$ when $x \to \infty$.

(2) For a very shallow sea, i.e. for h/λ small, the equation becomes

$$v^2 = gh \qquad (10.5,14)$$

for $\tanh x \to x$ when $x \to 0$. The propagation is here not dispersive.

To complete the investigation we have to find the paths of the particles of fluid. When we revert to the use of axes fixed relative to the earth the expression (10.5,4) for the stream function becomes

$$\psi = c \cos \nu(vt - x) \sinh \nu z. \qquad (10.5,15)$$

Let ξ, ζ be the displacements, in the directions OX, OZ respectively, of a particle of fluid. Then

$$\frac{d\xi}{dt} = u = -\frac{\partial \psi}{\partial z} = -c\nu \cos \nu(vt - x) \cosh \nu z$$

$$\frac{d\zeta}{dt} = w = \frac{\partial \psi}{\partial x} = c\nu \sin \nu(vt - x) \sinh \nu z.$$

---

† There is an approximation here (covered by the linearisation) for we neglect the influence of the motion on the pressure over the small difference of level $a \cos \nu x$.

Now we have restricted the investigation to the case where the amplitude of the wave is very small and this implies that the orbits of the particles are very small. Accordingly we may regard x and z as constants when we integrate the last equations with respect to t. Hence

$$\begin{aligned} \xi &= -\frac{c}{v}\sin \nu(vt - x)\cosh \nu z \\ \zeta &= -\frac{c}{v}\cos \nu(vt - x)\sinh \nu z. \end{aligned} \tag{10.5,16}$$

When $(vt - x)$ is eliminated we get

$$\frac{\xi^2}{\left(\frac{c}{v}\right)^2\cosh^2 \nu z} + \frac{\zeta^2}{\left(\frac{c}{v}\right)^2\sinh^2 \nu z} = 1 \tag{10.5,17}$$

so the orbit is an ellipse whose horizontal and vertical semi-axes are $c/v \cosh \nu z$ and $c/v \sinh \nu z$ respectively. At the sea bed the vertical axis is zero while at the free surface the ratio of the axes is $\tanh \nu h$ which is very nearly unity unless the wavelength is greater than the depth. For a deep sea the orbits are circles at and near the free surface and the radius at the free surface is equal to the amplitude of the wave (see equation (10.5,6)).

We shall next consider the influence of surface tension on wave propagation and we can solve the problem very easily by making use of the Principle of Energy,† very much as in the treatment of oscillations in a U tube (see § 10.2). Let us suppose that there is a regular wave train of wavelength λ and amplitude a. Then, whether surface tension is present or not, the kinetic energy in a rectangular element of sea of length λ, depth h and unit width is equal to the potential energy in the same element (reckoned from the undisturbed state as datum). Also the stream function satisfies the same kinematic conditions as before (but condition (c) is not satisfied when surface tension is present (see Example 2 below)). Hence the stream function is given by (10.5,15) while c/a is fixed by the kinematic condition (10.5,6). The kinetic energy in a vertical column of unit width and length dx is

$$dT = \tfrac{1}{2}\rho \int_0^h (u^2 + w^2)\, dz\, dx$$

and we find after a little reduction that

$$u^2 + w^2 = \tfrac{1}{2}c^2\nu^2[\cos 2\nu(vt - x) + \cosh 2\nu z].$$

Hence $$dT = \tfrac{1}{4}\rho c^2\nu^2\left[h \cos 2\nu(vt - x) + \frac{1}{2\nu}\sinh 2\nu h\right] dx$$

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† An alternative and direct treatment is given in Example 2 below.

and the total kinetic energy in the element is

$$T = \tfrac{1}{4}\rho c^2 \nu^2 \int_0^\lambda \left[h \cos 2\nu(vt - x) + \frac{1}{2\nu} \sinh 2\nu h \right] dx$$

$$= \frac{\pi}{4} \rho c^2 \sinh \frac{4\pi h}{\lambda}$$

since $\nu = 2\pi/\lambda$. When we substitute the value of c given by (10.5,6) we get

$$T = \frac{\pi}{2} \rho v^2 a^2 \coth \frac{2\pi h}{\lambda}. \tag{10.5,18}$$

For the same element the gravitational potential energy is

$$V_g = \tfrac{1}{2} g\rho \int_0^\lambda (z - h)^2 \, dx$$

where z is given by (10.5,1). Hence

$$V_g = \tfrac{1}{4} g\rho a^2 \lambda. \tag{10.5,19}$$

The energy due to the surface tension is σ per unit area of surface. Let ds be an element of the arc of the curve (10.5,1) (with t constant). Then

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dz}{dx}\right)^2\right\}} = 1 + \frac{1}{2}\left(\frac{dz}{dx}\right)^2$$

since dz/dx is very small. Thus the additional energy associated with the extension of the surface caused by the wave is

$$V_s = \tfrac{1}{2}\sigma \int_0^\lambda \left(\frac{dz}{dx}\right)^2 dx$$

$$= \tfrac{1}{4}\sigma a^2 \nu^2 \lambda = \frac{\pi^2 \sigma a^2}{\lambda}. \tag{10.5,20}$$

Hence the total potential energy is

$$V = V_g + V_s = \frac{a^2}{4\lambda}(g\rho\lambda^2 + 4\pi^2\sigma). \tag{10.5,21}$$

By (10.5,18) and (10.5,21) the equation

$$T = V$$

yields

$$v^2 = \frac{g\lambda}{2\pi}\left(1 + \frac{4\pi^2\sigma}{g\rho\lambda^2}\right) \tanh\left(\frac{2\pi h}{\lambda}\right). \tag{10.5,22}$$

When σ is zero we recover the formula (10.5,12). If g were zero and σ finite we should have

$$v^2 = \frac{2\pi\sigma}{\rho\lambda} \tanh\left(\frac{2\pi h}{\lambda}\right) \tag{10.5,23}$$

and this gives the phase velocity of capillary waves. In the usual case where λ/h is very small this becomes

$$v^2 = \frac{2\pi\sigma}{\rho\lambda}. \tag{10.5,24}$$

Example 1. *Find the minimum phase velocity for waves in a deep sea*

When λ/h is small (10.5,22) yields

$$v^2 = \frac{g\lambda}{2\pi} + \frac{2\pi\sigma}{\rho\lambda}.$$

This is a minimum when

$$\lambda = 2\pi\sqrt{\frac{\sigma}{g\rho}}$$

and

$$v_{\min} = \sqrt[4]{\frac{4g\sigma}{\rho}}.$$

Waves shorter than this critical value of λ are sometimes defined as *ripples.*

Example 2. *Alternative investigation of the influence of surface tension*

The pressure difference across a surface is equal to the surface tension multiplied by the total curvature of the surface and the greater pressure is on the concave side. For the wave (10.5,3) the total curvature is just the curvature of the profile and this is d^2z/dx^2 when dz/dx is small, as here. (When d^2z/dx^2 is positive the surface is concave upwards.) Hence the pressure just inside the fluid is reduced by $\sigma(d^2z/dx^2)$ and equation (10.5,10) becomes

$$v\left(\frac{\partial\psi}{\partial z}\right)_{z=h} + \frac{p(h) - p_0}{\rho} - \frac{\sigma}{\rho}\frac{d^2z}{dx^2} = 0.$$

Therefore equation (10.5,11) becomes

$$vcv \cosh vh + a\left(g + \frac{\sigma v^2}{\rho}\right) = 0.$$

When c/a is eliminated from this and the kinematic equation (10.5,6) the result is (10.5,22). This can be obtained from (10.5,12) by substituting $\{g + (\sigma v^2/\rho)\}$ for g.

10.6 Water Waves in General

The theory of waves given in § 10.5 is restricted in several respects. Thus, the fluid is treated as perfect, the depth is uniform and the amplitude a of the waves so small that the squares and products of a/h and a/λ can be neglected. In this section we shall describe trochoidal waves of finite amplitude without entering fully into the theory and give a brief account of other matters related to wave motions in liquids.

When the depth h is very great (i.e. when λ/h is small) the paths of fluid particles at and near the surface are circles, according to the theory developed in § 10.5 and, in these circumstances, the radius of the orbit at a depth l beneath the free surface is

$$r = a \exp\left[-\frac{2\pi l}{\lambda}\right]. \qquad (10.6,1)$$

For, when νz is very large, we may substitute $\frac{1}{2}e^{\nu z}$ for cosh νz and sinh νz while $\frac{1}{2}e^{\nu h}$ may be substituted for sinh νh. Then, by (10.5,16) and (10.5,6) the radius of the orbit is

$$\begin{aligned} r &= -\frac{c}{v}\cosh \nu z = \frac{a\cosh \nu z}{\sinh \nu h} = a\exp\left[-\nu(h-z)\right] \\ &= a\exp\left[-\frac{2\pi l}{\lambda}\right] \quad \text{since} \quad l = h - z. \qquad (10.6,2 \end{aligned}$$

These features of the motion are retained in the theory of trochoidal waves[†] according to which the coordinates x, z at any time t of a particle of fluid are given by

$$\left.\begin{aligned} x &= x_0 + a\exp\left[-\frac{2\pi l}{\lambda}\right]\sin\frac{2\pi(vt - x_0)}{\lambda} \\ z &= z_0 + a\exp\left[-\frac{2\pi l}{\lambda}\right]\cos\frac{2\pi(vt - x_0)}{\lambda} \end{aligned}\right\} \qquad (10.6,3)$$

where x_0, z_0 are the coordinates of the centre of the circular path of the particle and serve to identify the particle. This is an example of the Lagrangean kinematic specification of a fluid motion (see § 2.2). As before, a is the amplitude of the waves at the free surface, λ is the wavelength, v is the velocity of propagation, while l is the distance of the centre of the orbit of the particle beneath the centres of the orbits of the particles lying in the free surface. It can be shown that the equations of motion and the condition of constant pressure at the free surface are *exactly* satisfied when

$$v^2 = \frac{g\lambda}{2\pi} \qquad (10.6,4)$$

and it will be noted that this is in exact agreement with the theory of § 10.5 when the depth of the fluid is very great. The displacements given by (10.6,3) exactly satisfy the requirement for continuity but they do not represent an irrotational motion. Hence these waves could not be generated by the application only of pressures at the surface of an ocean of perfect fluid at rest. They could be generated by suitably arranged surface pressures in a sea already possessing a horizontal velocity of drift, suitably graded with depth, in a direction opposed to that of the wave propagation.[‡] It readily follows from equations (10.6,3) when we put $l = 0$ that the profile of the free surface at any instant is a *trochoid.* The tracing point of the curve is at the distance a from the centre of a circle of radius $\lambda/2\pi$ which rolls beneath a horizontal straight line situated at the height $\lambda/2\pi$ above the centres of the orbits. Similarly, the locus at any instant of all the particles whose orbits have centres at a fixed level z_0 is a trochoid and the pressure is constant on this trochoid. The curvature of a trochoidal wave is numerically greater at the crests than at the troughs and this accords with observations on ocean waves. In the extreme case where $a = \lambda/2\pi$ the profile of the surface becomes a cycloid with cusps pointing vertically upward. Waves approximating to this shape can be seen near a shelving shore and curling over at the cusp then occurs. However, cuspid waves do not appear to exist in the open sea but a wedge-shaped wave crest is often seen and the total internal angle of the wedge is about 120 degrees.

[†] First published by Gerstner of Prague in 1802 and rediscovered independently by Rankine. For further details the reader may consult Lamb's *Hydrodynamics*, Chapter IX.

[‡] See Lamb, *loc. cit.*

We have stated that trochoidal waves could be generated by pressure acting at the free surface in a sea initially possessing a horizontal retrograde velocity suitably graded with depth. This indicates that irrotational waves of finite amplitude (which could be generated by the action of suitably distributed surface pressures on a sea initially at rest) are associated with a current in the direction of propagation. The drift velocity is greatest at the surface and quickly falls off as distance from the surface increases; moreover, the ratio of the drift velocity to the orbital velocity tends to zero as a/λ tends to zero and we then have the irrotational waves of small amplitude investigated in § 10.5. When the drift velocity is not zero it must be balanced by a return current somewhere else when conditions are steady.

Hitherto we have considered only waves in a layer of fluid of constant depth. However, the theory of § 10.5 can be applied with fair approximation when the depth is variable, provided that the slope of the bed is small in all directions. This enables us to explain the observed fact that the wave fronts always wheel round near a shelving shore so that they become parallel to the water's edge as they approach it. Consider a regular train of waves approaching the shore. Then the periodic time T is everywhere the same but both λ and v change with the depth h. When we equate the values of v^2 given by (10.1,4) and (10.5,12) we get

$$\frac{2\pi\lambda}{gT^2} = \tanh\left(\frac{2\pi h}{\lambda}\right) \tag{10.6,5}$$

which establishes a relation between h and λ since g and T are constant. It will be found that a reduction of h must be associated with a reduction of both λ and h/λ; since $v = \lambda/T$ the phase velocity also falls as h is reduced. In the extreme case where h/λ is very small we have

$$v^2 = gh \tag{10.6,6}$$

while

$$\lambda = T\sqrt{gh}. \tag{10.6,7}$$

Suppose now that the regular wave train impinges on a shelving shore whose line of steepest slope is inclined to the direction of propagation of the waves in deep water (see Fig. 10.6,1). Then h is small where the wave is near the shore and v is also small there. Consequently those parts of the wave front which are far from the shore advance faster than those near the shore and the front sweeps round, tending to become parallel to the shore. A most important property of a shelving beach is that it causes the dissipation of the energy of the wave motion since the waves break, leading to turbulent motion which is then destroyed by viscosity; the mechanical energy is converted into heat. In contrast, a sea wall in deep water causes reflection of the waves with comparatively little dissipation of energy.

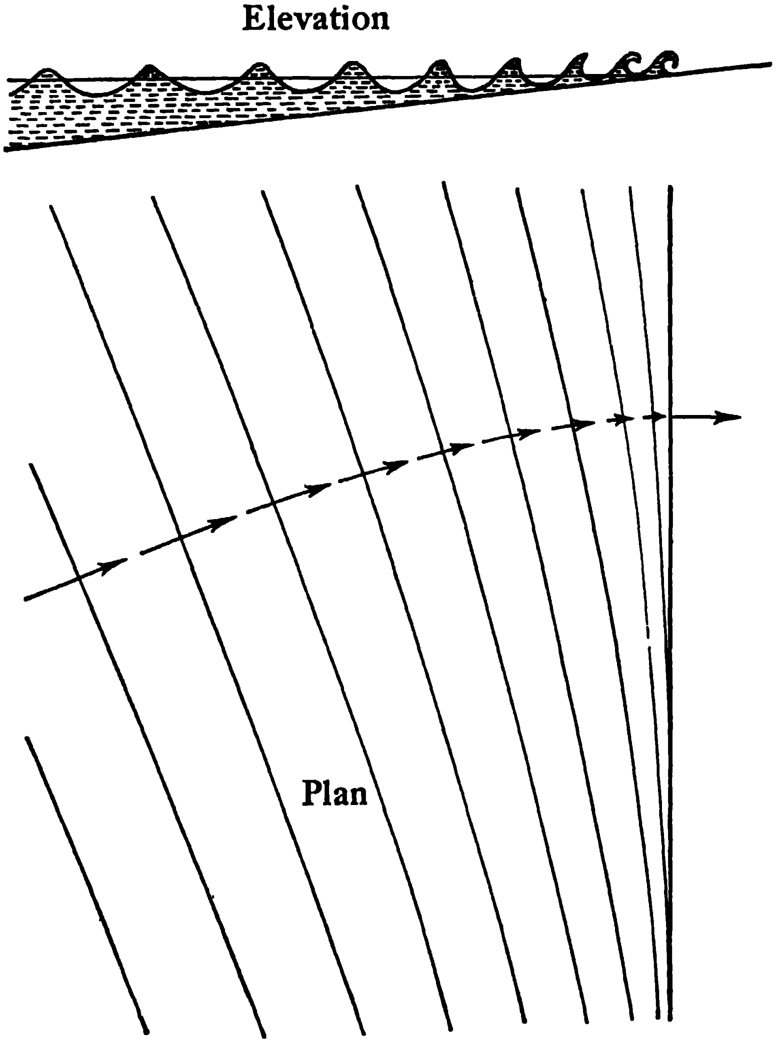

Fig. 10.6,1. Plan and elevation showing refraction and decrease of wavelength as waves approach a sloping beach.

A very brief account of *solitary waves* will now be given. These were first discovered by Scott Russell in canals[1] and were also extensively studied by him on the model scale. The solitary wave is a *single hump* on the surface which is propagated in a uniform channel without change of shape or height (in the absence of viscosity). Let h be the constant depth of the fluid where undisturbed and a the greatest elevation of the free surface in the wave, so the total depth at the summit is $(h + a)$. Scott Russell deduced from his observations and experiments that the velocity of propagation of the wave is

---

[1] *Report on Waves.* British Association Report for 1844. The theory is given in Chapter IX of Lamb's *Hydrodynamics.* The solitary wave is an example of a pulse.

$$v = \sqrt{g(h + a)} \tag{10.6,8}$$

and this is confirmed by the theory subsequently developed. According to the theory the profile of the hump is given by

$$z = a \operatorname{sech}^2 \left(\frac{x}{2b}\right) \tag{10.6,9}$$

where

$$b^2 = \frac{h^2(h + a)}{3a}. \tag{10.6,10}$$

The paths of the particles are finite segments of parabolic arcs, the axis of the parabola being vertical and the vertex above, but the particles at the bottom describe segments of a straight line. The total horizontal displacement is the same for all the particles and is in the direction of propagation. Thus the resultant effect of the wave is to move the fluid onward horizontally through a constant distance. As Scott Russell demonstrated in his experiments, the wave can be initiated in a uniform layer of fluid at rest by suddenly opening a sluice gate so as to allow fluid from a small reservoir with a raised free surface to flow into the layer. The total volume of the wave hump is equal to the volume of fluid added while the horizontal displacement of the particles is equal to this volume divided by the cross-sectional area of the channel.

A very important aspect of sea water waves is their destructive effect on structures but it is beyond our scope to discuss this here.[†] The bending and other stresses imposed by waves on ships are of fundamental importance in Naval Architecture and the reader should refer to treatises on this subject for further information.

10.7 Ship Waves and Ship Resistance

A ship floats on the water and is propelled through it; the relative motion of ship and water is associated with changes of velocity accompanied by changes in the level of the free surface in the immediate neighbourhood of the hull and wave motion at the free surface is accordingly set up. The existence of the waves implies that the ship continually communicates energy to the water and must therefore itself experience a resistance. This would be true even if the water were a perfect fluid, in contrast with the case of a deeply immersed body where the resistance would be zero in a perfect fluid. A completely but not deeply immersed body will disturb the surface to some extent and will therefore experience some resistance attributable to wave-making.

[†] See Chapter VII of *Waves and Tides* by Russell and Macmillan.

Since wave-making resistance is present even when the fluid is perfect, we shall begin the investigation with the case where the viscosity and compressibility are zero and the fluid uniform. We shall apply dimensional analysis and assume that the resistance F depends on the typical linear dimension l of the family of geometrically similar ships considered, on the relative speed V, density ρ of the fluid and acceleration g due to gravity. The conditions for similarity require that the draughts fore and aft bear constant ratios to the length at the water-line. All motions of rotation are absent and the velocity vector makes a fixed angle (normally zero) with the fore-and-aft plane of symmetry of the hull. It will be assumed that the resistance F is balanced by the pull of a tow rope and that the propelling apparatus of the ship is out of action, so it is legitimate to postulate that a completely steady state has been reached. In accordance with the usual procedure of dimensional analysis we assume that

$$F = kl^aV^b\rho^cg^d. \tag{10.7,1}$$

The corresponding dimensional relation is

$$MLT^{-2} \mathrel{\triangleq} L^a(LT^{-1})^b(ML^{-3})^c(LT^{-2})^d$$

and the indicial equations are

$$\begin{aligned} &(M) \quad 1 = c \\ &(L) \quad 1 = a + b - 3c + d \\ &(T) \; -2 = -b - 2d. \end{aligned}$$

There is an indeterminacy and it is convenient to express the values of the indices in terms of $n = -d$. We then obtain

$$\begin{aligned} a &= 2 - n, \quad & b &= 2 + 2n \\ c &= 1, & d &= -n \end{aligned}$$

so that (10.7,1) becomes

$$F = k\rho V^2l^2\left(\frac{V^2}{lg}\right)^n. \tag{10.7,2}$$

In accordance with the usual argument (see § 4.9) we conclude that

$$\frac{F}{\rho V^2l^2} = f\left(\frac{V^2}{lg}\right) \tag{10.7,3}$$

where the function f cannot be determined by dimensional analysis alone. We conclude that, when all the geometric conditions for similarity are satisfied, the non-dimensional resistance coefficient $F/\rho V^2l^2$ is a function of the non-dimensional quantity V^2/lg which is the *Froude number*. It is clear from equation (10.7,3) that we can deduce the resistance of a ship from that of its model only when the Froude number has the same value for the ship and the model. We shall assume that g is the same for the ship and the model and then the condition for equality of the Froude numbers becomes effectively that $V/\sqrt{l}$ shall have the same value for the ship and the model. Let quantities with accents refer to the model while the corresponding unaccented quantities refer to the ship. Then for a valid comparison we must have

$$\frac{V'}{\sqrt{l'}} = \frac{V}{\sqrt{l}} \tag{10.7,4}$$

and then, by (10.7,3),

$$\frac{F'}{\rho' V'^2 l'^2} = \frac{F}{\rho V^2 l^2}$$

or

$$\frac{F'}{\rho' l'^3} = \frac{F}{\rho l^3} \tag{10.7,5}$$

on account of (10.7,4). When, in addition, $\rho' = \rho$ we may write

$$\frac{F'}{\Delta'} = \frac{F}{\Delta} \tag{10.7,6}$$

where Δ is the displacement of the ship and Δ' is that of the model, since for similar bodies Δ is proportional to l^3. Equations (10.7,4) and (10.7,6) together constitute *Froude's Law of Comparison* (see also Example 2 of § 4.4).

In the foregoing investigation we have neglected both the viscosity and the surface tension of the fluid. It is unnecessary to go through the whole process of dimensional analysis again since we can obtain the result at once by applying the Pi Theorem (see § 4.8). We have to use two non-dimensional quantities, one containing the viscosity μ and the other the surface tension σ. Obviously the Reynolds number (see § 4.10)

$$R = \frac{\rho V l}{\mu} = \frac{V l}{\nu} \tag{10.7,7}$$

contains μ and is non-dimensional while the Weber number

$$W = \rho \frac{V^2 l}{\sigma} \tag{10.7,8}$$

is non-dimensional and contains σ. Hence we now have in place of equation (10.7,3)

$$\frac{F}{\rho V^2 l^2} = f\left(\frac{V^2}{lg}, \frac{Vl}{\nu}, \frac{\rho V^2 l}{\sigma}\right). \tag{10.7,9}$$

For the full-scale ship surface tension is negligible and this is usually also true for the model. When we neglect the Weber number we get

$$\frac{F}{\rho V^2 l^2} = f\left(\frac{V^2}{lg}, \frac{Vl}{\nu}\right) \tag{10.7,10}$$

It is not possible in tank tests to ensure that both the Froude and Reynolds numbers for the model have the same values as for the ship. It is usual to make the tank test at the correct Froude number and to attempt to estimate the influence of viscosity by a process of analysis (see Example 1 of § 4.10) based on the assumption that the total resistance is the sum of the wave-making and skin-frictional resistances, each quite independent of the other. However, this assumption cannot be justified because the presence of the boundary layer (see § 6.2) effectively modifies the size and shape of the hull and, with this, the wave-making resistance, while the thickness of the boundary layer depends on the Reynolds number. Thus, even if we admit as an approximation that the frictional resistance is uninfluenced by the Froude number, it is certainly untrue that the wave-making resistance is independent of the Reynolds number. No entirely satisfactory treatment of this question has yet been devised.

It is a fact established by experiment that the wave-making resistance of a ship increases rapidly with speed but the increase is not of the simple and regular type which could be represented by a 'power law', i.e. resistance proportional to a positive power of the speed. This can be explained in a general way if it be assumed that the resultant wave system of the ship is mainly due to two sources of disturbance, one situated near the bow and the other near the stern; these are the regions where the cross-sectional area of the hull changes rapidly and the approximately parallel 'middle body' is relatively ineffective in generating waves. Now if the speed is such that the wave trains generated at the bow and stern reinforce each other, a large amount of energy will be 'radiated' in the waves and the resistance will be high. However, if the speed is such that the two wave systems largely annul each other by interference, the resistance will be low. Hence the curve of resistance on a base of speed shows considerable waviness, though the slope never in practice becomes negative.

10.8 Water Hammer

Whenever the rate of flow of a fluid in a pipe is varied pressure changes arise due to the inertia of the fluid. It is a matter of everyday observation that when a valve in a pipe conveying water is rather suddenly closed a hammering noise is produced. This is a manifestation of a sudden and large increase of pressure and is the origin of the name 'water hammer'. Sometimes the hammering persists with regular periodicity unless action is taken to damp out the periodic motion. Especially in long pipe lines, water hammer is potentially destructive and arrangements must be made to limit the pressure rise, as by the provision of 'surge tanks' (see below).

When the rate of change of velocity of flow in a pipe conveying liquid is small it may be adequate to treat the liquid as incompressible and the pipe as rigid. In general, however, it is necessary to take account of the compressibility of the fluid while the lack of rigidity of the pipe wall complicates the phenomena. As a consequence of the elasticity of the fluid and of the pipe wall, pressure changes are propagated by waves and not instantaneously. It is sufficiently accurate in this context to assume that the pressure waves are propagated relative to the fluid with a fixed velocity c which is, however, somewhat less than the velocity of sound in the fluid; the reduction in the wave velocity is caused by the elasticity of the pipe wall. Since in practice c is enormously greater than the velocity of flow there is no appreciable error in taking c to be the velocity of the pressure waves relative to the pipe. In accordance with the definition given in § 9.8, even the most severe waves occurring in water hammer are *weak* shock waves but the presence of the elastic pipe walls modifies their velocity.

It is instructive to begin the theoretical investigation with the case where the fluid is incompressible and the pipe rigid; we shall at first also take the fluid to be inviscid. The extension of Bernoulli's theorem expressed by equation (3.4,18) provides a complete solution to any problem of accelerated flow in a pipe when the foregoing assumptions are adopted. As a simple case, take a uniform horizontal pipe of length l filled with fluid of density ρ and suppose that the velocity of flow is varied by operating a valve at the outlet end of the pipe. Let p_0 be the pressure at the upstream end of the pipe while p is the pressure just upstream of the valve. Equation (3.4,8) yields

$$p - p_0 = -\rho l \frac{dq}{dt} \tag{10.8,1}$$

and the last equation could be obtained at once by applying Newton's Second Law of Motion to a column of fluid of length l and unit sectional area. Since the fluid is at present regarded as frictionless, the pressure just upstream from the valve would be p_0 if the flows were steady, so

$$\rho l \left(-\frac{dq}{dt}\right)$$

is the pressure caused by the acceleration of the flow. This pressure is positive when the valve is being closed so that the rate of flow decreases with time. The more rapid the closure the greater is the pressure rise and for instantaneous closure the pressure rise would be infinite. However, the assumptions of this simple theory are invalid for instantaneous closure (which is itself an unattainable abstraction) and, as we shall see, the pressure rise is in fact finite. If the fluid were viscous (but still incompressible) there would be an additional rise of pressure at the valve corresponding to the reduced frictional loss of pressure over the pipe as the velocity falls.

Now let us examine the pressure changes on the basis of wave propagation and take first the theoretically simple case of instantaneous and complete valve closure. Then, as explained above, a wave travels along the pipe in the upstream direction with velocity c. The fluid in front of the wave has velocity q while that behind it is at rest. Take a column of fluid of unit sectional area with its axis parallel to the axis of the pipe. Then in time dt fluid in the column which initially had the velocity q is brought to rest over a length $c\,dt$. Consequently the momentum lost in time dt is $\rho qc\,dt$ and this must be equal to $(p - p_0)\,dt$ by the Principle of Momentum (see § 3.2) for the net retarding force is $(p - p_0)$ and its impulse is $(p - p_0)\,dt$. Accordingly we obtain the equation

$$p - p_0 = \rho qc. \tag{10.8,2}$$

Next suppose that the velocity is suddenly reduced from q_1 to q_2 and let the fluid be of density ρ_1 where it is unaffected by the wave but of density ρ_2 after the wave has passed. The loss of momentum in time dt is $(\rho_1 q_1 - \rho_2 q_2)c\,dt$ and the Principle of Momentum now yields

$$p - p_0 = (\rho_1 q_1 - \rho_2 q_2)c. \tag{10.8,3}$$

For a liquid there is little error in taking $\rho_1 = \rho_2 = \rho$ and equation (10.8,3) becomes

$$p - p_0 = \rho(q_1 - q_2)c. \tag{10.8,4}$$

In the above elementary theory we have supposed the velocity to be uniform across the pipe at any section and the wave front to be a plane normal to the axis of the pipe. Provided that the wave front advances without change of form the argument remains valid for any shape of wave front. For a real fluid the velocity will not be constant across the section but equations (10.8,2) to (10.8,4) will remain valid if we replace q by the mean velocity v throughout the equations.

The pressure changes associated with water hammer are complicated by wave reflections at the ends of the pipe and at any junction or change of section. We shall content ourselves here with considering reflection at the ends of a uniform pipe and shall use the theory developed in § 10.3 but use c in place of a for the velocity of propagation. The aim is to develop a theory of general applicability and the abscissa x is measured along the pipe in whichever direction is convenient. Suppose that a single wave, which we shall call the *incident wave*, is moving in the positive sense of OX. Then, as explained in § 10.3, the displacement in this wave may be written

$$\xi = F_1(x - ct).$$

Suppose that the pipe is closed at the point whose abscissa is x_1. The wave reflected from this closed end will travel in the opposite sense to the incident wave. Hence the displacement in the reflected wave alone may be written

$$\xi = F_2(x + ct)$$

where the function F_2 is still to be determined from the condition that the total displacement is always zero when $x = x_1$. Accordingly, the complete expression for the displacement when both waves are present is

$$\xi = F_1(x - ct) + F_2(x + ct) \tag{10.8,5}$$

and the condition at the closed end is

$$F_2(x_1 + ct) = -F_1(x_1 - ct) \tag{10.8,6}$$

for all values of t. Then

$$F_2(x + ct) = F_2(x_1 + ct')$$

where

$$t' = t + \frac{x - x_1}{c}$$

and by (10.8,6)

$$F_2(x_1 + ct') = -F_1(x_1 - ct').$$

Therefore we have

$$\begin{aligned} F_2(x + ct) &= -F_1(x_1 - ct') \\ &= -F_1(2x_1 - x - ct) \end{aligned} \tag{10.8,7}$$

and equation (10.8,5) becomes

$$\xi = F_1(x - ct) - F_1(2x_1 - x - ct). \tag{10.8,8}$$

It is easy to verify that this makes ξ always zero when $x = x_1$. Next, by equation (10.3,1) the pressure is given by

$$\frac{p_0 - p}{K} = \frac{\partial \xi}{\partial x} = F_1'(x - ct) + F_1'(2x_1 - x - ct). \tag{10.8,9}$$

Hence when $x = x_1$, we have

$$\frac{p_0 - p}{K} = 2F_1'(x_1 - ct). \tag{10.8,10}$$

If the incident wave alone were present the pressure would be given by (10.8,10) with the factor 2 omitted. Hence the effect of a closed end is to *double* the pressure change at the closed end due to the incident wave alone. Next let us take the case where the pressure is always equal to p_0 when $x = x_0$ (pipe open to reservoir or atmosphere at this section). The expression for the displacement in the incident wave is again taken as

$$\xi = F_1(x - ct)$$

and the total displacement when both incident and reflected waves are present is now

$$\xi = F_1(x - ct) + F_1(2x_0 - x - ct) \tag{10.8,11}$$

for this gives

$$\frac{p_0 - p}{K} = F_1'(x - ct) - F_1'(2x_0 - x - ct) \tag{10.8,12}$$

and the expression on the right is always zero when $x = x_0$. We can summarise these results as follows:—

(a) The wave reflected from a closed end has the same pressure at the closed end as the incident wave but the displacement there is reversed. Thus the resultant pressure change at the closed end is twice that in the incident wave while the resultant displacement there is zero.

(b) The wave reflected from an open end has the same displacement there as for the incident wave but the pressure change is reversed. Thus the resultant pressure change is zero at the open end but the displacement there is twice that in the incident wave.

It should be noted that the value of the bulk modulus K to be used in (10.8,9) and onward is that of the fluid as modified by the influence of the elasticity of the pipe wall (see below).

In order to make clearer the implications of the foregoing theory of reflection let us suppose, for simplicity, that the incident wave is a pulse in which the pressure change is everywhere positive. Then the wave reflected from a closed end will likewise have a positive pressure change everywhere; consequently there will be augmentation of the pressure change where the incident and reflected waves overlap. However, when the same incident wave is reflected at an open end, the pressure change in the reflected wave will be negative and there will be some cancellation of pressure changes where the incident and reflected waves overlap. It is easy to construct diagrams showing the time history of the pressure changes by making use of the rules giving the influence of closed and open ends in reflection and it assists matters to have the original incident wave reproduced on several strips of tracing cloth; when reflection at an open end occurs the strip is turned over its lengthwise axis so as to reverse the pressure changes.

A question still to be examined is the value of the effective bulk modulus for the fluid in the pipe. We shall denote this by K and use K_0 for the true bulk modulus of the fluid. When the pressure in the pipe rises the wall is subject to additional stresses giving rise, in general, to both circumferential and longitudinal strains. The longitudinal strains will be associated with longitudinal displacements of the pipe wall. However, we shall assume here for simplicity that this longitudinal displacement has no influence on the wave propagation in the fluid and we shall also assume that there is no tensile or compressive longitudinal load in the pipe wall at any time. In such circumstances Lamé's theory of thick circular cylinders† gives the following expression for the circumferential extensional strain associated with an excess of internal over external pressure amounting to p

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† See, for example, Morley's *Strength of Materials*, Chapter XI.

$$e_c = \frac{[D^2 + 2(1 + \eta)T(D + T)]p}{2ET(D + T)} \tag{10.8,13}$$

where D is the bore, T the wall thickness, E is Young's modulus for the material of the wall and η is Poisson's ratio for this. The fractional increase of the sectional area of the bore is accurately

$$(1 + e_c)^2 - 1$$

and this reduces to $2e_c$ when the square of the very small quantity e_c is neglected. Suppose now that a pair of pistons are fitted into the pipe at unit distance apart and let thrusts be applied producing the pressure p. Also suppose at first that the fluid is incompressible so that the volume between the pistons is constant. Then the pistons will move towards one another through the distance $2e_c$ since the product of sectional area and length remains constant. But when the fluid is compressible there will be an additional contraction equal to p/K_0. Hence the total contraction is

$$e = 2e_c + \frac{p}{K_0}. \tag{10.8,14}$$

But if the pipe were rigid and the fluid had the effective bulk modulus K we should have

$$e = \frac{p}{K}.$$

Hence

$$\frac{p}{K} = \frac{p}{K_0} + 2e_c \quad \text{or} \quad \frac{1}{K} = \frac{1}{K_0} + \frac{2e_c}{p}$$

$$\frac{1}{K} = \frac{1}{K_0} + \frac{D^2 + 2(1 + \eta)T(D + T)}{ET(D + T)}. \tag{10.8,15}$$

This equation shows that K is necessarily less than K_0. Finally, the velocity of propagation of the waves is

$$c = \sqrt{\frac{K}{\rho}}. \tag{10.8,16}$$

Strictly speaking the isentropic values of K_0 and E should be used in equation (10.8,16) but these differ little from the isothermal values.

The development of excessive pressure when a valve at the outlet end of a pipe line is closed quickly can be prevented to a large extent by providing the liquid with an escape route of some kind. An air vessel, see Fig. 10.8,1, provides such a means of escape because, as the pressure rises, the air above the liquid is compressed into a smaller volume thus allowing liquid to enter. Air vessels have been applied to pipe lines but their chief application is to reciprocating pumps where they are connected to the discharge pipe adjacent to the pump. For long pipe lines a *surge tank* is generally used to reduce the violence of pressure surges.[1] The general lay-out of a typical system is shown in Fig. 10.8,2, while some varieties of surge tank are sketched in Fig. 10.8,3.

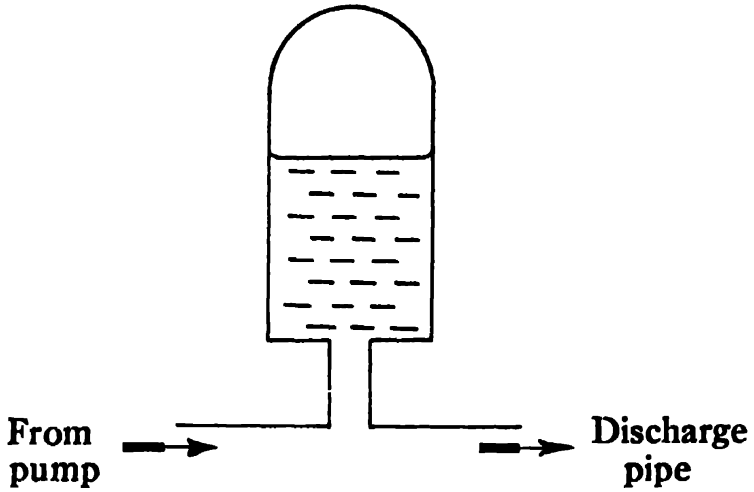


Fig. 10.8,1. 'Air vessel'.

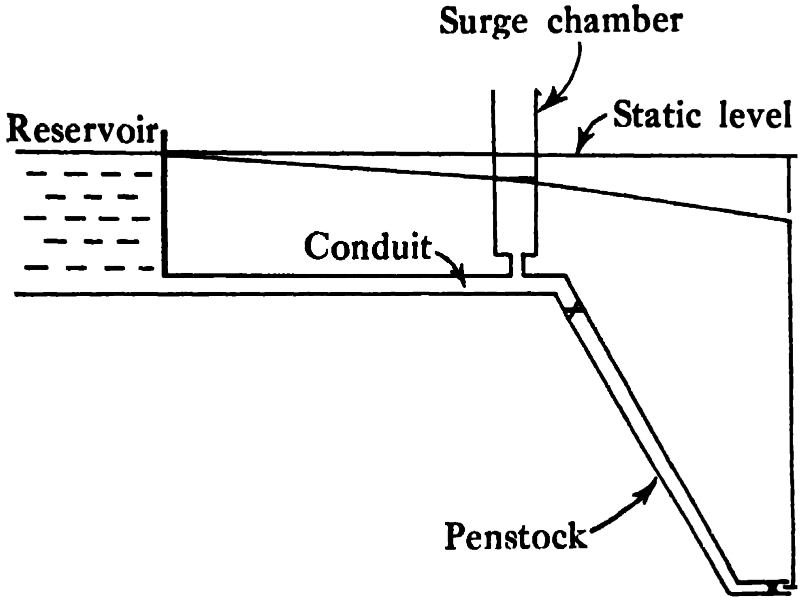
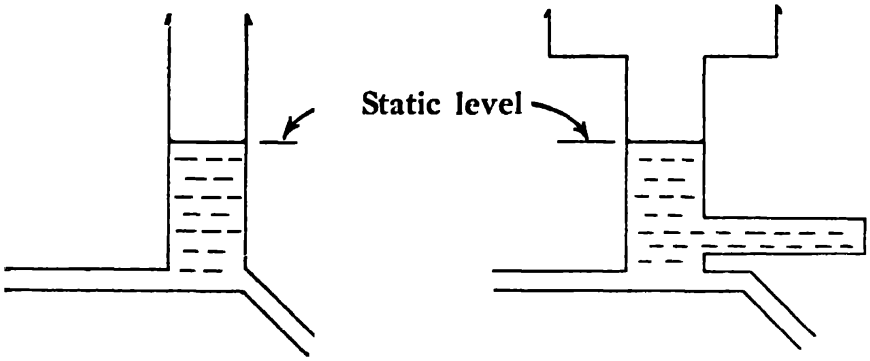
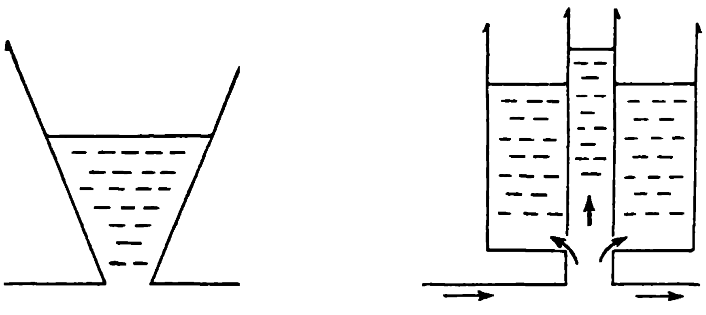


Fig. 10.8,2. Layout of conduit, surge chamber and penstock.

Two important characteristics of the system comprising the pipe line and surge tank are the periodic time and the rate of damping of the oscillations in the system. Now the system is essentially a U tube with the cross-sectional area relatively enormous in the supply reservoir and the periodic time can be calculated by the method given in § 10.2 since the resistance in the pipes has very little influence on the periodic time. In practice the flow in the system is turbulent and this complicates the calculation of the rate of damping; however, this can be found approximately by the method now to be explained. We consider one complete oscillation of the system beginning and ending with a state of rest and equate the total work done (energy dissipated) to the reduction in the potential energy. In order to calculate the energy dissipated we assume that the motion is an *undamped* oscillation whose amplitude is the mean of the amplitudes at the beginning and end of the complete oscillation; it is here that the inexactitude of the method enters but in most instances the error is small. The method can be applied for any law of resistance but we shall now assume that the pressure loss per unit length of pipe is kv^2 where k will in general vary along the pipe and v is the mean velocity at the section considered. The liquid is treated as incompressible and the pipe rigid. Take one complete half swing (beginning and ending with rest) so that there is no reversal of flow and assume at present that the amplitude of the volume displacement is constant and equal to V. Then the volume displacement at any instant may be taken as $V \sin \omega t$ and the instantaneous flux is $\omega V \cos \omega t$. If A is the cross-sectional area of the pipe the local velocity of flow is



Simple Variable Section



Conical Differential

Fig. 10.8,3. Types of surge chamber.

$$\frac{\omega V}{A} \cos \omega t$$

and the loss of pressure over the length ds of pipe is

$$\frac{k\omega^2 V^2\, ds \cos^2 \omega t}{A^2}.$$

The work done against the resistance in time dt is obtained by multiplying this pressure loss by the volume displacement in time dt, namely $dt\omega V \cos \omega t$. The work done for a half swing is therefore

$$dW = \frac{k\omega^3 V^3\, ds}{A^2} \int_{-\pi/(2\omega)}^{\pi/(2\omega)} \cos^3 \omega t\, dt = \frac{4k\omega^2 V^3\, ds}{3A^2}.$$

The work done in a complete oscillation is twice this and the energy dissipated in the whole system per cycle is therefore

$$W = \frac{8\omega^2 V^3}{3} \int \frac{k\, ds}{A^2} \tag{10.8,17}$$

where the integral covers the whole length of that part of the system where the fluid is oscillating. At the extremity of a swing the energy of the system is entirely potential and its value is (see equation 10.2,8)

$$\tfrac{1}{2} g\rho V^2 \left(\frac{\sin \phi_B}{A_B} + \frac{\sin \phi_C}{A_C}\right).$$

Suppose that in one complete oscillation the volume amplitude falls from V_1 to V_2. Then the loss of energy is

$$W = \tfrac{1}{2} g\rho (V_1^2 - V_2^2) \left(\frac{\sin \phi_B}{A_B} + \frac{\sin \phi_C}{A_C}\right).$$

When we equate this to the energy dissipated in a complete cycle of amplitude $\frac{1}{2}(V_1 + V_2)$ we get from (10.8,17) after a little rearrangement

$$\frac{V_1 - V_2}{(V_1 + V_2)^2} = \frac{2\omega^2 \displaystyle\int \frac{k\, ds}{A^2}}{3g\rho \left(\dfrac{\sin \phi_B}{A_B} + \dfrac{\sin \phi_C}{A_C}\right)}. \tag{10.8,18}$$

It follows from this equation that $(V_1 - V_2)/(V_1 + V_2)$ becomes progressively smaller as the oscillation subsides, i.e. the logarithmic rate of damping decreases with successive swings. If the resistance had been proportional to the velocity (laminar flow) the calculation would have yielded a constant value of $(V_1 - V_2)/(V_1 + V_2)$, i.e. a constant logarithmic rate of damping. When the supply end of the pipe line is connected to a very large reservoir the area A_B will be so great that the term depending on it in equation (10.8,18) is negligible.

EXERCISES. CHAPTER 10

1. Plane sinusoidal sound waves are propagated in the same direction OX. Then, provided that their amplitudes are small, the displacements ξ may be added. Two waves of the same amplitude a and slightly different frequencies are superposed. Show that the resulting motion at a given point consists of an oscillation whose frequency is the mean of the frequencies of the superposed waves and whose amplitude varies periodically with a frequency equal to the difference of the frequencies (*Phenomena of beats*).
2. Find the group velocity for capillary waves in a fluid of constant depth h (see equation (10.5,23)).
$$\left(\text{Answer } w = \frac{3}{2}v + \frac{2\pi^2\sigma h}{\rho\lambda^2 v}\,\mathrm{sech}^2\left(\frac{2\pi h}{\lambda}\right)\right)$$
3. A tube is of conical form and is bent into a U tube with vertical limbs. The total length of the column of uniform liquid is l and the areas at the free surfaces are A_1, A_2. Find ω^2 for the free oscillations under gravity.
$$\left(\text{Answer } \omega^2 = \frac{g}{l}\left(\sqrt{\frac{A_2}{A_1}} + \sqrt{\frac{A_1}{A_2}}\right)\right)$$
4. A column of liquid of total length $\frac{1}{2}l$ stands in one limb of a symmetrical U tube under the influence of a constant pressure applied to the other limb and the lower end of the column is exactly at the middle of the U. Show that the natural frequency is the same as that for a column of length l symmetrically situated in the tube. Hence prove that in this latter configuration the pressure at the middle of the U is constant during a free oscillation of the liquid. (Note. The bore of the limbs may not be constant and they need not be vertical but there must be complete symmetry about a vertical plane through the lowest point of the U.)
5. A perfectly symmetrical U tube, as described in Exercise 4, has a third limb lying in the plane of symmetry through the lowest point of the U. Show that in one free oscillation of the system the liquid in the third limb is at rest and that the frequency is the same as that for the U tube without the third limb. Show further that in the other mode of free motion a very thin rigid septum may be supposed to exist dividing the third limb and U tube into exactly equal halves. Hence show how to calculate the frequency in this free mode of oscillation.
6. For the general 3-limbed tube (see example in § 10.2) let $\mu_C = \mu_A$ and $\sigma_C = \sigma_A$. Show that the two natural frequencies are obtained from $\omega_1^2 = \sigma_A/\mu_A$ and $\omega_2^2 = (\sigma_A + 2\sigma_B)/(\mu_A + 2\mu_B)$. Apply these formulae to the symmetric 3-limbed tube of Exercise 5 and verify that the results are in accord with the statements in that Exercise.
7. A uniform U tube has vertical limbs open to atmosphere and a horizontal middle part. The right and left parts are filled with liquids of density ρ_1, ρ_2 respectively while the liquid columns meet in the horizontal part of the tube and are of lengths l_1, l_2 respectively in the state of equilibrium, measured from the junction of the liquids. Find ω for the free oscillation under gravity.
$$\left(\text{Answer } \omega^2 = \frac{g(\rho_1 + \rho_2)}{\rho_1 l_1 + \rho_2 l_2}\right)$$
8. The arrangement is the same as in the last Exercise except that the liquids of density ρ_1 and ρ_2 are separated by a column of liquid of density ρ_3 and length l_3 lying entirely in the horizontal part of the tube. Find ω.
$$(\text{Answer } \omega^2 = g(\rho_1 + \rho_2)/(\rho_1 l_1 + \rho_2 l_2 + \rho_3 l_3))$$

9. A tube of constant cross section is divided into two chambers by a thin membrane. The pressure in one chamber is p_1 and in the other is $p_1 + \Delta p$, where $\Delta p/p_1$ is small. The membrane is burst at time $t = 0$, but since Δp is small the waves that in consequence are propagated into the two chambers can be taken as sound waves and the flow conditions can be taken as isentropic. Show that at time t an expansion wave will have travelled a distance at into the chamber initially at the higher pressure and a compression wave of equal strength will have moved in the opposite direction an equal distance into the chamber initially at the lower pressure. Show that in the portion of tube between these wave fronts the air has a velocity in the direction of motion of the compression wave equal to $\frac{1}{2}\frac{a}{\gamma}\frac{\Delta p}{p_1}$ and the pressure is $p_1 + \frac{\Delta p}{2}$.

10. If in the tube of the previous question the compression wave is reflected from a closed end of the tube show that the pressure at that end becomes $p_1 + \Delta p$.

11. An element of a wave of finite amplitude will travel at the local speed of sound a relative to the local stream velocity, u. Thus, for a finite disturbance moving in the positive direction of the x axis, say, the speed of propagation of an element is

$$c = a + u.$$

If the flow conditions are assumed to be isentropic show that

$$c = a_0\left\{1 + \frac{\gamma + 1}{\gamma - 1}\left[\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{2\gamma}} - 1\right]\right\}$$

where a_0 and p_0 are the speed of sound and pressure, respectively, in the undisturbed fluid. Hence deduce that the shape of the disturbance changes as it is propagated so that the gradient $\partial p/\partial x$ for the disturbance tends to become steeper when it is negative and flatter when it is positive.

Note that a region of negative $\partial p/\partial x$ will eventually steepen to a sharp front of compression, but in a real fluid this front is stabilised by the viscous and heat conduction effects brought into play by the associated large velocity and temperature gradients. The front is then a shock wave across which the flow changes are non-isentropic.

12. Find the pressure at a point in a sea of uniform depth h at a point x,z (axes fixed to earth) at the instant t when the surface wave is of amplitude a and length λ (use $\nu = 2\pi/\lambda$).

$$\left(\text{Answer } p = p_0 + g\rho(h - z) + \frac{ag\rho \cosh \nu z \cos \nu(vt - x)}{\cosh \nu h}\right.$$

where v is given by (10.5,12.)

13. Apply the method given at the end of § 10.8 to find the rate of damping of the oscillations in a U tube when the pressure drop per unit length is kv (laminar flow). Show that the ratio of successive swings is now constant.

$$\left(\text{Answer. } \frac{V_1 - V_2}{V_1 + V_2} = \frac{\pi\omega \int \frac{k\,ds}{A}}{2g\rho\left(\frac{\sin\phi_B}{A_B} + \frac{\sin\phi_C}{A_C}\right)} = r, \text{ say.}\right.$$

$$\left.\text{Then } \frac{V_2}{V_1} = \frac{1 - r}{1 + r} = 1 - 2r, \text{ approx.}\right)$$

14. When the flow is laminar, the free oscillations in a tube will be governed by the equation

$$a\ddot{V} + b\dot{V} + cV = 0$$

where V is the volume displacement. Show that

$$b = \int \frac{k\,ds}{A}$$

where k is the resistance coefficient used in the previous Exercise while A and s have the same meanings as in § 10.8.

15. A uniform rigid pipe is filled with incompressible fluid and is open to atmosphere (pressure p_0) at its downstream end, $x = x_0$. The displacement of the fluid is given by $\xi = kt + \xi_0 \sin \omega t$ where k, ξ_0 and ω are constant. Find the pressure at any point. (Answer $p = p_0 - \rho\omega^2(x_0 - x)\xi_0 \sin \omega t$.)

16. A uniform pipe is closed at $x = x_1$ and the incident wave is the infinite train with displacement $\xi = \xi_0 \sin \omega(x - ct)$. Find the displacement in the complete motion and show that the pressure $(p_0 - p)$ is equal to that at the closed end in the incident wave (alone) multiplied by a factor f which is independent of time.

(Answer $\xi = 2\xi_0 \sin \omega(x - x_1) \cos \omega(x_1 - ct)$.
$f = 2 \cos \omega(x - x_1)$.)

17. A uniform pipe is open at $x = x_0$ and the incident wave is the infinite train with displacement $\xi = \xi_0 \sin \omega(x - ct)$. Find the displacement and pressure $(p - p_0)$ in the complete motion. Show that the complete displacement is equal to that at the open end in the incident wave (alone) multiplied by a factor f which is independent of time.

(Answer $\xi = 2\xi_0 \cos \omega(x - x_0) \sin \omega(x_0 - ct)$.
$p - p_0 = 2K\omega\xi_0 \sin \omega(x - x_0) \sin \omega(x_0 - ct)$.
$f = 2 \cos \omega(x - x_0)$.)

18. Water ($K_0 = 2{\cdot}0$ GN/m^2) is contained in a steel pipe of 100 mm bore and 3 mm wall thickness (assume $E = 200$ GN/m^2 and $\eta = 0{\cdot}27$). If the density of the water is 1000 kg/m^3, find the velocity of sound a in the water in bulk and the velocity of propagation c in the pipe.

(Answer $a = 1{,}411$ m/s; $c = 1{,}212$ m/s)

CHAPTER 11

FORCES AND MOMENTS ON BODIES IMMERSED IN A STREAM OF FLUID

11.1 Introduction

At various relevant points in previous chapters we have referred to the forces such as the lift and drag experienced by a body immersed in a stream of fluid; it is now convenient and desirable to review as a whole the subject of these forces as well as the moments experienced. In what follows we shall assume that variations in hydrostatic pressures and forces can be ignored and therefore they will not be explicitly considered†. Unless otherwise stated the discussion will be confined to an incompressible fluid.

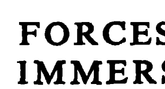


Fig. 11.1,1.

For completeness we will begin by repeating some of the salient points of § 1.2 but with a convenient change of notation. If we consider a small plane element of surface δA of the body or of an imaginary surface in the fluid, then the fluid on one side of the element will act on that element with a force which in general will have a component P normal to the element into the fluid region considered and a component S in the plane of the elements (Fig. 11.1,1).

The limits of the ratios $P/\delta A$ and $S/\delta A$ as δA tends to zero are referred to as the *normal* and *shear stress*, respectively. The fundamental property of a fluid is that it cannot sustain a shear stress and remain at rest and hence the force on an element of surface in a fluid at rest is always normal to the element. When the fluid is in motion, however, shear stresses become evident, and we can regard the viscosity of a fluid as a measure of its capacity for developing such shear stresses. More specifically we find that the stresses due to viscosity (viscous stresses) are simply related to the rates of change of shape of fluid elements in the motion as manifest in rates of elongation of line elements and rates of change of angle between such elements. These rates of change of shape are usually referred to as *rates of strain*.

Thus, suppose we have cartesian axes (x, y, z) and suppose the surface element δA at a point Q in the fluid is taken parallel to the yz plane. Then the normal and shear stress on δA can be resolved into three components parallel to the three axes which may be written

$$p_{xx},\ p_{xy},\ p_{xz}$$

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† A discussion of hydrostatic pressures and forces will be found in Chapter 1.

where the convention is adopted that the first suffix refers to the direction of the normal to δA and the second suffix refers to the direction of the component.

Similarly, if the element δA were taken parallel to the zx or xy planes at the point Q, corresponding sets of stress components would be obtained, namely,

$$p_{yx},\ p_{yy},\ p_{yz}$$

and $$p_{zx},\ p_{zy},\ p_{zz},\ \text{respectively.}$$

The complete array of nine stress components is referred to as the stress tensor for the point in the fluid considered and the general term can be written $p_{\alpha\beta}$. The tensor is readily shown to be symmetric, i.e.

$$p_{\alpha\beta} = p_{\beta\alpha}.$$

We see that p_{xx}, p_{yy} and p_{zz} are normal stress components, whilst p_{yz}, p_{zx} and p_{xy} are shear stress components.

If we now consider the rate at which a fluid element is changing shape in the neighbourhood of the point, we find that this can be related to a similar array or tensor of nine components (see Ch. 2, App. I), namely

$$e_{xx} = \frac{\partial u}{\partial x},\quad e_{yy} = \frac{\partial v}{\partial y},\quad e_{zz} = \frac{\partial w}{\partial z},\quad e_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right),$$

$$e_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),\quad \text{and}\quad e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),$$

where u, v and w are the velocity components along the axes.

It is not difficult to see that e_{xx} is the rate of extension of a line element through the point of unit length and parallel to the x axis, whilst $2e_{yz}$ is the rate of change of the angle between two line elements moving with the fluid which at the instant under consideration pass through the point and lie parallel to the axes of y and z.

Now we have learnt that in inviscid flow there are no shear stresses and it can readily be shown that the normal stress on a surface at a point is then independent of the orientation of the surface (see § 3·3). We can then write

$$p_{xx} = p_{yy} = p_{zz} = -p,$$
$$p_{yz} = p_{zx} = p_{xy} = 0,$$

where p is the static pressure.

We have also learnt that, in the case of a simple two-dimensional shear flow of a viscous fluid moving parallel to the x axis where u is a function of y only, we have in addition to the normal stress equal to $-p$ a shear stress

$$\tau = p_{xy} = \mu\frac{du}{dy} = 2\mu e_{xy},$$

where μ is the coefficient of viscosity (see § 1·4).

If we now seek to generalise these relations by assuming that the stress components are linear functions of the rate of strain components, we find that the most general relations consistent with the assumption that the medium is isotropic and incompressible are†

$$p_{xx} = -p + 2\mu e_{xx}, \quad p_{yy} = -p + 2\mu e_{yy}, \quad p_{zz} = -p + 2\mu e_{zz},$$

$$p_{yz} = 2\mu e_{yz}, \quad p_{zx} = 2\mu e_{zx}, \quad \text{and} \quad p_{xy} = 2\mu e_{xy}. \qquad (11.1,1)$$

These relations are found to be of wide validity and the general analysis of the flow of a viscous incompressible fluid is based on them. It will be noted that the normal pressure on a plane surface element at a point is no longer independent of the orientation of the surface. However, for any set of axes, the equation of continuity is

$$e_{xx} + e_{yy} + e_{zz} = 0$$

and therefore

$$p_{xx} + p_{yy} + p_{zz} = -3p, \qquad (11.1,2)$$

i.e., the mean of the normal stresses on three mutually perpendicular surface elements at a point is constant and equal to $-p$.

In general, therefore, a body in a stream of fluid will experience at all points of its surface normal and shear stresses, and these will result in a force acting through the C.G. of the body and a moment about an axis through the C.G. This force and moment can be resolved each into three components along the three axes, and for aircraft the following notation is usually adopted for the components:

	Axis component		
	x	y	z
Force	X	Y	Z
Moment	L	M	N

The convention is to take the x axis forwards in the plane of symmetry, usually—but not necessarily—in the direction of flight (or opposite to the undisturbed stream direction), the y axis normal to the plane of symmetry of the aircraft in the starboard direction, and the z axis downwards. The axes form a right handed system as illustrated in Fig. 11.1,2, and the sign conventions are those associated with such a system.

If the flight direction is in the plane of symmetry and the x axis is taken along it, then the axes are referred to as wind axes. The force component in the negative x direction is then referred to as the *drag*, whilst that in the negative z direction is called the *lift*.

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† These equations can be conveniently presented in tensor notation in the form

$$p_{\alpha\beta} = -p \, . \, \delta_{\alpha\beta} + 2\mu e_{\alpha\beta}$$

$$\text{where} \quad \delta_{\alpha\beta} = 0, \quad \text{if } \alpha \neq \beta$$
$$= 1, \quad \text{if } \alpha = \beta.$$

The other force and moment components are named as follows:

Y side-force (+ve to right or starboard)
L rolling moment (+ve port wing up, starboard wing down)
M pitching moment (+ve nose-up)
N yawing moment (+ve nose turning to right or starboard).

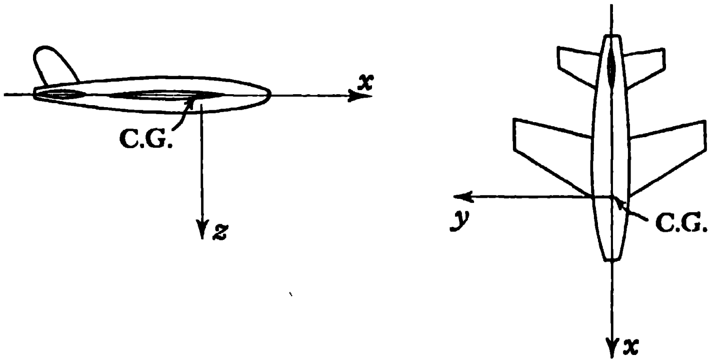


Fig. 11.1,2.

It is often convenient to express the force and moment components in non-dimensional form as coefficients, and these forms have been listed in § 4.16 (see also the Table of § 11.7).

11.2 The Generation of Lift and Drag

We have noted that an inviscid fluid sustains no shear stresses. If such a fluid is in steady motion relative to a body immersed in it, at a speed which is sub-critical for the body (see Chapter 9) and uniform far upstream, then the body experiences no force due to the motion of the fluid in the absence of circulation about it but it does in general experience a moment.

To be more specific, let us consider a wing section in the two-dimensional, steady flow of a perfect fluid. In the absence of viscous stresses the fluid slips past the wing surface and there is no mechanism for the dissipation of mechanical energy and no work is done, hence the wing can experience no drag. It could experience a force normal to the main stream direction, i.e. a lift, without violating the condition that no work can be done but, as we shall see, lift can only occur if there is circulation round the wing. If we consider a very large circuit enclosing the wing, then by Kelvin's Theorem (see § 3.7) the circulation round it must remain constant and, since this circulation was initially zero before the insertion of the wing into the flow, it must remain zero. The flow being irrotational, it follows that there can be no circulation round the wing and therefore no lift on it. The flow is then completely reversible in the sense that the streamline pattern would be unchanged were the flow reversed in direction. Typical flow patterns for a flat plate and wing at incidence in a perfect fluid are illustrated in Fig. 11.2,1.

It will be seen that there are in each case two dividing streamlines meeting the plate or wing surface at the stagnation points, S and S', and the flow negotiates without separation the sharp leading and trailing edges of the plate and the sharp trailing edge of the wing, implying the development of infinite suction there.

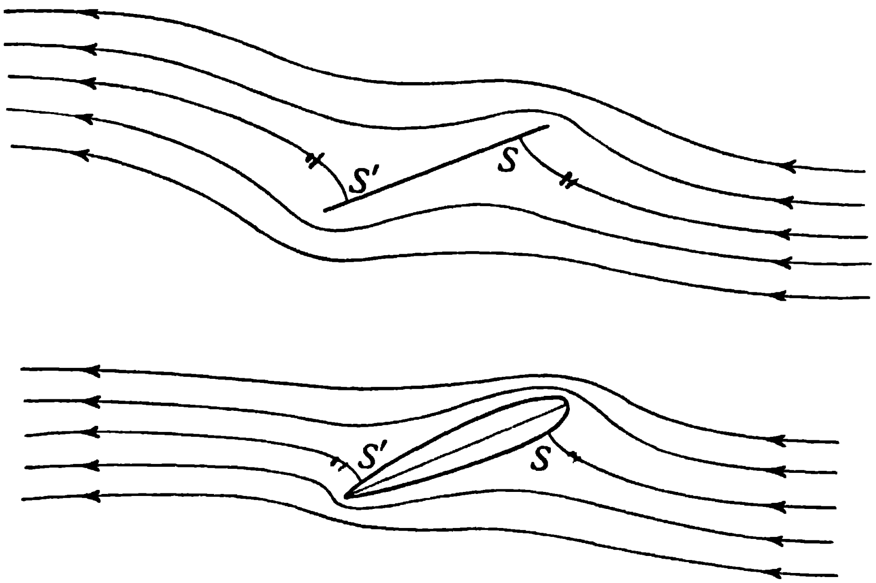


Fig. 11.2,1. Typical streamline patterns for a flat plate and a wing section at incidence with no circulation in a steady stream of inviscid fluid.

The question then arises, how does circulation and therefore lift, as well as drag, develop for a wing in a real fluid? The answer to this question lies in the viscosity of the real fluid and the consequent development of the boundary layers. As we have seen in Chapter 6, these are associated with a skin friction drag, which is the resultant in the drag direction of the shear stresses at the surface, and a boundary layer pressure drag (or form drag) which results from the change in the pressure distribution produced by the effective modification of the wing section shape due to the boundary layer. The sum of the skin friction and boundary layer pressure drags is referred to as the boundary layer drag or profile drag. The determination of this drag is described in § 6.21.

To see how the circulation develops, consider what happens to the flow pattern just after the wing section is inserted in the stream. In the first instant the flow pattern is that of an inviscid fluid, as illustrated in Fig. 11.2,1. Within a very short time, however, the boundary layers on both surfaces develop and the retarded fluid of the lower surface boundary layer then has not the full energy of the inviscid fluid required to negotiate the pressure rise from the infinite suction at the trailing edge to the rear stagnation point S' on the upper surface. In consequence the boundary layer separates from the lower surface at the trailing edge, and on the upper surface a flow is induced from the stagnation point S' towards the trailing edge, that is, in the reverse direction to that of the inviscid flow there. The result is the formation of an eddy on the upper surface in the region of the trailing edge as sketched in Fig. 11.2,2.

This flow pattern does not, however, persist as it is unstable and the eddy is swept downstream from the wing. But the total circulation round a large circuit embracing wing and eddy must by Kelvin's Theorem remain zero. Since the eddy contains vorticity, a circuit round it alone would have a non-zero circulation $-\Gamma$, say, it therefore follows that the wing must be left with circulation Γ. The effect of the circulation about the wing would be to bring the stagnation point S' nearer to the trailing edge, but the above discussion shows that in viscous flow eddies would continue to develop and be swept downstream until the circulation round the wing is such that the upper and lower surface boundary layers merge smoothly at the trailing edge and neither is required to flow round the edge. The corresponding inviscid flow would be that where the circulation is just sufficient to bring S' to the trailing edge so that the velocity and lift-loading are zero there. These final patterns of inviscid and real flow are illustrated in Fig. 11.2,3.

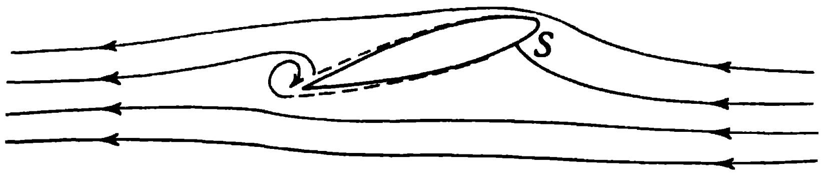
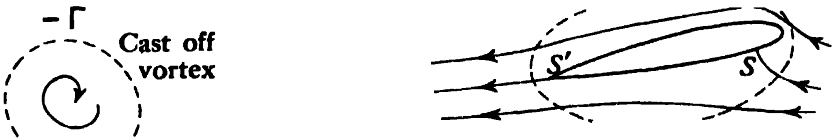
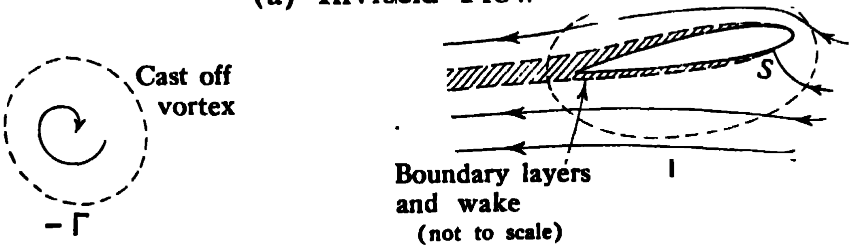


Fig. 11.2,2. Formation of eddy (starting vortex) during the process of development of boundary layer and circulation about a wing at incidence in a real fluid.



(a) Inviscid Flow



(b) Viscous Flow

Fig. 11.2,3. Flow pattern with final (steady) circulation round a wing for inviscid and viscous fluids.

There are certain small but significant differences between these final flows for inviscid and viscous fluids. In the former case we have at the trailing edge full stagnation pressure, but in the latter case the pressure there is normally only slightly above the undisturbed stream pressure. This is because, as explained in § 6.21 and §6.23, the boundary layers and wake have the effect, as far as the main flow is concerned, of modifying the shape of the wing to the extent of increasing its ordinates by the displacement thickness of the boundary layer and merging this modified shape into a 'tail' formed by the displacement thickness of the wake. At the trailing edge therefore the pressure in the real fluid is that of a flow following a relatively smooth contour, not departing greatly from the undisturbed stream direction. There are correspondingly smaller differences in pressure distribution elsewhere on the wing between the inviscid and viscous fluid cases. In consequence, the circulation and therefore the lift for the real fluid is always a little less (10–20%) than that required to bring the rear stagnation point to the trailing edge in inviscid flow. Nevertheless, the inviscid flow with this circulation provides a valuable theoretical model for the real fluid case. The theorem concerning this close correspondence is known as the *Kutta-Joukowsky* theorem, after the two scientists who independently discovered it. The condition for the circulation in inviscid flow to be such as to bring the stagnation point S' to the trailing edge is known as the *Kutta-Joukoswky condition.*

We see in the light of this discussion that not only is viscosity the cause of boundary layer drag, but it is also the cause of the development of circulation and therefore of lift. However, whilst the magnitude of the drag is strongly dependent on the magnitude of the viscosity of the medium, the circulation and lift are only slightly dependent on the magnitude of the viscosity.

We have seen that the eddy or eddies that are shed downstream have a total circulation equal and opposite to that of the wing. It is convenient to think of them as one eddy or vortex to which the name *starting vortex* or *cast-off vortex* is frequently applied. After the relatively short time required for it to be swept downstream several chord lengths its influence on the flow about the wing section becomes negligible and the flow round the latter can be regarded as steady if the incidence is kept constant and the undisturbed flow is otherwise steady. We cannot entirely forget the cast-off vortex, however, as we shall see when we come to consider the lift of wings of finite span. We should note that if for any reason the flow round the wing is changing, as for example, due to changing incidence or changing main flow direction, then a continuous stream of such vorticity will be shed from the wing trailing edge and the reaction back on the wing of this vorticity can be most important.

11.3 Relation between Lift and Circulation

In the previous paragraph we have referred to the intimate association of lift and circulation. We will now examine in more detail the nature of this association.

Consider first a wing section in inviscid flow with positive lift. The average pressure over the upper surface must then be less than the average pressure over the lower surface and therefore the mean velocity must be higher over the upper surface than over the lower surface. If we then consider the circulation round a circuit enclosing the wing section and approximating to its surface, it follows that this circulation must be non-zero. Thus we infer that the presence of a lifting force implies some circulation round the wing and vice-versa. The same argument and inference apply to a wing in a real fluid provided the circuit round the wing is taken outside the boundary layers over the wing surfaces and normal to the flow direction in the wake behind the wing. Since the flow outside the boundary layers and wake is irrotational and since under steady conditions the net vorticity shed into the wake from the wing surface must be zero, it follows that for all such circuits the circulation is constant and is equal to the integral of the vorticity taken over the cross-sectional areas of the boundary layers on the wing. This is otherwise obvious since this integral

$$= \int\int \left(\frac{\partial u}{\partial n}\right) dn\, ds,$$

where s is taken along the wing surface and n is taken normal to the wing surface and the limits of n are 0 and δ where δ is the boundary layer thickness. This double integral immediately reduces to the single integral

$$\oint u_1\, ds$$

where u_1 is the velocity just outside the boundary layer, and the integral is taken in an anti-clockwise direction round the wing.† The integral can obviously be identified as the circulation round the wing. This argument again illustrates the point that the development of circulation round a wing in a real fluid is a consequence of the development of the boundary layers on the wing surfaces.

Now consider a circular cylinder of radius a in an otherwise uniform stream of inviscid fluid but with positive circulation round the cylinder, as illustrated in Fig. 11.3,1. The x axis is taken opposite to the undisturbed stream V.

Without circulation the circumferential velocity at a point P on the cylinder surface is $2V \sin \theta$, if V is the undisturbed stream velocity, whilst the circulation Γ provides a further contribution $\Gamma/2\pi a$ (see § 2.8, Ex. 1, and § 2.10). Here θ is measured from the x axis and is positive if anti-clockwise, and the circumferential velocity is quoted as positive if in the direction of increasing θ.

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† The convention adopted here is that the circulation round a plane circuit is measured in a clockwise direction about the positive direction of the axis normal to the plane of the circuit. With cartesian axes x and y defined in the usual way in the plane, this convention results in the circulation being taken in an anti-clockwise direction as viewed by the reader.

Applying Bernoulli's equation we have, if p is the static pressure at P and p_1 the static pressure of the undisturbed stream,

$$p + \frac{\rho}{2}\left(\frac{\Gamma}{2\pi a} + 2V \sin\theta\right)^2 = p_1 + \tfrac{1}{2}\rho V^2 = H, \text{ say.}$$

Hence

$$p = H - \frac{\rho\Gamma^2}{8\pi^2 a^2} - 2\rho V^2 \sin^2\theta - \frac{\rho V \sin\theta\Gamma}{\pi a}.$$

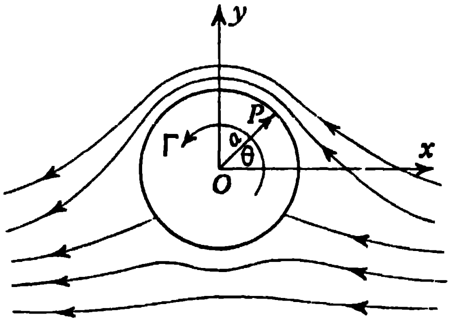

Fig. 11.3,1. Flow past a circular cylinder with circulation in inviscid flow.

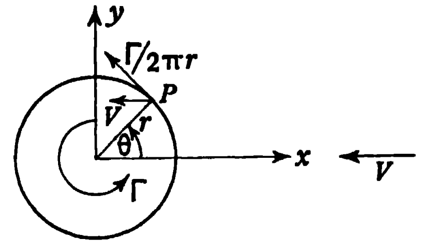

Fig. 11.3,2. Vortex of strength Γ in a stream of velocity V.

Thus, the force components per unit span on the cylinder are

$$X' = \text{drag} = \int_0^{2\pi} pa \cos\theta \, d\theta = 0,$$

$$Y' = \text{lift} = \int_0^{2\pi} pa \sin\theta \, d\theta = \int_0^{2\pi} \Gamma \frac{\rho V \sin^2\theta \, d\theta}{\pi}$$

$$= \rho V \Gamma. \tag{11.3,1}$$

This very simple relation between lift per unit span and circulation is in fact of general validity whatever the shape of the cylinder. To see this consider first a vortex of strength Γ held fixed at a point O in an otherwise uniform stream of velocity V, the direction of which is opposite to the axis Ox (see Fig. 11.3,2).

Consider the fluid which at some instant occupies a circle of radius r centred on O. At a point P on the circle the velocity is q, say, where q has components

$$-V - \frac{\Gamma}{2\pi r}\sin\theta, \quad \frac{\Gamma}{2\pi r}\cos\theta.$$

By Bernoulli's equation, if p is the static pressure at P,

$$p + \tfrac{1}{2}\rho q^2 = \text{const.} = H, \text{ say,}$$

so that

$$p = H - \tfrac{1}{2}\rho\left[\frac{\Gamma^2}{4\pi r^2}\cos^2\theta + \left(V + \frac{\Gamma}{2\pi r}\sin\theta\right)^2\right].$$

Thus the pressures acting on the circumference of the circle exert a force on it with components per unit span

$$\begin{aligned} X'_p &= -\int_0^{2\pi} pr \cos\theta \, d\theta \\ &= -\int_0^{2\pi} \left\{ H - \tfrac{1}{2}\rho\left[\frac{\Gamma^2}{4\pi r^2}\cos^2\theta + \left(V + \frac{\Gamma}{2\pi r}\sin\theta\right)^2\right]\right\} r\cos\theta \, d\theta \\ &= 0, \end{aligned} \tag{11.3,2}$$

and

$$\begin{aligned} Y'_p = -\int_0^{2\pi} pr \sin\theta \, d\theta &= \tfrac{1}{2}\rho \int_0^{2\pi} \frac{2V\Gamma}{2\pi}\sin^2\theta \, d\theta \\ &= \rho V\Gamma/2. \end{aligned} \tag{11.3,3}$$

Likewise the rate of transport of momentum into the circle has components

$$X'_M = -\int_0^{2\pi} \rho V \cos\theta \left(V + \frac{\Gamma}{2\pi r}\sin\theta\right) r \, d\theta = 0 \tag{11.3,4}$$

and

$$Y'_M = \int_0^{2\pi} \rho V \cos\theta \frac{\Gamma}{2\pi r} \cos\theta r \, d\theta = \rho V\Gamma/2. \tag{11.3,5}$$

Hence, the fluid outside the circle acts on the fluid inside with a force per unit span which has no component in the x direction but has a component in the y direction $= Y'_p + Y'_M = \rho V\Gamma$. Since the motion is steady, this must be balanced by the force with which the vortex acts on the fluid inside the circle, and hence in turn the vortex experiences a force per unit span in the y direction (a lift) equal to $\rho V\Gamma$, and a force equal and opposite to this must be applied to the vortex to keep it fixed in position.

We have seen that the circulation Γ round a wing is equal to the integral of the vorticity over its boundary layers. Viewed from a long distance away, this vorticity may be regarded as concentrated into a single vortex of strength Γ at the centre of gravity of the vorticity distribution, which is usually close to the point a quarter of the chord aft of the leading edge. For a large circle centred on this point the above analysis applies, and hence we can conclude that the lift per unit span on the wing must be equal to $\rho V\Gamma$. Thus, the theorem that

$$\text{Lift per unit span} = \rho V\Gamma \tag{11.3,6}$$

is quite general and independent of the shape of the section.

This relation is frequently referred to as the *Kutta-Joukowsky relation.*

If there are other bodies or discrete vortices present, then this relation is still found to hold for the force on a vortex if by V we understand the resultant velocity at the position of the vortex due to all these other bodies and vortices and the main stream, and the lift acts normal to the direction of this resultant velocity.

For yet another example of this theorem consider a slightly cambered plate at a small incidence in an otherwise uniform stream, as illustrated in Fig. 11.3,3.

We can regard the vorticity, that in a real fluid is distributed in the boundary layers of the plate, as located in the form of a sheet (see § 2.11) along the plate itself in an inviscid fluid, its strength being defined by $\gamma(x)$, where the x axis is taken as before in a direction opposite to the undisturbed stream direction. The vorticity thus acts in a sense as roller bearings permitting the different velocities of flow on upper and lower surfaces of the plate.

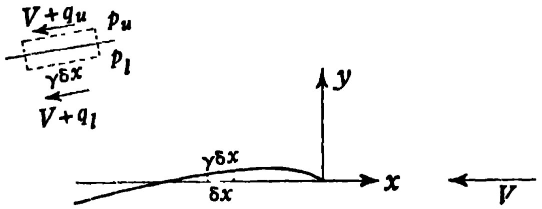

Fig. 11.3,3. Circulation and lift on a cambered plate.

Write the velocity magnitude and pressure over the upper surface as $V + q_u$, p_u, respectively, and likewise the velocity magnitude and pressure over the lower surface as $V + q_l$, p_l, respectively, where V is the magnitude of the undisturbed stream velocity. Then consider a small element δx of the plate and enclose it with a quasi-rectangular circuit as shown in Fig. 11.3,3, for which the upper and lower sides lie very close to the plate and the other two small sides are normal to the plate. The circulation round this circuit must equal $\gamma\delta x$, and hence it follows that

$$(V + q_u)\,\delta x - (V + q_l)\,\delta x = \gamma\,\delta x.$$

Hence we deduce that

$$\gamma = (q_u - q_l). \tag{11.3,7}$$

It follows that the velocities over the upper and lower surfaces can be written

$$\left.\begin{aligned} V + q_u &= V + \left(\frac{q_u + q_l}{2}\right) + \frac{\gamma}{2} \\ \text{and}\qquad & \\ V + q_l &= V + \left(\frac{q_u + q_l}{2}\right) - \frac{\gamma}{2}. \end{aligned}\right\} \tag{11.3,8}$$

The effect of the local vorticity distribution is therefore to increase the local upper surface velocity above the local mean by $(\gamma/2)$ and to decrease the local lower surface velocity below the mean by the same amount.

If we now apply Bernoulli's Theorem we have

$$p_l + \tfrac{1}{2}\rho(V + q_l)^2 = p_u + \tfrac{1}{2}\rho(V + q_u)^2$$

and hence

$$p_l - p_u = \rho(q_u - q_l)\left(V + \frac{q_u + q_l}{2}\right),$$

$$= \rho\gamma V\left[1 + \frac{q_u + q_l}{2V}\right]. \qquad (11.3,9)$$

Since we have assumed the plate camber and incidence as small, the perturbations in flow introduced by the plate are small, and hence we can neglect $(q_u + q_l)/2V$ as compared with unity in equation (11.3,9).† It then follows that the lift per unit span is given by

$$L' = \int_0^c (p_l - p_u)\, dx = \int_0^c \rho V\gamma\, dx$$

$$= \rho V\Gamma \qquad (11.3,10)$$

where Γ is the total circulation round the plate $= \int_0^c \gamma\, dx$.

It will be clear from our previous discussion that, although this result has been proved within the limitations and assumptions of a small perturbation analysis, it is in fact exact.

Circulation can be readily developed about a circular cylinder in a uniform stream by rotating the cylinder, and it then follows that it will experience a force normal to the direction of the stream and proportional to the circulation developed. This is the so-called *Magnus effect*, and a practical application of this effect that has been considered is to the propulsion of ships (the Flettner rotor). The swerve of tennis balls and golf balls when spun by being sliced derives from a similar effect.

11.4 Lift, Drag and Pitching Moment Characteristics of Aerofoil Sections

The geometry of a wing section is generally defined in terms of its thickness distribution and its camber line. For convenience we generally take its chord line as the straight line joining the centre of curvature of its nose and the trailing edge, and if we take the x axis as coincident with the chord line and the y axis as normal to it, then the camber line is defined by the mean of its y ordinates $(y_u + y_l)/2$, and its thickness distribution is defined by the algebraic difference between upper and lower surface ordinates, $y_u - y_l$, as illustrated in Fig. 11.4,1.†

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† Although $\left(\frac{q_u}{V}\right)$ and $\left(\frac{q_l}{V}\right)$ are of the first order in terms of the mean slope of the section relative to the x axis, and therefore small compared with unity, they are of opposite sign and nearly equal in magnitude, and we find that $\left(\frac{q_u + q_l}{2V}\right)$ is at most of the order of the square of the mean slope of the section.

Important geometrical parameters are the ratio of the maximum thickness to the chord (t/c), the position of this maximum thickness aft of the nose or leading edge as a fraction of the chord, the maximum camber as a percentage of the chord and its position aft of the nose as a ratio of the chord, the nose radius of curvature as a percentage of the chord, and the angle between the upper and lower surfaces at the trailing edge (trailing edge angle). A typical aircraft wing section might have a thickness chord ratio of 0·1, with the maximum thickness position at 0·4c, a maximum camber of 0·01c (1% camber), maximum camber position at 0·5c, a nose radius of 1·5% of the chord, and a trailing edge angle of 10°. However, in general, the aerodynamic requirements will largely determine the values of the geometrical parameters chosen, and, since these requirements may vary from those of a modern high speed aircraft to those of the blading in a compressor stage or a cascade in the corner of a duct, a considerable number of different types of sections have been developed. For further details the reader is referred to Abbott and Doenhoff.[1]

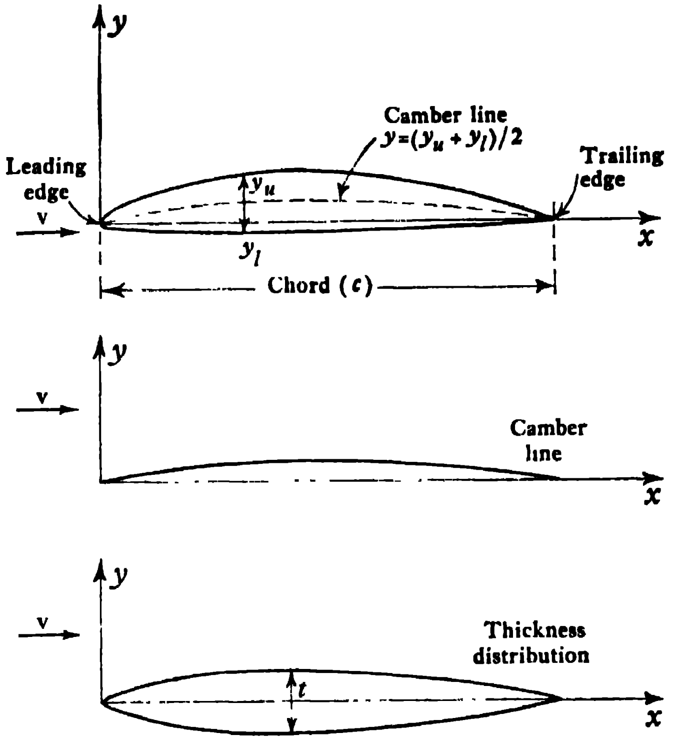


Fig. 11.4,1.

A typical variation of pressure distribution on a wing section of moderate thickness with incidence is illustrated in Fig. 11.4,2.

It will be seen that with increase of incidence the suction increases over the upper surface, particularly towards the leading edge, and with it the lift increases. However, at a sufficiently high incidence the adverse pressure gradient following the peak approaches a value for which boundary layer separation develops (see also § 6.2), and with further increase of incidence flow separation rapidly spreads over the upper surface, the peak suction falls and over the region of separated flow the pressure becomes more nearly constant over the surface whilst the trailing edge pressure and the lift fall. The wing is then said to be *stalled*. The variations of lift coefficient (C_L) and drag coefficient (C_D) with incidence (α) are illustrated in Fig. 11.4,3. Here $C_L = L'/\frac{1}{2}\rho V^2 c$, $C_D = D'/\frac{1}{2}\rho V^2 c$.

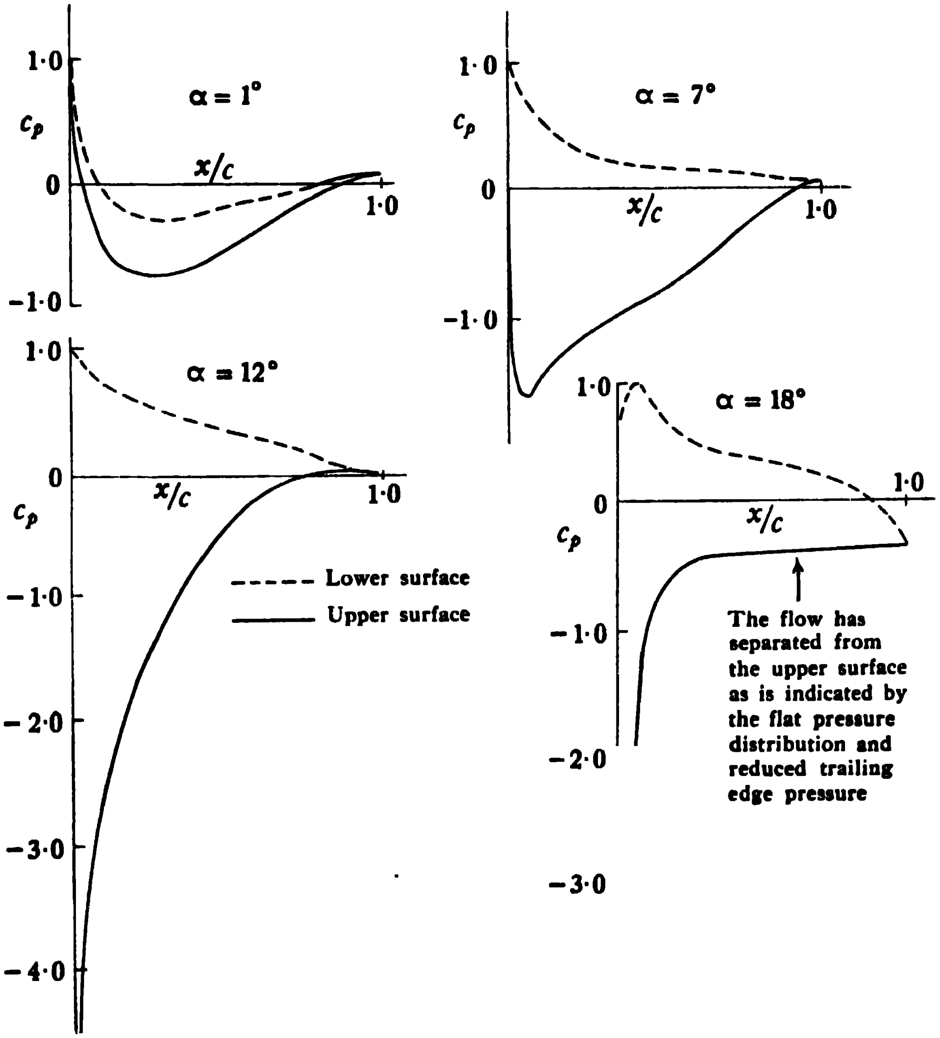


Fig. 11.4,2. Sketches of typical pressure distributions on a wing of 12% thickness and 2% camber.

It will be seen that for quite a wide range of incidence the variation of C_L with incidence is practically linear, a typical value of $dC_L/d\alpha$ being about 5·8 per radian. The incidence for which $C_L = 0$, usually denoted α_0, is a function of the wing camber,

$$\alpha_0 \text{ (in deg.)} = -\text{camber (in \%), approx.} \qquad (11.4,1)$$

The incidence for maximum lift and the maximum lift coefficient attained ($C_{L\max}$) are functions of t/c, camber, nose radius and Reynolds number. For values of t/c greater than about 10%, Reynolds numbers greater than about 10^6 and with moderate camber values of $C_{L\max}$ of the order of 1·4 may normally be expected; for such sections at the stall flow separation of the turbulent boundary layer tends to develop from the rear although there may be a small laminar separation bubble near the nose. For thinner sections, particularly those with relatively small nose radii of curvature, more extensive separation of the laminar boundary layer from close to the nose is likely, and the value of $C_{L\max}$ may be about 1·0. At sufficiently low Reynolds numbers, for which separation of the laminar boundary layer is more likely to occur than transition, the distinction between thick and thin wings is less marked and the value of $C_{L\max}$ is generally low. In the stalled state the value of C_L remains at about 0·7 — 0·8 up to quite large angles of incidence. For a fuller discussion reference may be made to Duncan.[1]

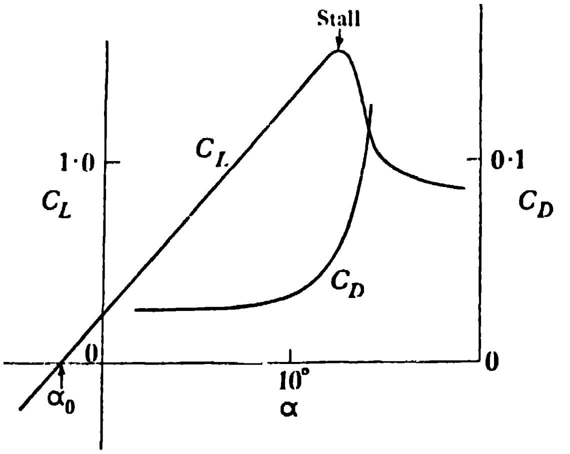


Fig. 11.4,3. Typical variation of lift and drag coefficients for a wing section of 12% thickness and 2% camber.

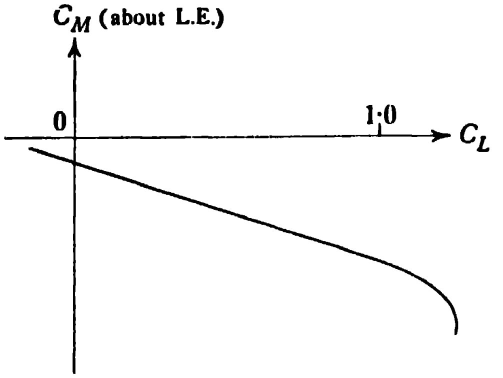


Fig. 11.4,4. Typical variation of pitching moment coefficient (measured about the leading edge) with lift coefficient.

The drag coefficient at small incidences when there is no flow separation is typically that of a good streamline shape and is very largely due to skin friction drag (see § 6.2). For incidences that are not too close to the stall the drag coefficient is almost constant, increasing only slightly with increase of incidence. However, as the stall is approached and flow separation develops from the upper surface the form drag begins to increase rapidly with increase of incidence and the drag coefficient then displays a correspondingly rapid increase.

The variation of pitching moment (C_M) about the leading edge with C_L is illustrated in Fig. 11.4,4. Here $C_M = M'$(i.e. moment per unit span)$/\frac{1}{2}\rho V^2 c^2$.

Again we note that for the range of C_L for which C_L varies linearly with incidence we get a corresponding linear relation between C_M and C_L. Whatever point on or near the chord line we choose to take as a reference point about which to measure C_M we find over a wide range of C_L a relation of the form

$$C_M = C_{M_0} + BC_L. \tag{11.4,2}$$

The quantity C_{M_0} is the same for all reference points and is referred to as the *pitching moment coefficient for zero lift*. It is mainly determined by the camber of the wing section, and its independence of the reference point is due to the fact that it is, in coefficient form, the pure couple acting on the section when the net lift is zero. A cambered wing set at zero lift would be set at a negative incidence and hence would tend to sustain a down load at the front and an up load at the rear, giving a negative moment. We find that

$$C_{M_0} = -2{\cdot}5 \times \text{camber} \tag{11.4,3}$$

approximately.

In contrast to C_{M_0} the quantity $B = dC_M/dC_L$ is a function of the position of the reference point. If the reference point is taken on the chord line and is very close to the leading edge an increase of incidence and therefore of C_L would be accompanied by a change of moment tending to reduce the incidence, i.e. the pitching moment would become more nose-down or negative, using the sign conventions of § 11.1. Hence B would be negative for such a reference point. With the reference point far back from the leading edge, however, an increase of incidence would be accompanied by a nose-up (positive) moment increment and then B would be positive. It follows that there must be one reference point on the wing chord for which B is zero. This point is referred to as the *aerodynamic centre*. Measured about the aerodynamic centre the pitching moment is constant and independent of change of incidence or lift; it must therefore be the point through which lift due to incidence change acts. We find that this point is usually very close to but slightly ahead of the quarter chord point, a result which, as we shall see, is predicted by theory. The importance of this point is apparent when the stability of a pivoted wing subjected to a small change of incidence from a trimmed state is being considered, since the quantity $(-dC_M/dC_L)$ is a measure of the restoring moment that would initially develop, and is therefore a measure of what is known as the *static stability* of the wing. But this quantity is readily seen to be the distance the aerodynamic centre is aft of the pivot as a fraction of the chord. More generally the concept of aerodynamic centre plays a vital part in the stability analysis of an aircraft (see Duncan[1]).

It is important to note that the aerodynamic centre is not the same as the centre of pressure. The latter is the point on the chord line through which the resultant force on the wing acts and this includes the effect of the basic couple represented by C_{M_0}.

Thus, suppose—referred to some point O on the chord line—we have

$$C_M(O) = C_{M_0} + BC_L,$$

If we neglect the moment due to the drag then referred to some other point P on the chord line

$$C_M(P) = C_{M_0} + \left[B + \frac{OP}{c}\right]C_L.$$

The point P is the centre of pressure if

$$C_M(P) = 0$$

and hence

$$\frac{OP}{c} = -\frac{C_{M_0}}{C_L} - B = -\frac{C_M(O)}{C_L} \tag{11.4,4}$$

Similarly, referred to a point A on the chord line

$$C_M(A) = C_{M_0} + \left[B + \frac{OA}{c}\right]C_L$$

and A is the aerodynamic centre if

$$dC_M(A)/dC_L = 0$$

and hence

$$\frac{OA}{c} = -B. \tag{11.4,5}$$

We see from equations (11.4,4) and (11.4,5) that

$$AP/c = -C_{M_0}/C_L. \tag{11.4,6}$$

11.5 Lift and Pitching Moment in Steady Two-Dimensional Potential Flow

11.5.1 Introductory Remarks

It is instructive now to examine the predictions of inviscid potential flow theory for the characteristics of aerofoil sections, making use of the Kutta-Joukowsky theorem (see § 11.2) to fix the circulation required. We shall see that these predictions are in reasonable agreement with the experimental results obtained in real fluids and described in the last paragraph, the small differences between experiment and theory being readily explained by the effective profile changes produced by the boundary layers.

11.5.2 Flow about a Flat Plate set at Incidence

We have learnt (§ 2.15) how to transform a circle into a straight line of finite length by the Joukowsky transformation. We shall use this to determine the flow about a flat plate at incidence with the circulation required by the Kutta-Joukowsky condition so that the flow streams smoothly from both surfaces at the trailing edge.

Consider a circle C of radius a with centre O at the origin of the cartesian coordinates x, y. Then the transformation

$$\zeta = z + \frac{a^2}{z}$$

where $z = x + iy$, and $\zeta = \xi + i\eta$, transforms C into the straight line $A'B'$ whose length $c = A'B' = 4a$.

The point P on the circle for which $x = a \cos \theta$ is transformed into P' for which $\xi = 2a \cos \theta$. In particular the point $A(\theta = 0)$ and $B(\theta = \pi)$ transform into A' and B'.

Now suppose the circle represents a cylinder with circulation Γ round it in an otherwise uniform stream of velocity V at a direction α to the x axis as illustrated in Fig. 11.5,1.

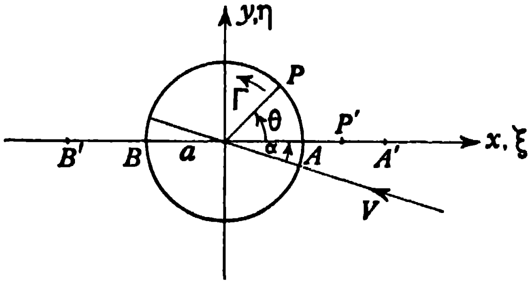


Fig. 11.5,1.

Then the velocity on the circle is (see § 11.3)

$$q = 2V \sin(\theta + \alpha) + \frac{\Gamma}{2\pi a}. \tag{11.5,1}$$

Now B is a singular point of the transformation and hence if we want the velocity at B' for the corresponding flow past the plate not to be infinite (in accordance with the Kutta-Joukowsky condition) we must adjust the circulation round C so that the velocity at B is zero.

Hence from (11.5,1), since $\theta = \pi$ at B, we must have

$$\Gamma = 4\pi a V \sin \alpha = \pi c V \sin \alpha \tag{11.5,2}$$

and with this circulation the velocity at P is

$$\begin{aligned} q &= 2V[\sin(\theta + \alpha) + \sin \alpha] \\ &= 4V \sin\left(\alpha + \frac{\theta}{2}\right) \cos \frac{\theta}{2}. \end{aligned} \tag{11.5,3}$$

The velocity at P' must then be given by

$$q' = q \Big/ \left|\frac{d\zeta}{dz}\right|. \tag{11.5,4}$$

But

$$\begin{aligned} \left|\frac{d\zeta}{dz}\right| &= \left|1 - \frac{a^2}{z^2}\right| = \left|\frac{(z - a)(z + a)}{z^2}\right| \\ &= \frac{AP \,.\, BP}{a^2} = 4 \sin \frac{\theta}{2} \cos \frac{\theta}{2}. \end{aligned}$$

Hence

$$q' = V\left[\sin \alpha \cot \frac{\theta}{2} + \cos \alpha\right], \tag{11.5,5}$$

i.e. q' = component of V parallel to the plate plus component of V normal to the plate times cot $\theta/2$.

If suffix u denotes the upper surface and suffix l the lower surface, then

$$\theta_l = 2\pi - \theta_u,$$

and it follows that if θ now denotes θ_u

$$\left.\begin{aligned} &q_u' = V\left[\sin\alpha\cot\frac{\theta}{2} + \cos\alpha\right], \quad q_l' = V\left[-\sin\alpha\cot\frac{\theta}{2} + \cos\alpha\right], \\ &\text{and} \qquad \frac{q_u' + q_l'}{2} = V\cos\alpha. \end{aligned}\right\} \qquad (11.5,6)$$

The vortex sheet strength on the plate is

$$\begin{aligned} \gamma &= q_u' - q_l' \\ &= 2V\sin\alpha\cot\frac{\theta}{2}. \end{aligned} \qquad (11.5,7)$$

Thus γ tends to infinity as P' tends to the leading edge ($\theta = 0$).

From Bernoulli's equation we have for the loading on the plate

$$\begin{aligned} p_l - p_u &= \tfrac{1}{2}\rho(q_u'^2 - q_l'^2) = \tfrac{1}{2}\rho(q_u' - q_l')(q_u' + q_l') \\ &= 2\rho V^2\sin\alpha\cos\alpha\cot\frac{\theta}{2}. \end{aligned} \qquad (11.5,8)$$

Hence, the normal force per unit span is

$$N' = \int_{B'}^{A'} (p_l - p_u)\, d\xi,$$

and, noting that $d\xi = -2a\sin\theta\, d\theta$, we find that

$$\begin{aligned} N' &= 4a\rho V^2\sin\alpha\cos\alpha\int_0^\pi \sin\theta\cot\frac{\theta}{2}\, d\theta \\ &= 4\pi a\rho V^2\sin\alpha\cos\alpha. \end{aligned} \qquad (11.5,9)$$

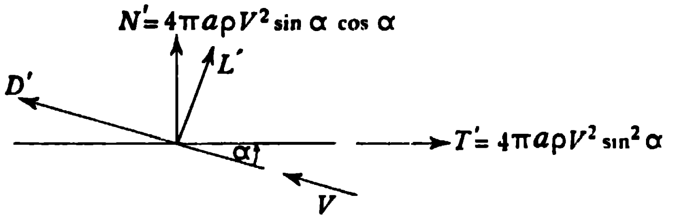
Hence the lift per unit span is

$$\left.\begin{aligned} &L' = N'\cos\alpha = 4\pi a\rho V^2\sin\alpha\cos^2\alpha, \\ &\text{and the drag per unit span is} \\ &D' = N'\sin\alpha = 4\pi a\rho V^2\sin^2\alpha\cos\alpha. \end{aligned}\right\} \qquad (10.5,10)$$

At first sight this would appear to contradict our general theorem that in inviscid potential flow

$$L' = \rho V\Gamma = 4\pi a\rho V^2\sin\alpha \text{ and } D' = 0.$$

The answer to this apparent contradiction lies in the fact that at the leading edge there is an infinite velocity and consequently an infinite suction which, acting on the infinitesimally thin edge, produces a force T' say, in the plane of the plate and directed forwards of magnitude $4\pi a\rho V^2\sin^2\alpha$. To obtain this result we must find the value of this force for a section with a rounded leading edge and then let the section thickness and leading edge radius tend to zero. Alternatively, we can obtain it by considering the balance of pressure and flux of momentum for a small circle centred on the plate leading edge.



$N' = 4\pi a\rho V^2 \sin\alpha\cos\alpha$
L'
D'
α
$T' = 4\pi a\rho V^2 \sin^2\alpha$
V

Fig. 11.5,2.

If we take into account this suction force, as it is called, then we see from Fig. 11.5,2 that the lift and drag forces (normal to and along the direction of V) are given by

$$\left.\begin{aligned} L' &= N'\cos\alpha + T'\sin\alpha = 4\pi a\rho V^2\sin\alpha \\ \text{and}\qquad D' &= N'\sin\alpha - T'\cos\alpha = 0 \end{aligned}\right\} \tag{11.5,11}$$

as required.

We see that the lift coefficient is

$$\left.\begin{aligned} C_L &= \frac{L'}{\frac{1}{2}\rho V^2 c} = 2\pi\sin\alpha \\ \text{and}\qquad dC_L/d\alpha &= 2\pi\cos\alpha \doteqdot 2\pi, \text{ when } \alpha \text{ is small.} \end{aligned}\right\} \tag{11.5,12}$$

As we have already noted, this theoretical value for $dC_L/d\alpha$ is not quite attained in practice, but the difference is readily explained by the effects of the boundary layers on the external flow (see § 6.21 and § 6.23).

11.5.3 Flow Past a Slightly Cambered Plate. Thin Aerofoil Theory

We will now consider a slightly cambered plate at a small angle of incidence. We have seen that we can regard the plate as a distribution of vorticity in the form of a vortex sheet of strength $y(x)$, say, such that the plate is a streamline of the combined flow due to the uniform stream of velocity V and the vortex sheet.

Take the x axis along the chord of the plate and the origin at the leading edge as illustrated in Fig. 11.5,3. We define a coordinate θ by

$$x = -\frac{c}{2}(1-\cos\theta) = -c\sin^2\frac{\theta}{2}, \tag{11.5,13}$$

so that $\theta = 0$ corresponds to the leading edge, and $\theta = \pi$ corresponds to the trailing edge.

Suppose that the velocity components induced at the point P' on the plate by the vorticity are u and v, which are taken to be small compared with V.

Then the slope of the resulting flow at P' relative to the x axis

$$= \frac{V \sin \alpha + v}{-V \cos \alpha + u} \approx -\left(\frac{V\alpha + v}{V}\right)$$

$= -(\alpha + v/V)$, neglecting squares and higher order terms of quantities assumed small.

Hence

$$(dy/dx)_{P'} = -(\alpha + v/V), \tag{11.5,14}$$

where $(dy/dx)_{P'}$ denotes the slope of the surface at P'. We must now relate v to the vortex sheet strength γ. Within the order of accuracy of the theory

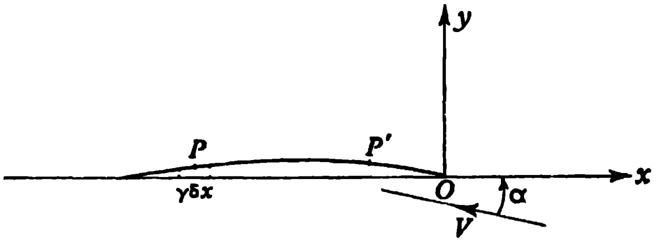


Fig. 11.5,3.

we have that a vortex element $\gamma \delta x$ at P will produce an upwash (or upward component of velocity normal to the x axis) at P'

$$= \frac{\gamma(x)\delta x}{2\pi(x' - x)}.$$

Hence

$$v = \int_{-c}^{0} \frac{\gamma(x)\,dx}{2\pi(x' - x)}, \tag{11.5,15}$$

and therefore

$$\int_{-c}^{0} \frac{\gamma(x)\,dx}{2\pi(x' - x)} = -V\left[\left(\frac{dy}{dx}\right)_{P'} + \alpha\right]. \tag{11.5,16}$$

If we change our variable from x to θ using (11.5,13) this equation becomes

$$\frac{1}{2\pi}\int_{0}^{\pi} \frac{\gamma(\theta)\sin\theta\,d\theta}{(\cos\theta' - \cos\theta)} = -V\left[\left(\frac{dy}{dx}\right]_{\theta'} + \alpha\right]. \tag{11.5,17}$$

We have seen that for the flat plate

$$\gamma(\theta) = 2V \sin \alpha \cot \frac{\theta}{2}$$

and following Glauert† we assume that the expression for γ for the curved plate contains a similar term plus additional terms expressible as a Fourier series. Thus, we suppose

$$\frac{\gamma(\theta)}{2V} = A_0 \cot \frac{\theta}{2} + \sum_{1}^{\infty} A_n \sin n\theta \tag{11.5,18}$$

and we determine A_0, A_1, etc. by satisfying the integral equation (11.5,17).

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† H. Glauert, *Elements of Aerofoil and Airscrew Theory*, 2nd Edition, p. 88, 92 (Camb. Univ. Press, 1948).

As we shall see, the Kutta-Joukowsky condition is equivalent to $\gamma = 0$ at the trailing edge and this condition is automatically satisfied by the assumed form for γ.

The left hand side of equation (11.15,17) after substitution for $\gamma(\theta)$ from equation (11.5,18) becomes

$$\frac{V}{\pi}\int_0^\pi \left\{ \frac{A_0(1+\cos\theta) + \frac{1}{2}\sum_1^\infty A_n[\cos(n-1)\theta - \cos(n+1)\theta]}{\cos\theta' - \cos\theta} \right\} d\theta.$$

We now make use of a well known theorem,† viz.

$$\int_0^\pi \frac{\cos n\theta\, d\theta}{\cos\theta - \cos\theta'} = \frac{\pi \sin n\theta'}{\sin\theta'}, \qquad (11.5,19)$$

and the above expression therefore becomes

$$V\left\{-A_0 + \frac{1}{2}\sum_1^\infty A_n\left[\frac{\sin(n+1)\theta' - \sin(n-1)\theta'}{\sin\theta'}\right]\right\}$$
$$= V\left\{-A_0 + \sum_1^\infty A_n \cos n\theta'\right\}.$$

Hence, from equation (11.5,17) we can write

$$\frac{v(\theta')}{V} = -\left(\frac{dy}{dx}\right)_{\theta'} - \alpha = -A_0 + \sum_1^\infty A_n \cos n\theta'. \qquad (11.5,20)$$

Dropping the accent of θ' we have therefore

$$-\left(\frac{dy}{dx}\right)_\theta = (\alpha - A_0) + \sum_1^\infty A_n \cos n\theta. \qquad (11.5,20)$$

If we now multiply by $\cos n\theta$, with $n = 0, 1$, etc. in turn and integrate for θ from 0 to π, in the usual way, we get the following expressions for A_0, A_1, etc.

$$A_0 = \alpha + \frac{1}{\pi}\int_0^\pi \left(\frac{dy}{dx}\right)_\theta d\theta \qquad (11.5,21)$$

$$\left.\begin{aligned} A_1 &= -\frac{2}{\pi}\int_0^\pi \left(\frac{dy}{dx}\right)_\theta \cos\theta\, d\theta \\ A_n &= -\frac{2}{\pi}\int_0^\pi \left(\frac{dy}{dx}\right)_\theta \cos n\theta\, d\theta. \end{aligned}\right\} \qquad (11.5,22)$$

We see that all the coefficients except A_0 are independent of incidence and are functions only of the shape of the plate.

We can now determine the lift, loading and velocity distributions, as well as the pitching moment in terms of the coefficients and therefore in terms of the section shape and incidence.

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† H. Glauert. *loc. cit.* p. 92.

The lift per unit span

$$L' = \rho V\Gamma = \rho V\int_{-c}^{0} \gamma\, dx = \rho V\int_{0}^{\pi}\gamma(\theta)\,\frac{c}{2}\sin\theta\, d\theta$$

$$= \rho V^2 c\int_0^{\pi}\left\{A_0(1+\cos\theta) + \sum_1^{\infty} A_n \sin n\theta\sin\theta\right\} d\theta$$

$$= \rho V^2 c\left(\pi A_0 + \frac{\pi A_1}{2}\right).$$

Hence

$$C_L = 2L'/\rho V^2 c = \pi(2A_0 + A_1). \qquad (11.5,23)$$

Therefore from (11.5,21) and (11.5,22) it follows that

$$C_L = 2\pi\alpha + 2\int_0^{\pi}\left(\frac{dy}{dx}\right)_{\theta}(1-\cos\theta)\, d\theta, \qquad (11.5,24)$$

and hence $dC_L/d\alpha = 2\pi$, as for the flat plate,

and writing $C_L = 2\pi(\alpha - \alpha_0)$,

where α_0 is the angle of zero lift, we see that

$$\alpha_0 = -\frac{1}{\pi}\int_0^{\pi}\left(\frac{dy}{dx}\right)_{\theta}(1-\cos\theta)\, d\theta. \qquad (11.5,25)$$

Using Bernoulli's equation as before, we have that the loading distribution is given by

$$p_l - p_u = \frac{\rho}{2}(q_u^2 - q_l^2),$$

where q now denotes the velocity adjacent to the plate surface, and suffices u and l denote upper and lower surfaces respectively. Hence

$$p_l - p_u = \frac{\rho}{2}(q_u - q_l)(q_u + q_l)$$

$$= \rho V\gamma, \qquad (11.5,26)$$

since $q_u - q_l = \gamma$, and to our order of approximation $q_u + q_l = 2V$.

From (11.5,18) it follows that γ and therefore the loading is zero and not infinite at the leading edge when $A_0 = 0$ and this happens, from (11.5,21), at the incidence

$$\alpha = -\frac{1}{\pi}\int_0^{\pi}\left(\frac{dy}{dx}\right)_{\theta} d\theta. \qquad (11.5,27)$$

This incidence is sometimes referred to as the *ideal angle of attack*, because at all other incidences the infinite suction at the leading edge and the subsequent large positive pressure gradient of inviscid flow, although not attained in practice, nevertheless correspond in practice to conditions in which flow breakaway either partial or complete is liable to occur.

We note also that the Kutta-Joukowsky condition requires zero loading at the trailing edge and therefore γ must be zero there. This requirement is automatically satisfied by the assumed form for γ (equation 11.5,18).

We see that

$$q_u = V + \frac{\gamma}{2}, \quad \text{and} \quad q_l = V - \frac{\gamma}{2}$$

$$\left.\begin{aligned} &\text{and hence} \quad q_u = V\left[1 + A_0 \cot\frac{\theta}{2} + \sum_1^\infty A_n \sin n\theta\right] \\ &\text{and} \qquad\quad q_l = V\left[1 - A_0 \cot\frac{\theta}{2} - \sum_1^\infty A_n \sin n\theta\right]. \end{aligned}\right\} \tag{11.5,28}$$

$$\left.\begin{aligned} &\text{Therefore} \qquad dq_u/d\alpha = V \cot \theta/2 \\ &\text{and} \qquad\qquad dq_l/d\alpha = -V \cot \theta/2. \end{aligned}\right\} \tag{11.5,29}$$

The anti-clockwise pitching moment per unit span about the leading edge is

$$\begin{aligned} M' &= \int_{-c}^{0} (p_l - p_u)x\,dx = \int_{-c}^{0} \rho V\gamma x\,dx \\ &= -\tfrac{1}{2}\rho V^2c^2\int_0^\pi (1 - \cos\theta)\left\{A_0(1 + \cos\theta) + \sum_1^n A_n \sin n\theta \sin\theta\right\} d\theta. \end{aligned}$$

Noting that $\sin\theta(1 - \cos\theta) = \sin\theta - \frac{1}{2}\sin 2\theta$, we readily deduce that the only terms in the Fourier series which contribute to the integral are the first and second, $n = 1$ and 2. Hence

$$\begin{aligned} M' &= -\tfrac{1}{2}\rho V^2c^2\int_0^\pi\left\{A_0(1 - \cos^2\theta) + A_1\sin^2\theta - \frac{A_2}{2}\sin^2 2\theta\right\} d\theta \\ &= -\frac{\pi}{4}\rho V^2c^2\left(A_0 + A_1 - \frac{A_2}{2}\right). \end{aligned}$$

The moment coefficient is

$$\begin{aligned} C_M &= 2M'/\rho V^2c^2 \\ &= -\frac{\pi}{2}\left(A_0 + A_1 - \frac{A_2}{2}\right). \end{aligned} \tag{11.5,30}$$

From (11.5,23) and the fact that of the coefficients A_0, A_1, etc. only A_0 depends on α (equations 11.5,21 and 11.5,22) we see that we can write

$$C_M = -\frac{C_L}{4} + C_{M_0} \tag{11.5,31}$$

where C_{M_0} is a constant for the section

$$= \frac{\pi}{4}(A_2 - A_1). \tag{11.5,32}$$

We can identify C_{M_0} as the pitching moment coefficient for zero lift.

Further,

$$dC_M/dC_L = -1/4$$

and hence the aerodynamic centre is 0·25*c* behind the leading edge, i.e. at the so-called quarter chord point.

11.5.4 The Blasius Theorems

We will now discuss two important theorems due to Blasius, which will enable us to consider the general problem of the forces and moments on any shaped section in steady flow.

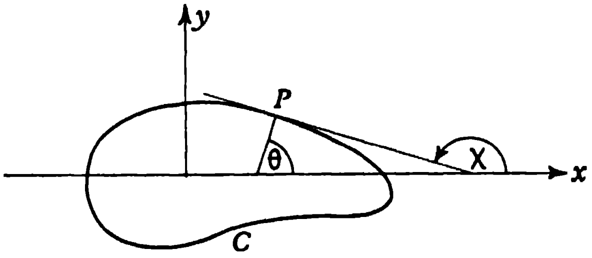

Fig. 11.5,4.

Let C denote the section and P a point on the surface (Fig. 11.5,4) and let θ now denote the angle the normal at P makes with the x axis and χ the angle the tangent at P makes with the x axis. Then, if the components of the force per unit span on C are X' and Y'

$$Y' + iX' = -\oint_C p \sin \theta \, ds - i \oint_C p \cos \theta \, ds$$

where the integrals are taken round the contour C and ds is an element of length along the contour.

But by Bernoulli's equation

$$p = -\tfrac{1}{2}\rho q^2 + \text{const.},$$

where q is the resultant velocity at the surface, and hence

$$\begin{aligned} Y' + iX' &= \tfrac{1}{2}\rho \oint_C q^2(\sin \theta + i \cos \theta) \, ds \\ &= -\tfrac{1}{2}\rho \oint_C q^2(\cos \chi - i \sin \chi) \, ds \\ &= -\tfrac{1}{2}\rho \oint_C q^2(\cos \chi - i \sin \chi)^2(\cos \chi + i \sin \chi) \, ds. \end{aligned}$$

But

$$q(\cos \chi - i \sin \chi) = -dw/dz,$$

where w is the complex potential (see § 2.14).

Also $(\cos\chi + i\sin\chi)\,ds = dx + i\,dy = dz.$

Hence

$$Y' + iX' = -\tfrac{1}{2}\rho\oint_C \left(\frac{dw}{dz}\right)^2 dz$$

which can also be written

$$X' - iY' = \tfrac{1}{2}\rho i\oint_C \left(\frac{dw}{dz}\right)^2 dz. \qquad (11.5,33)$$

Now suppose, as would normally be the case, that $(dw/dz)^2$ has no singularities outside the contour C and that it can be expanded outside C in inverse powers of z, thus

$$\left(\frac{dw}{dz}\right)^2 = B_0 + \frac{B_1}{z} + \frac{B_2}{z^2} + \dots \qquad (11.5,34)$$

where the B's may be complex.

Then, by Cauchy's theorem

$$\oint_C \left(\frac{dw}{dz}\right)^2 dz = 2\pi i \times \text{sum of residues of } \left(\frac{dw}{dz}\right)^2 \text{at its poles inside } C$$
$$= 2\pi i B_1.$$

Hence

$$X' - iY' = -\pi\rho B_1. \qquad (11.5,35)$$

As an example of the use of this formula consider the circular cylinder with circulation Γ in an otherwise uniform stream V in the direction of Ox reversed. Then

$$w = V\left(z + \frac{a^2}{z}\right) + \frac{i\Gamma}{2\pi}\ln z$$

and

$$\frac{dw}{dz} = V - \frac{Va^2}{z^2} + \frac{i\Gamma}{2\pi z}.$$

Hence $B_1 = Vi\Gamma/\pi$, and therefore $Y' = \Gamma\rho V$, and $X' = 0$.

Now consider the moment per unit span on C about the origin. This is M' where

$$M' = -\oint_C p\sin\theta x\,ds + \oint_C p\cos\theta y\,ds = \oint_C p(x\cos\chi + y\sin\chi)\,ds$$
$$= \oint_C p(x\,dx + y\,dy) = -\oint_C \tfrac{1}{2}\rho q^2(x\,dx + y\,dy).$$

But $\mathscr{R}\oint_C z\left(\frac{dw}{dz}\right)^2 dz = \mathscr{R}\oint_C (u^2 - v^2 - 2iuv)(x + iy)(dx + i\,dy)$

where $\mathscr{R}$ denotes the 'real part of', and u and v are the velocity components. Therefore

$$\mathscr{R}\oint_C z\left(\frac{dw}{dz}\right)^2 dz = \oint [(u^2 - v^2)(x\,dx - y\,dy) + 2uv(x\,dy + y\,dx)].$$

But since C is a streamline

$$dx/u = dy/v$$

and therefore

$$\mathscr{R}\oint_C z\left(\frac{dw}{dz}\right)^2 dz = \oint[(u^2 - v^2)(x\,dx - y\,dy) + 2v^2x\,dx + 2u^2y\,dy]$$

$$= \oint_C (u^2 + v^2)(x\,dx + y\,dy).$$

Hence

$$M' = -\mathscr{R}\oint_C \tfrac{1}{2}\rho z\left(\frac{dw}{dz}\right)^2 dz. \tag{11.5,36}$$

Again, making use of Cauchy's theorem, we have finally from equation (11.5,34)

$$M' = -\mathscr{R}\left[\frac{\rho}{2}B_2 2\pi i\right]$$

$$= \pi\rho\mathscr{I}(B_2) \tag{11.5,37}$$

where $\mathscr{I}$ denotes the 'imaginary part of'.

11.5.5 Aerodynamic Properties of General Aerofoil Sections

By means of a suitable conformal transformation it is possible to transform a circle into an aerofoil section, and the parameters of the transformation determine the geometrical characteristics of the section.

Consider the circle illustrated in Fig. 11.5,5, with centre at C and radius b.

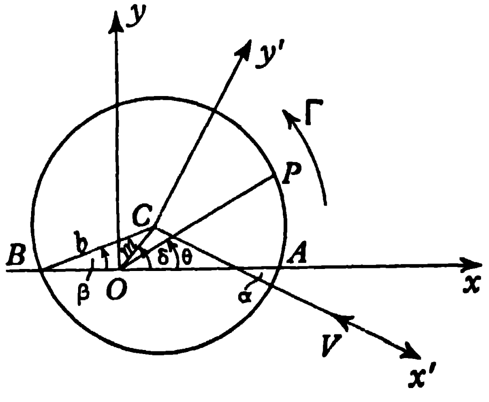

Fig. 11.5,5.

Suppose the undisturbed velocity past the cylinder represented by the circle is V at an angle α to the x axis. The origin of the axis system is at O where $OC = m$, say, and the angle between OC and the x axis is δ, say. The x axis cuts the circle at points A and B and the angle CBO is denoted by β.

If axes Cx' and Cy' are taken through C with Cx' in the direction of V reversed and if Γ is the circulation round the circle, then the complex potential for the flow is

$$w = V\left(z' + \frac{b^2}{z'}\right) + \frac{i\Gamma}{2\pi}\ln z',$$

and hence

$$\frac{dw}{dz'} = V\left(1 - \frac{b^2}{z'^2}\right) + \frac{i\Gamma}{2\pi z'}$$

Now we suppose that the transformation is such that B transforms into the trailing edge of the aerofoil and in consequence B must be a singularity of the transformation. If an infinite velocity at the trailing edge is to be avoided in accordance with the Kutta-Joukowsky condition, it follows that in the flow past the circle the circulation must be such that the velocity at B (and hence dw/dz') must be zero.

At B, $z' = -be^{i(\alpha+\beta)}$ and hence from the above expression for dw/dz'

$$V[1 - e^{-2i(\alpha+\beta)}] = \frac{i\Gamma}{2\pi b} e^{-i(\alpha+\beta)}$$

and therefore $$\Gamma = 4\pi bV \sin(\alpha + \beta). \tag{11.5,38}$$

Suppose further that the transformation from circle to aerofoil section is of the form

$$\zeta = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots,$$

where a_1, a_2, etc. can be complex.

Then, since

$$z' = (z - me^{i\delta})e^{i\alpha}$$

it follows that

$$\frac{dw}{d\zeta} = \frac{dw}{dz'}\frac{dz'}{dz}\frac{dz}{d\zeta} = \frac{\left[V\left(1 - \dfrac{b^2}{z'^2}\right) + \dfrac{i\Gamma}{2\pi z'}\right]e^{i\alpha}}{\left(1 - \dfrac{a_1}{z^2} - \dfrac{2a_2}{z^3}\dots\right)}$$

$$= \left\{Ve^{i\alpha} - \frac{Vb^2}{z^2}\left(1 - \frac{m}{z}e^{i\delta}\right)^{-2} e^{-i\alpha} + \frac{i\Gamma}{2\pi z}\left(1 - \frac{m}{z}e^{i\delta}\right)^{-1}\right\}$$

$$\times \left[1 - \frac{a_1}{z^2} - \frac{2a_2}{z^3} - \dots\right]^{-1}$$

$$= Ve^{i\alpha} + \frac{i\Gamma}{2\pi z} + \frac{1}{z^2}\left[-Vb^2e^{-i\alpha} + \frac{i\Gamma}{2\pi}me^{i\delta} + a_1Ve^{i\alpha}\right] + \dots \tag{11.5,39}$$

Hence if we write, as in equation (11.5,34),

$$\left(\frac{dw}{d\zeta}\right)^2 = B_0 + \frac{B_1}{z} + \frac{B_2}{z^2} + \dots$$

we have

$$B_0 = V^2e^{2i\alpha}, \quad B_1 = \frac{iV\Gamma}{\pi}e^{i\alpha}.$$

and

$$B_2 = -2V^2b^2 + \frac{mi\Gamma V}{\pi}e^{i(\alpha+\delta)} + 2a_1V^2e^{2i\alpha} - \frac{\Gamma^2}{4\pi^2}. \tag{11.5,40}$$

Using the first of the Blasius theorems (equation 11.5,35) we have

$$X' - iY' = -\pi\rho B_1 = -\rho i V\Gamma e^{i\alpha}$$

and hence

$$\left.\begin{aligned} X' &= \rho V\Gamma \sin\alpha \\ Y' &= \rho V\Gamma \cos\alpha \end{aligned}\right\} \qquad (11.5,41)$$

and

i.e., the resultant force L' is $\rho V\Gamma = 4\pi\rho V^2 b \sin(\alpha + \beta)$ normal to the stream direction, as expected.

The second Blasius theorem (equation 11.5,37) gives the moment per unit span about O as

$$M'(O) = \pi\rho\mathscr{I}\left[2a_1V^2e^{2i\alpha} + \frac{iV\Gamma}{\pi}me^{i(\alpha+\delta)}\right].$$

If we write

$$a_1 = a^2e^{2i\gamma}, \quad \text{say,} \qquad (11.5,42)$$

then

$$M'(O) = 2\pi a^2\rho V^2 \sin 2(\alpha + \gamma) + \rho V\Gamma m \cos(\alpha + \delta). \qquad (11.5,43)$$

The moment about C is

$$\begin{aligned} M'(C) &= M'(O) - L'm\cos(\alpha + \delta) \\ &= M'(O) - \rho V\Gamma m \cos(\alpha + \delta) \\ &= 2\pi a^2\rho V^2 \sin 2(\alpha + \gamma). \end{aligned} \qquad (11.5,44)$$

We see that $L' = 0$, when $\alpha = -\beta$ and then

$$M'(C) = 2\pi a^2\rho V^2 \sin 2(\gamma - \beta).$$

It follows that $C_{M_0} = 0$ only if $\gamma = \beta$, i.e. the coefficient a_1 of the transformation must be of the form $a^2e^{2i\beta}$.

A classic transformation is the Joukowsky transformation (see § 2.15)

$$\zeta = z + \frac{a^2}{z} \qquad (11.5,45)$$

where a is now the distance OB. Therefore $a_1 = a^2 = OB^2$, and $\gamma = 0$.

It can be readily shown for this transformation (see, for example, Glauert[1]) for small β and $b - a = \varepsilon a$, where ε is small, that

(1) the chord of the aerofoil $\doteqdot 4a$

(2) the thickness of the aerofoil $\doteqdot \varepsilon[4a \sin\theta + 2a \sin 2\theta]$

(3) the ordinate of the camber line $\doteqdot 2a\beta \sin^2\theta$

(4) the trailing edge angle $= 0$.

Here θ is the angle OP makes with the x axis. It follows from (2) that the maximum thickness position corresponds to $\theta = 60°$, and the thickness chord ratio is

$$t/4a \doteqdot 3\sqrt{3}\varepsilon/4.$$

Hence the assumption of small ε implies that the wing is thin.

From (3) we see that the maximum camber position corresponds to $\theta = \pi/2$, and the ratio of the maximum camber to the chord $\doteqdot \beta/2$. The assumption of small β implies that the camber is small.

The lift coefficient is then

$$\left.\begin{aligned} C_L &= L'/\tfrac{1}{2}\rho V^2 4a \doteqdot 2\pi \sin(\alpha + \beta) \\ &\doteqdot 2\pi(\alpha+\beta) \\ \text{and} \quad dC_L/d\alpha &= 2\pi. \end{aligned}\right\}\dagger \tag{11.5,46}$$

From equation (11.5,44)

$$M'(C) \doteqdot 2\pi a^2 \rho V^2 2\alpha$$

and

$$C_M, \text{ about } C, = \frac{M'(C)}{\tfrac{1}{2}\rho V^2 16a^2} = \frac{\pi\alpha}{2}, \tag{11.5,47}$$

whilst the pitching moment coefficient at zero lift is

$$C_{M_0} = -\frac{\pi\beta}{2}. \tag{11.5,48}$$

Hence

$$\begin{aligned} C_M, \text{ about } C, &= C_{M_0} + \frac{\pi}{2}(\alpha+\beta) \\ &= C_{M_0} + \frac{C_L}{4}. \end{aligned} \tag{11.5,49}$$

Since C is very closely the mid-point of the chord it follows that the lift acts through a point very close to the quarter-chord point.

It will be clear that the analysis can be refined to take a more complete account of the effect of profile shape on the aerodynamic characteristics, and for details of later developments in potential flow theory the reader is referred to Lighthill[1], Goldstein[2] and Sells[3].

11.5.6 First Order Theory for Velocity Distribution on a Thin Symmetrical Wing Section at Zero Incidence

The following analysis provides a first order approximation to the velocity distribution on a thin symmetrical wing section. It is based on the derivation of an equivalent source distribution subject to the assumption that the disturbance velocities produced by the wing are small, so that squares and products and higher order terms of the ratios of these velocities to the undisturbed stream velocity can be neglected. The analysis therefore breaks down in the region of a stagnation point.

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† A more accurate analysis would show a slight dependence of $dC_L/d\alpha$ on thickness chord ratio, the former increasing slightly with the latter. In a real fluid however this effect is swamped by the effects of viscosity and in particular by the tendency for the boundary layers to thicken and even separate with increase of wing thickness/chord ratio.

We assume that the equivalent source distribution is distributed along the chord line and is of strength $\sigma(x)$ per unit length. Consistent with the assumption that the disturbances introduced by the wing (and therefore by the equivalent source distribution) are small is the assumption that each source element $\sigma(x)\,\delta x$ on an element of chord length δx causes the whole of the local deviation of the flow from the undisturbed stream direction. Thus if v is the velocity component normal to the chord line at the element of wing surface corresponding to the element δx (see Fig. 11.5,6)

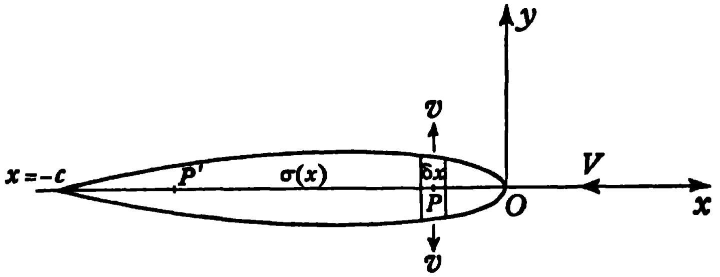

Fig. 11.5,6.

$$\sigma(x)\,\delta x = 2v\,\delta x$$

and hence

$$\sigma(x) = 2v. \tag{11.5,50}$$

But to the same order of accuracy

$$\frac{dy}{dx} = -\frac{v}{V}$$

where y is the local ordinate of the section, and V is the undisturbed stream velocity which is directed against the x axis.

Hence if $y = -f(x)$ defines the shape of the section

$$\sigma(x) = 2Vf'(x).\dagger \tag{11.5,51}$$

Now write, as before,

$$x = -\frac{c}{2}(1-\cos\theta)$$

and suppose that we can express $f(x)$ as a Fourier series in θ, thus

$$f(x) = \frac{c}{2}\sum_{1}^{\infty} A_n \sin n\theta. \tag{11.5,52}$$

The contribution to the velocity component in the x direction at P', where $x = x'$, say, due to the source element $\sigma(x)\,\delta x$ at P

$$= \frac{\sigma(x)\,\delta x}{2\pi(x'-x)}, \text{ to the first order,}$$

$$= \frac{f'(x)2V\,\delta x}{2\pi(x'-x)}.$$

---

† See § 2.8 Ex. 2, for an alternative proof of this relation.

Hence the total velocity increment in the x direction at P' due to the source distribution is

$$u(P') = \frac{2V}{2\pi}\int_{-c}^{0} \frac{f'(x)\,dx}{(x'-x)} = \frac{V}{\pi}\int_{0}^{\pi} \frac{\sum_{1}^{\infty} nA_n \cos n\theta\, d\theta}{\cos\theta - \cos\theta'}$$

$$= V\sum_{1}^{\infty} \frac{nA_n \sin n\theta'}{\sin\theta'}, \qquad (11.5,53)$$

(see equation 11.5,19).

From (11.5,52) we derive the following expression for A_n

$$A_n = \frac{4}{\pi c}\int_{0}^{\pi} f[x(\theta)] \sin n\theta\, d\theta \qquad (11.5,54)$$

Hence, if we are given the shape and therefore $f[x(\theta)]$ for the section, we can determine A_n and therefore the distribution of u. The reverse process is also possible, i.e. given the distribution of u the corresponding shape can be found. This was in fact one of the earliest methods[1,2] adopted for the design of low-drag wings, i.e. wings with far back maximum suction so that the boundary layers could remain laminar over a considerable extent of the wing surface (see § 6.2 and 6.13).

To a fairly close order of approximation the thickness and camber effects of a wing section can be regarded as additive, and so the two distributions of velocity perturbation can be evaluated separately and then added together. Thus, a combination of the analysis of this section with that of § 11.5,3 can provide a first order estimate of the velocity distribution over a thin cambered wing. For more accurate methods of analysis reference should be made to Goldstein.[2]

11.6 Wings of Finite Span in Steady Flow

11.6.1 Introductory Remarks

We have seen how a wing of infinite span develops a circulation round it, equal to the integral of the vorticity in its boundary layers, and how this circulation is exactly balanced by the cast off vortex far downstream (see Fig. 11.2,3). This vorticity is referred to as the bound vorticity, as it is carried with the wing and is readily related to the lift loading on the wing (see equation 11.5,26).

Now for simplicity consider an unswept wing of finite span and of constant cross-section. This similarly develops bound vorticity, circulation and lift, but these must tend to zero as the wing tips are approached, since there can be no discontinuities of pressure in the air adjacent to the wing tips. However, vortices can neither begin nor end in a fluid (see § 2.11), and the total vortex strengths approaching a point must be equal to the total vortex strengths leaving it. It follows that if the circulation at spanwise station y from the centre line of the wing is $\Gamma(y)$, say, then from an element of span δy a vortex of strength $(-d\Gamma(y)/dy)\,\delta y$ must trail from the wing downstream, or we can say a vortex of strength $(d\Gamma/dy)\,\delta y$ approaches the wing from downstream (see Fig. 11.6,1). We recall the sign convention that vorticity and circulation are positive if clockwise about the positive direction of the appropriate axis. Hence, where $d\Gamma/dy$ is positive, as would normally be the case on the port wing with positive lift, the trailing vorticity is clockwise about the positive x direction; where $d\Gamma/dy$ is negative, as on the starboard wing, the trailing vorticity is anti-clockwise about the positive x axis.

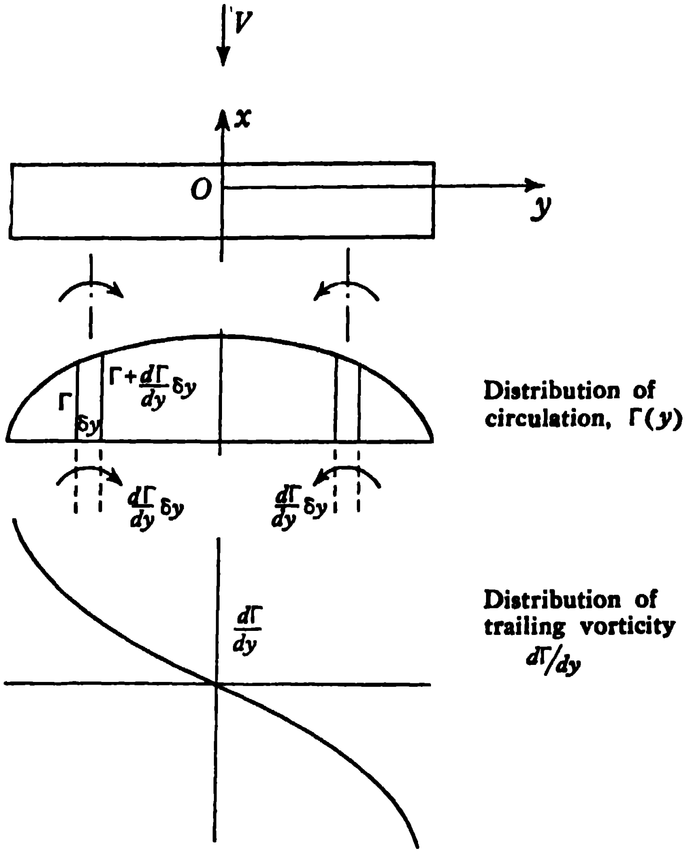
V
x
O
y
Γ
δy
$\Gamma+\frac{d\Gamma}{dy}\delta y$
$\frac{d\Gamma}{dy}\delta y$
$\frac{d\Gamma}{dy}\delta y$
$\frac{d\Gamma}{dy}$
Distribution of circulation, Γ(y)
Distribution of trailing vorticity
$d\Gamma/dy$

Fig. 11.6,1. Distribution of circulation and trailing vorticity for a wing of finite span.

Thus we have a sheet of trailing vortices behind the wing and the intensity of this sheet is strongest towards the tips where $|d\Gamma/dy|$ is greatest. As the sheet progresses downstream it tends to roll up into two discrete vortices and the distance between these vortices is somewhat smaller than the wing span. The bound vortices, trailing vortices and cast off vortex form a continuous closed system.

We can get a simple physical picture of the generation of trailing vortices by considering the flow and pressure distribution near the tips of a wing of finite span at an incidence at which it is developing lift. The lift implies that the pressures over the lower surface must in general be higher than those over the upper surface. Consequently air tends to be pushed round the wing tips from the lower surface to the upper and at the same time the pressure must fall on the lower surface from the centre out to the tips, whilst conversely the pressure must fall on the upper surface from the tips to the centre. Thus, the air tends to move outwards towards the tips on the lower surface and inwards towards the centre on the upper surface. This motion imposed on the general motion of the air relative to the wing causes the flow to spiral round from the lower surface to the upper, at the same time creating a difference of velocity direction between that just above and that just below the wing wake, and hence a trailing vortex sheet develops which is strongest in intensity near the tips (Fig. 11.6,2).

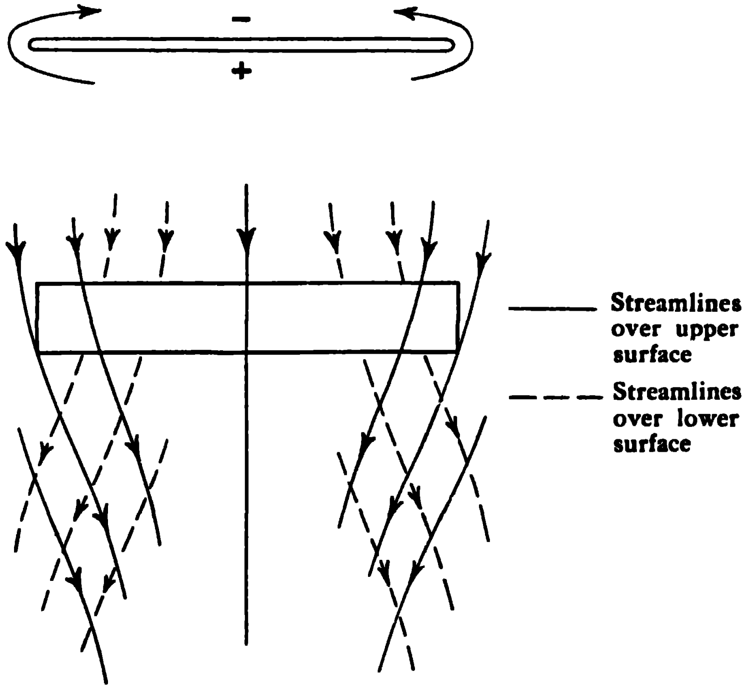
-
+
Streamlines over upper surface
Streamlines over lower surface

Fig. 11.6,2. Sketch illustrating formation of a trailing vortex sheet from a wing of finite span with positive lift.

In general, whatever the plan form of the wing, the essential characteristics of the trailing vortex pattern and its formation are similar. The difference in pressure between lower and upper surfaces induces a spiral motion from the lower surface to the upper surface, and this motion persists in the form of a trailing vortex sheet the strength of which is determined by the rate of change of circulation about the wing with spanwise distance. Since the flow over the wing will have in general spanwise as well as streamwise components, the vortex sheet equivalent to the boundary layers on the wing surface will likewise have spanwise and streamwise components.

Hence, if the wing is thin and of small camber, then as far as its lift is concerned it can be regarded as equivalent to a vortex sheet. This concept is a generalisation to three dimensions of the thin aerofoil theory discussed in § 11.5,3. Thus, suppose the perturbation velocity components are (u, v, w), with the x axis taken opposite to the direction of the undisturbed stream velocity V, the y axis in the spanwise direction normal to the plane of symmetry of the wing, and the z axis is downward (see Fig. 11.6,3). We suppose further that u/V, v/V, and w/V are small quantities, so that squares, products and higher order terms involving them can be neglected. Then the equivalent vortex sheet will be of strength γ, say, and will have components (γ_x, γ_y), such that

$$\left.\begin{aligned} \gamma_x &= (v_u - v_l) \\ \text{and} \qquad \gamma_y &= -(u_u - u_l) \end{aligned}\right\} \qquad (11.6,1)$$

where suffix u refers to the velocity component over the upper surface and suffix l to the velocity component over the lower surface (see Fig. 11.6,3).

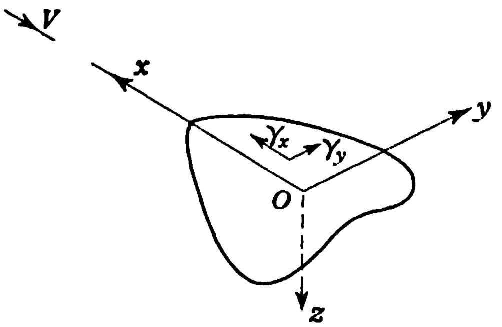

Fig. 11.6,3.

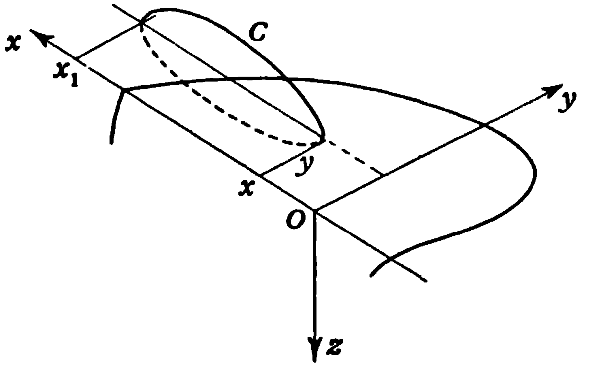

Fig. 11.6,4.

The pressure difference between upper and lower surface is by Bernoulli's equation

$$p_l - p_u = \tfrac{1}{2}\rho\{(-V + u_u)^2 + v_u^2 + w^2\} - \tfrac{1}{2}\rho\{(-V + u_l)^2 + v_l^2 + w^2\}$$

where w is the velocity component in the z direction, which is the same at the wing for both upper and lower surface since we are disregarding the thickness of the wing.

With the assumption that the perturbation velocities are small, we have

$$\begin{aligned} p_l - p_u &= -\rho V(u_u \quad u_l) \\ &= \rho V \gamma_y. \end{aligned} \qquad (11.6,2)$$

Thus, the loading is related only to the component γ_y, which we refer to as the *bound vorticity.** Since the flow outside the vortex sheet and its wake is irrotational

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

and hence from equation (11.6,1)

$$\partial \gamma_y / \partial y = -\partial \gamma_x / \partial x. \qquad (11.6,3)$$

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* This term is commonly used but it should be noted that γ is strictly the strength of the equivalent vortex sheet and therefore has the dimensions of vorticity × length.

Consider as in Fig. 11.6,4 a spanwise station y and a datum $x = x_1$, say, ahead and in the plane of the wing. Then the quantity $\Gamma(x, y)$ defined by

$$\Gamma(x, y) = \left[\int_x^{x_1} \gamma_y \, dx\right] \tag{11.6,4}$$

is the circulation round any circuit C that pierces the wing at the point x and is threaded by the wing leading edge and lies in a plane parallel to Oxz. It is also evidently the integral of the bound vorticity from the point x to the leading edge at the station y.

Now from (11.6,3)

$$\begin{aligned}\frac{\partial\Gamma(x, y)}{\partial y} &= \frac{\partial}{\partial y}\int_x^{x_1} \gamma_y \, dx \\ &= \int_x^{x_1} \frac{\partial \gamma_y}{\partial y}\, dx = \int_x^{x_1} - \frac{\partial \gamma_x}{\partial x}\, dx \\ &= \gamma_x(x, y),\end{aligned} \tag{11.6,5}$$

since $\gamma_x(x_1, y)$ must be zero, as there can be no vorticity ahead of the wing Hence we see that the vortex sheet component in the x direction at a point on the surface is the rate of change with respect to y of the circulation round a circuit piercing the wing at the point and threaded by the wing leading edge. In particular at the trailing edge we again obtain

$$\gamma_x(y) = \frac{d\Gamma(y)}{dy} \tag{11.6,6}$$

where $\Gamma(y)$ is the circulation about the wing section defined by the spanwise station y.

We refer to γ_x as the *trailing vorticity*. Hence, over the wing we have both bound and trailing vorticity, but the loading is determined only by the former (equation 11.6,2). In the wake there can be no pressure discontinuities, and therefore there is only trailing vorticity.

The downwash (w) at any point on the wing surface is determined by the distribution of bound and trailing vorticity over the wing and wake, and the resulting streamline direction at the point must conform to the wing surface. Hence the problem of determining the lifting characteristics of a wing surface is, subject to the assumption of small perturbations, that of determining the vorticity distribution in the form of a vortex sheet over wing and wake for which this boundary condition at the wing surface is satisfied.

It will be noted that it is the presence of trailing vorticity over the wing and wake that distinguishes the flow over a wing surface in three dimensions from that over a two-dimensional wing section. Both trailing and bound vorticity contribute to the downwash over the wing and elsewhere, but far downstream in the wake the *induced downwash*, as it is called, is entirely due to the trailing vorticity; that due to the bound vorticity is negligible there.

The extent of the trailing vortex system behind the wing and its associated downwash field clearly increases under steady conditions at a rate equal to the wing speed. Hence energy must be continuously expended at a steady rate in the process of generating the system. This implies work that is required from the propulsive units, and is equivalent therefore to a contribution to the drag. This drag contribution is referred to as the *induced drag* or *vortex drag*. It is manifest in the pressure distribution over the wing.

We shall not pursue the general subject of wing theory further; the reader who wishes to do so is referred to Robinson and Laurmann.[1] We shall confine ourselves here to presenting the simpler developments of the theory applicable to unswept wings of large aspect ratio and slender wings of very small aspect ratio.

11.6.2 Unswept Wings of Large Aspect Ratio. The Lanchester-Prandtl Lifting Line Theory

If the mean chord of the wing is small compared with the span, then an approximate but effective analysis can be developed on the assumption that the bound vorticity can be treated as a single line vortex, or *lifting line*. In practice it is found that this analysis leads to results of acceptable accuracy, provided the aspect ratio, i.e. the ratio of the span to the mean chord, is greater than about 4.

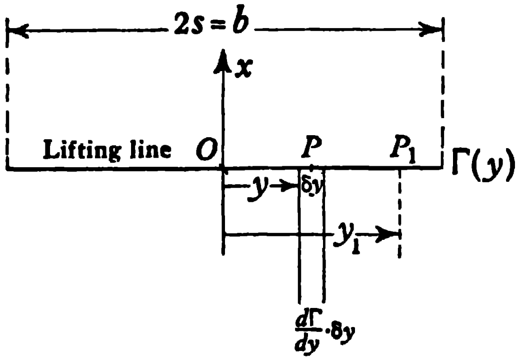


Fig. 11.6,5.

Since the lift due to incidence on a thin wing section acts, according to theory, through the point a quarter of the chord aft of the leading edge (the quarter-chord point), and in practice it acts through a point very close to this point, the lifting line is generally taken to coincide with the quarter chord point locus.

At any spanwise station y along the lifting line the circulation and therefore the strength of the lifting line is a function of y, $\Gamma(y)$, say, and associated with it we have a trailing vortex system of strength $d\Gamma/dy$ per unit span.

Now consider the downwash induced at the lifting line by the trailing vortex system (Fig. 11.6,5).

The semi-infinite trailing vortex of strength $(d\Gamma/dy)\delta y$ of the spanwise element δy at P will induce, normal to the xy plane, a downwash at P_1, where $y = y_1$, which is just half of that due to an infinite vortex of the same strength, and hence this downwash is

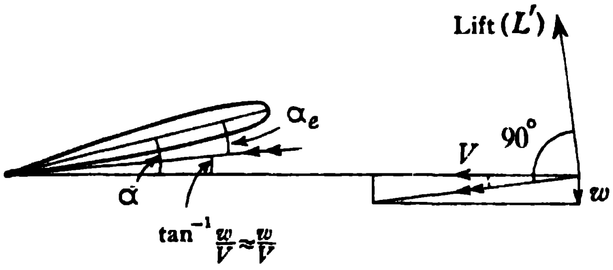
$$\delta w = \frac{1}{4\pi}\frac{d\Gamma}{dy}\,\delta y/(y_1 - y).$$

The total downwash at P_1 is therefore

$$w(y_1) = \frac{1}{4\pi}\int_{-s}^{s} \frac{\frac{d\Gamma}{dy}\,dy}{(y_1 - y)} \tag{11.6,7}$$

where s = wing semi-span = $b/2$.

Fig. 11.6,6. The change of effective incidence due to the downwash induced by the trailing vortices.



The assumption is now made that w can be taken as the mean induced downwash in the neighbourhood of the wing at the spanwise station P_1 and that w/V is small. It is then argued that the spanwise element of wing at P_1 behaves as if it were in two-dimensional flow, but with the effective incident flow the resultant of V and w. Thus the local effective angle of incidence is (see Fig. 11.6,6)

$$\alpha_e(y_1) = \alpha - \frac{w(y_1)}{V}$$

$$= \alpha - \frac{1}{4\pi}\int_{-s}^{s} \frac{\frac{d\Gamma}{dy}\,dy}{(y_1 - y)} \tag{11.6,8}$$

where α and α_e are measured from the local no-lift angle.

Hence, the local lift coefficient is

$$C_l(y_1) = a_\infty \alpha_e(y_1) \tag{11.6,9}$$

where a_∞ = rate of change of lift with incidence of the local wing section in two-dimensional flow (or on a wing of infinite aspect ratio).

Now the lift per unit span is

$$L' = \tfrac{1}{2}\rho V^2 c C_l(y_1) = \rho V \Gamma(y_1)$$

and therefore

$$\Gamma(y_1) = \tfrac{1}{2}cVC_l(y_1)$$

$$= \frac{c}{2}\,V a_\infty \alpha_e(y_1), \text{ from equation (11.6,8).}$$

It follows that

$$w(y_1) = V[\alpha(y_1) - \alpha_e(y_1)] = V\alpha(y_1) - \frac{2\Gamma(y_1)}{a_\infty c}.$$

Hence

$$\frac{1}{4\pi}\int_{-s}^{s} \frac{\frac{d\Gamma}{dy}\,dy}{y_1 - y} = \left(V\alpha - \frac{2\Gamma}{a_\infty c}\right)_{y=y_1}. \tag{11.6,10}$$

This in an integral equation for determining $\Gamma(y)$. To solve this equation put

$$y = -s\cos\theta, \tag{11.6,11}$$

so that $\theta = 0$ at the port tip and $\theta = \pi$ at the starboard tip. We assume that $\Gamma(\theta)$ can be expressed as a Fourier sine series, thus

$$\Gamma(\theta) = 4sV\sum_1^\infty \gamma_n \sin n\theta. \tag{11.6,12}$$

Then

$$\frac{d\Gamma}{dy} = \frac{d\Gamma}{d\theta}\bigg/\frac{dy}{d\theta} = 4V\sum_1^\infty n\gamma_n\frac{\cos n\theta}{\sin\theta}$$

and the left hand side of equation (11.6,10) becomes

$$\begin{aligned} w(\theta_1) &= \frac{V}{\pi}\int_0^\pi \sum_1^\infty \frac{n\gamma_n\cos n\theta}{(\cos\theta - \cos\theta_1)}\,d\theta \\ &= V\sum_1^\infty \frac{n\gamma_n\sin n\theta_1}{\sin\theta_1}, \end{aligned} \tag{11.6,13}$$

where we have made use of equation (11.5,19).

Hence, equation (11.6,10) becomes

$$\sum_1^\infty \frac{n\gamma_n\sin n\theta_1}{\sin\theta_1} = \alpha(\theta) - \frac{8s}{a_\infty c}\sum_1^\infty \gamma_n\sin n\theta_1.$$

Therefore

$$\sum_1^\infty (n\mu + \sin\theta)\gamma_n\sin n\theta = \mu\alpha\sin\theta \tag{11.6,14}$$

where we have written $\mu = a_\infty c/8s$, and we have dropped the suffix 1 for θ. We note that in general μ and α are functions of θ. This is a key equation in lifting line theory.

The lift on the wing is

$$\begin{aligned} L = \int_{-s}^{s} \rho V\Gamma\,dy &= 4\rho V^2s^2\int_0^\pi \sin\theta\sum_1^\infty \gamma_n\sin n\theta\,d\theta \\ &= 2\pi\rho V^2s^2\gamma_1, \end{aligned} \tag{11.6,15}$$

and if we denote the wing area by S the wing lift coefficient is

$$C_L = 2L/\rho V^2S = \pi A\gamma_1 \tag{11.6,16}$$

where A is the aspect ratio $= b/c = 4s^2/S$.

Reverting to Fig. 11.6,6 we see that, since the effective incident stream is the resultant of V and the downwash w, the lift vector L', which must act normal to the direction of this resultant, is inclined backwards through the angle w/V relative to the normal to the undisturbed stream direction. Hence it provides a drag component per unit span

$$L'w/V$$

and this is the local value of the *induced drag* or *vortex drag* per unit span.

Therefore the total induced drag of the wing is

$$D_i = \int_{-s}^{s} \frac{L'w}{V}\, dy = \int_{-s}^{s} \rho w \Gamma\, dy. \tag{11.6,17}$$

From (11.6,12) and (11.6,13) we can write

$$D_i = 4\rho V^2 s^2 \int_0^{\pi} \left(\sum_1^{\infty} \gamma_n \sin n\theta \right) \left(\sum_1^{\infty} n\gamma_n \sin n\theta \right) d\theta$$

$$= \frac{\pi}{2} \rho V^2 b^2 \sum_1^{\infty} n\gamma_n^{\,2} \tag{11.6,18}$$

$$= \frac{\pi}{2} \rho V^2 b^2 (\gamma_1^{\,2} + 2\gamma_2^{\,2} + 3\gamma_3^{\,2} + \ldots)$$

$$= \frac{\pi}{2} \rho V^2 b^2 \gamma_1^{\,2} (1 + \delta), \tag{11.6,19}$$

where δ cannot be negative and is usually small.

Using (11.6,16) we have therefore

$$D_i = \frac{\pi}{2} \rho V^2 b^2 (1 + \delta) \frac{C_L^{\,2}}{\pi^2 A^2}$$

$$= \frac{\rho V^2 S}{2\pi A} (1 + \delta) C_L^{\,2} \tag{11.6,20}$$

since $b^2 = SA$.

Hence the induced drag coefficient is

$$C_{Di} = 2D_i/\rho V^2 S = \frac{C_L^{\,2}}{\pi A}(1 + \delta). \tag{11.6,21}$$

Thus, for a given value of the lift coefficient C_L the value of C_{Di} increases with reduction of aspect ratio. The quantity δ is a function of wing plan form and twist (i.e. variation of no-lift angle along the wing span). Its general order of magnitude can be gauged from the following specimen values for untwisted rectangular wings:—

A/a_∞	δ
0.5	0.019
1.0	0.049
1.75	0.088

Of particular interest is the wing for which $\delta = 0$ and the induced drag is a minimum. In that case $\gamma_2 = \gamma_3 = \ldots = 0$. Hence from (11.6,12)

$$\Gamma = 4sV\gamma_1 \sin\theta = 4sV\gamma_1 \left[1 - \left(\frac{y}{s} \right)^2 \right]^{\frac{1}{2}} \tag{11.6,22}$$

and therefore if Γ_0 denotes the value of Γ on the centre line ($y = 0$)

$$\frac{\Gamma^2}{\Gamma_0^{\,2}} + \frac{y^2}{s^2} = 1,$$

i.e. the distribution of circulation along the span is elliptic. Further, from (11.6,13)

$$\begin{aligned} w = V\gamma_1 &= \text{const. over the whole span} \\ &= VC_L/\pi A. \end{aligned} \tag{11.6,23}$$

The effective incidence from equation (11.6,8) is

$$\alpha_e = \alpha - \frac{w}{V} = \alpha - \frac{C_L}{\pi A}, \tag{11.6,24}$$

and since $C_L = a_\infty \alpha_e$

$$\alpha_e = \alpha \Big/ \left[1 + \frac{a_\infty}{\pi A}\right]. \tag{11.6,25}$$

But if a is the overall value of $dC_L/d\alpha$ (the lift curve slope)

$$C_L = a\alpha = a_\infty \alpha_e$$

and therefore

$$\left.\begin{aligned} a &= a_\infty \Big/ \left[1 + \frac{a_\infty}{\pi A}\right] \\ \text{or} \quad \frac{1}{a} &= \frac{1}{a_\infty} + \frac{1}{\pi A}. \end{aligned}\right\} \tag{11.6,26}$$

Hence the lift curve slope of the wing decreases with reduction of aspect ratio. This is a general result and follows directly from the fact that for a given lift there is an increase in induced downwash, which reduces the effective incidence as the span of the trailing vortex sheet is reduced with reduction of aspect ratio.

With elliptic loading we note from (11.6,14) that

$$(\mu + \sin\theta)\gamma_1 \sin\theta = \mu\alpha \sin\theta.$$

But

$$\gamma_1 = C_L/\pi A = a\alpha/\pi A = \left[\frac{a_\infty/\pi A}{1 + \dfrac{a_\infty}{\pi A}}\right]\alpha$$

and hence

$$\frac{(\mu + \sin\theta)a_\infty}{\pi A + a_\infty} = \mu$$

or

$$\mu = a_\infty \sin\theta/\pi A. \tag{11.6,27}$$

But $\mu = \dfrac{a_\infty c}{8s}$ and therefore

$$c = \frac{8s\sin\theta}{\pi A} = c_0 \sin\theta = c_0\left[1 - \frac{y^2}{s^2}\right]^{\frac{1}{2}} \tag{11.6,28}$$

where

$$c_0 = \frac{8s}{\pi A} = \frac{4}{\pi}\bar{c}$$

and $\bar{c}$ is the mean chord.

Thus the chord c like the circulation is distributed elliptically along the span, and c_0 is the chord at mid-span.

We thus have the striking result that the minimum induced drag coefficient for a given lift coefficient is obtained with an untwisted wing of elliptic plan form with the value of a_∞ constant along the span, and then $C_{D_i} = C_L^2/\pi A$.

For the more general case of non-elliptic loading we find that, instead of (11.6,24), we can write

$$\alpha - \alpha_\infty = \frac{C_L}{\pi A}(1 + \tau) \tag{11.6,29}$$

where τ is usually small for conventional plan forms. Thus, for an untwisted rectangular wing τ is found to have the following values:—

A/a_∞	τ
0·5	0·1
1·0	0·17
1·75	0·24

In the general case, given the plan form geometry, section characteristics and twist, the values of τ and δ are obtained as a result of solving equation (11.6,14). Having the values of τ and δ we can then obtain the induced drag and lift curve slope from equations (11.6,21) and (11.6,29). For further details of the general solution of equation (11.6,14) reference may be made to Glauert,[1] and Robinson and Laurmann.[2]

11.6.3 Slender Wing Theory

We repeat the warning that lifting line theory is applicable only to wings of moderate or large aspect ratios and zero or small sweep. For all other cases the details of the chordwise as well as the spanwise loading must be taken into account. Lifting surface theories, as they are called, of varying complexity and accuracy have been developed to deal with such cases and a description of some has been given by Robinson and Laurmann. The analysis as well as the numerical work involved in the application of such theories is inevitably considerably greater than that of lifting line theory. However, at the other end of the spectrum of plan form shapes, viz. the slender wing of very small aspect ratio, a welcome simplification of the analysis can again be made. We must confine ourselves here to a brief discussion of the essentials of the analysis, but it should be noted that the analysis is readily generalised to slender configurations of bodies and body-wing combinations in compressible and incompressible flow.

The basic idea of the theory was originally developed by Munk[3] when considering the aerodynamic forces on airship hulls, but its latent possibilities for dealing with all problems of slender configurations, and in particular slender wings, over a wide range of Mach numbers, were first exploited by R. T. Jones.[1] This basic idea is that for an elongated wing or body at a small angle of attack the flow pattern in any transverse plane, i.e. a plane substantially normal to the main stream direction, approximates near the body to that of two-dimensional incompressible flow. For a full discussion of the errors involved in this concept the reader is referred to Ward[2] and Adams and Sears.[3]

For incompressible flow this assumption is equivalent to saying that for such shapes the gradient of the streamwise perturbation velocity component with respect to x (the streamwise coordinate) are small near the body compared with the corresponding gradients of the lateral components with respect to the lateral (y and z) ordinates. In consequence the incompressible flow perturbation potential function equation, viz.

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

reduces to the Laplace equation in planes parallel to the yz plane, i.e.

$$\phi_{yy} + \phi_{zz} = 0.\dagger \tag{11.6,30}$$

A class of solutions to this equation can be written

$$\phi = \phi_1[y,z;x] + \phi_2(x), \tag{11.6,31}$$

where ϕ_1 is the solution of the two-dimensional incompressible flow in the transverse plane $x = \text{const.}$, and ϕ_2 is a function of x only. We note, however, that since the boundary conditions for ϕ_1 will in general vary with x, ϕ_1 is implicitly a function of x.

The perturbation velocity components are then

$$\left.\begin{aligned} u &= -\phi_x = -\phi_{1x} - \phi_{2x}, \\ v &= -\phi_y = -\phi_{1y}, \\ w &= -\phi_z = -\phi_{1z}. \end{aligned}\right\} \tag{11.6,32}$$

The pressure coefficient at any point is then given, within the assumptions of small perturbation theory, by‡ (see equation 9.11,7)

$$C_p = -2u/V = 2(\phi_{1x} + \phi_{2x})/V, \tag{11.6,33}$$

where V is the undisturbed stream velocity.

However, since ϕ_2 is a function of x only, its contribution to the pressure coefficient is a constant for any given transverse section. Hence it follows that the transverse forces and moments on the configuration are independent of ϕ_2, and for the purpose of evaluating them ϕ_2 can be ignored. For the determination of pressure distribution, however, ϕ_2 cannot be neglected. We shall confine ourselves here to considering the aerodynamic characteristics of the plate wing of slender plan form, for which ϕ_2 need not be taken into account (and is in any case zero).† For more general problems of wings and bodies of finite thickness distribution reference should be made to the authors already cited.

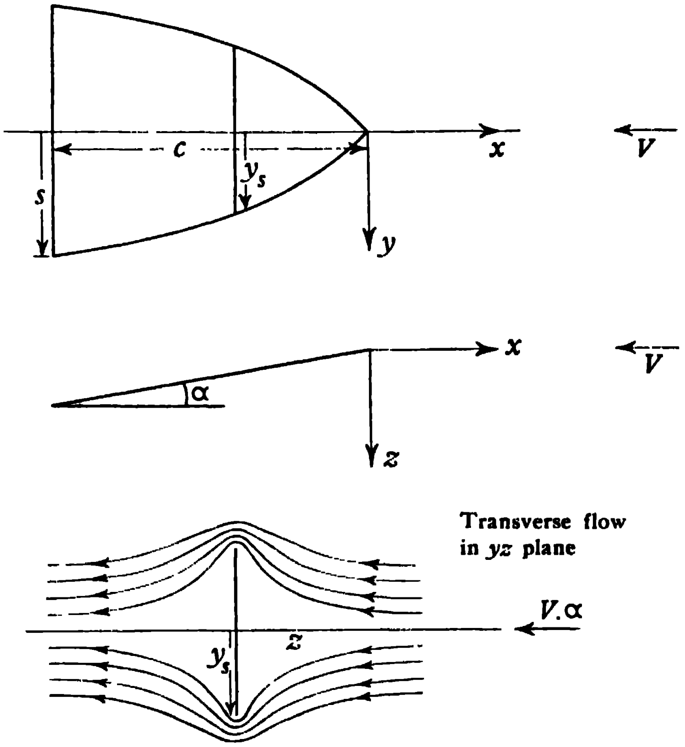


Fig. 11.6,7. Slender flat plate aerofoil at incidence in a uniform stream.

For simplicity consider a flat slender wing of root chord c and maximum span $b = 2s$, at a small incidence α in a main stream of velocity V in the plane of symmetry of the wing (Fig. 11.6,7). Take the x axis forward in the opposite direction to V, the y axis to starboard, and the z axis downward. Denote the semi-span of the wing at a distance x from the apex by y_s, then we shall assume that y_s increases monotonically from the apex to the trailing edge. We shall assume that the trailing edge is unswept. The theory then leads to the statement that near the wing the flow in the transverse section at the station is that past a plane lamina of span $2y_s$ in an otherwise uniform stream moving normal to the lamina with velocity $V\alpha$.

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† It is found that ϕ_2 can be obtained from the equivalent source distribution, $q(x)$ arising from the distribution of cross-sectional area S as a function of x, i.e.

$$q(x) = V\,dS/dx.$$

For this latter problem we have that at the surface of the lamina (see § 2.15 Ex. 1)

$$\phi_1 = \pm V\alpha(y_s^2 - y^2)^{\frac{1}{2}} \tag{11.6,34}$$

where the plus sign refers to the upper surface and the minus sign refers to the lower surface.† Hence from (11.6,33) the difference between the pressure coefficients on upper and lower surfaces is given by

$$\Delta C_p = C_{p_l} - C_{p_u} = 2C_{p_l} = -\frac{4}{V}(\phi_{1x})_l$$

$$\left.\begin{aligned} &= -\frac{4\alpha y_s}{(y_s^2 - y^2)^{\frac{1}{2}}}\frac{dy_s}{dx} \\ &= -\frac{4\alpha}{\sin\theta}\frac{dy_s}{dx} \\ \text{where} \qquad \theta &= \cos^{-1}(y/y_s). \end{aligned}\right\} \tag{11.6,35}$$

We see that the loading distribution is infinite at the leading edge, but the infinity is of the same kind as that obtained at the leading edge of a two-dimensional flat plate aerofoil at small incidences (cf. equation 11.5,8). The infinity does not lead to unacceptable values of total lift and pitching moment, but is regarded as a valid limit for potential flow associated with reducing the leading edge radius to zero.

The lift on a spanwise strip of width δx is

$$\delta L = \left[\tfrac{1}{2}\rho V^2 \int_{-y_s}^{y_s} -\frac{4\alpha}{\sin\theta}\frac{dy_s}{dx}\,dy\right]\delta x$$

and since $dy = -y_s \sin\theta\, d\theta$

$$\delta L/\tfrac{1}{2}\rho V^2 = -4\alpha\pi(dy_s/dx)y_s\,\delta x. \tag{11.6,36}$$

Therefore, the lift coefficient of the wing is

$$C_L = \frac{1}{\frac{1}{2}\rho V^2 S}\int_{-c}^{0}\frac{dL}{dx}\,\delta x = -\frac{2\alpha\pi}{S}\int_{b^2/4}^{0} d(y_s^2)$$

$$\left.\begin{aligned} &= \alpha\pi b^2/2S \\ &= \alpha\pi A/2, \end{aligned}\right\} \tag{11.6,37}$$

where A is the aspect ratio $= b^2/S$.

Hence

$$dC_L/d\alpha = \pi A/2. \tag{11.6,38}$$

Equation (11.6,38), which gives the theoretical lift curve slope of a wing for the limiting case as the aspect ratio tends to zero, is found to be in good agreement with experiment.

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† Note that with the axes chosen V is negative.

It can also be readily inferred that the lift on a chordwise strip of width δy distant y from the centre line is

$$\delta L = \tfrac{1}{2}\rho V^2 4\alpha\left(\frac{b^2}{4} - y^2\right)^{\frac{1}{2}} \delta y$$

and hence the spanwise distribution of loading on the wing is elliptic.

This result could also be obtained by noting (see § 2.12) that the circulation round such a strip in a plane parallel to the xz plane must equal the discontinuity in ϕ (and therefore in ϕ_1) at the trailing edge, i.e.

$$\Gamma(y) = 2\,|\phi_u|_{T.E.} = 2V\alpha\left(\frac{b^2}{4} - y^2\right)^{\frac{1}{2}}.$$

The lift on the strip is therefore

$$\begin{aligned}\delta L &= \Gamma(y)\rho V\,\delta y\\ &= 2\rho V^2\alpha\left(\frac{b^2}{4} - y^2\right)^{\frac{1}{2}}\delta y.\end{aligned}$$

We note that the total lift can be written

$$L = \rho V \oint_{T.E.} \phi_1\,dy \qquad (11.6,39)$$

where the integral is taken round a circuit enclosing the trailing edge.

Equation (11.6,39) is indeed quite general and is applicable to all cases of thin slender wings, whatever the plan form, camber and twist distribution, provided of course the basic assumption of small perturbations is not invalidated. We see from equation (11.6,36) that the lift on any portion of the wing between the planes $x = x_1$, and $x = x_2$, say, is equal to

$$\pi\rho V^2\alpha(y_{s2}^2 - y_{s1}^2)$$

where y_{s1} and y_{s2} are the corresponding semi-spans. This lift is therefore independent of the shape of the leading edge between the two planes.

For the case of a wing of triangular plan form it follows from equation (11.6,35) that the loading is constant along any straight line through the vertex,† since θ is constant along any such line. Hence, the centre of pressure for any elementary triangular element defined by two neighbouring values of θ is at the centroid of area for the element, i.e. on the line $x = -2c/3$, and therefore the overall centre of pressure is at the centroid of area of the plan form.

Since the spanwise loading distribution is elliptic it follows that far downstream the pattern of trailing vortices is identical with that far behind a lifting line of the same overall span and with the same elliptic loading.

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† This is a simple example of a conical flow, i.e. a flow in which there is a vertex such that along all rays through the vertex flow conditions are constant. However, we note from equation (11.6,36) that the Kutta-Joukowsky condition of zero loading at the trailing edge is not satisfied for such a flow.

But the induced drag times the forward speed represents the power required for the continuous generation of the trailing vortex system. Therefore, the induced drag coefficient of the slender wing must also be given by the formula (see equation 11.6,21 with $\delta = 0$)

$$C_{D_i} = C_L{}^2/\pi A. \tag{11.6,40}$$

Hence, from equation (11.6,37)

$$C_{D_i} = C_L\alpha/2. \tag{11.6,41}$$

It follows that the resultant force on the wing, in the absence of frictional drag, is along a direction halfway between the normal to the wing surface and the normal to the undisturbed air stream direction. But the pressures normal to the wing surface have a component coefficient in the drag direction equal to $C_L\alpha$. Hence there must exist a force in the plane of the wing whose component in the drag direction is equal to $-C_L\alpha/2$. Such a force can arise only from the infinite suction at the leading edge acting on the infinitesimally thin area of the edge, as in the case of the two-dimensional plate at incidence (cf. § 11.5,2). As in that case, this leading edge suction force can be regarded as the limit of the force component in the plane of the wing acting on a rounded leading edge as the radius of curvature of the leading edge is reduced to zero.

However, it should be noted that the magnitude of the suction force determined above refers strictly to inviscid potential flow. Separation of the boundary layer in the region of the leading edge, which may be expected in practice except over a small incidence range, can cause a reduction of the suction there and a consequent reduction of the suction force. Nevertheless, except at large incidences, such three-dimensional separation is not necessarily associated with any extensive 'dead air' regions of unsteady eddying flow as in two dimensions or for an unswept wing, but it usually takes the form of a vortex sheet which springs from each leading edge. Each sheet then rolls up into a streamwise vortex some distance above the wing surface and the flow beneath these vortices retains the full stream total head (see Fig. 11.6,8). Indeed, the vortices tend to increase the flow velocity over the surface beneath them and in consequence they increase the lift above the value to be expected without leading edge separation.

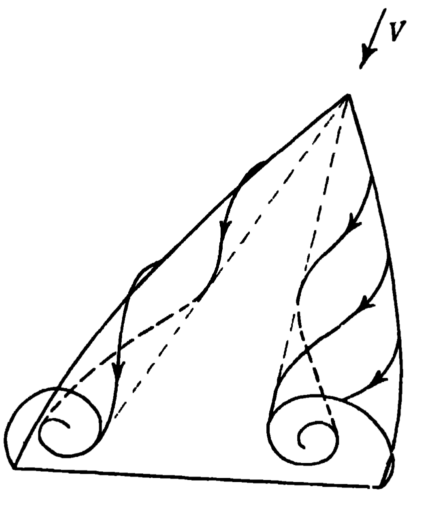

Fig. 11.6,8. Formation of vortex sheets from the leading edges of a slender wing and consequent rolling up into vortices.

The vortex sheets act in a sense like end plates as far as the flow over the wing is concerned, and in consequence produce a change in loading equivalent to that caused by an increase of aspect ratio. Indeed, if we compare the drag for the lift generated with that obtained without vortex sheet formation but with localised small scale separation and reattachment at the leading edge that is otherwise almost inevitable, then the flow with complete separation and vortex sheet formation can show up to advantage.

Thus we see that three-dimensional separation from the leading edge of a highly swept wing can be very different in character from separation in two dimensions and the former is not necessarily associated, as is the latter, with a large increase of drag and decrease of lift.

11.7 Aircraft Stability Derivatives

Consider an aircraft in flight with axes defined as in Fig. 11.1,2, and consider the resultant force and moment acting on it of aerodynamic origin and due to the propulsion units. Then any one of the components of the resultant force (X, Y, Z) or of the resultant moment (L, M, N) will be a function of the velocity components of the aircraft (U, V, W), its angular velocity components (p, q, r), the settings of its roll, pitch and yaw controls (i.e. aileron, elevator and rudder) denoted by ξ, η, ζ, respectively, and the rates of change of these quantities with time.

Thus, we can write

$$X = X[U, V, W, p, q, r, \xi, \eta, \zeta, \dot{U}, \text{etc.}]$$

and similarly for the other force and moment components.

Now suppose that the aircraft is initially in steady flight, so that these components of force balance its weight and thrust and the moment components are zero. Denote quantities in this initial condition by suffix 0. Then if the aircraft is slightly and momentarily disturbed from this initial condition small changes may result in U, V, W of amount (u, v, w), say, small angular velocity components p, q, r may develop and likewise small changes in the control settings ξ, η, ζ may be associated with the disturbance, as well as derivatives of all these quantities with respect to time. It follows that the associated change in X, to the first order in u, v, w, etc., will be given by

$$\delta X = X_u u + X_v v + X_w w + X_p p + X_q q + X_r r + \\ X_\xi \xi + X_\eta \eta + X_\zeta \zeta + X_{\dot{u}} \dot{u} + \ldots \quad (11.7,1)$$

where $X_u = \partial X/\partial u$, i.e. the rate of change of X with respect to u keeping all the other flight variables constant. Similar expressions describe the associated small changes in Y, Z, L, M and N.

Therefore, if we know the values of the quantities X_u, X_v, etc. and the nature of the initial disturbance, we can in principle, by applying Newton's Laws of Motion, determine the subsequent response of the aircraft and the time history of the force and moment components. We must make the proviso, however, that the departure of the aircraft's motion from the initial steady state must always be small, as otherwise equations such as 11.7,1 become inadequate to describe the changes in the force and moment components.

When we consider the stability of a dynamic system in a steady state, such as an aircraft in steady flight, we pose the question as to what will happen if it is slightly and momentarily disturbed from that state. If the resulting response is such that eventually the system returns to its initial steady state, then we say the system is stable; if on the other hand it eventually departs, either in an oscillatory manner or monotonically, from the initial state, then we say that it is *unstable*. If it remains in its disturbed condition without further change, then we say that it is *neutrally stable*. Thus, when considering the stability of an aircraft, initially in steady flight, we require to know how u, v, w, p, q, r, etc. change with time from the onset of the disturbance, and from this point of view it is normally sufficient to limit our attention to the behaviour of the disturbance components when they are small. Hence, for the stability analysis of an aircraft we must know the values of the quantities X_u, etc., and these important quantities are referred to as *stability derivatives*.

If the initial steady flight path is in the plane of symmetry of the aircraft, then it can readily be seen that the derivatives of X, Z and M with respect to v, p, r, ξ and ζ and their rates of change with time are all zero, as are the derivatives of Y, L and N with respect to u, w, q and η and their rates of change with time. For example, if X_v were not zero, then a small positive sideslip (to starboard) and an equal negative sideslip (to port) would result in equal but opposite force components in the x direction. But by symmetry such movements normal to the plane of symmetry can only result in the same change of force components in the plane of symmetry, and hence $X_v = 0$. Similarly, the other derivatives referred to above must be zero. Further, the derivatives with respect to $\dot{u}$, etc. $\dot{p}$, etc. $\dot{\xi}$, etc. are usually regarded as negligible for stability analysis, with the exception of $M_{\dot{w}}$. There is a time lag between a change of incidence of an aircraft wing and the corresponding downwash change in the region of the tail plane which causes this last derivative to be of some significance.

We are left finally with the derivatives of X, Z and M with respect to u, w, q and η (as well as $M_{\dot{w}}$), and those of Y, L and N with respect to v, p, r, ξ and ζ. The former group of derivatives are referred to as the *longitudinal group*, since they determine the motion in the plane of symmetry for small disturbances from initially steady flight in that plane, whilst the latter group are referred to as the *lateral group*, since they determine the lateral motion of the aircraft (i.e. rolling, yawing and sideslipping) as a result of the disturbance. With the initial conditions as specified above, i.e. steady flight in the plane of symmetry and small disturbances, there is then no coupling between the longitudinal and lateral motions in response to the disturbances and the two motions can be independently considered. We speak then of the longitudinal stability and of the lateral stability of the aircraft.

As with many other aerodynamic quantities it is convenient to use non-dimensional forms for the derivatives. The following Table† presents in standard British notation the derivatives in both dimensional and non-dimensional form and the quantities used to render them non-dimensional.

I Dimensional quantities	II Divisors	III Non-dimensional Quantities (Col. I ÷ Col. II)	IV Name
X Y Z	$\frac{1}{2}\rho V_1^2 S$	C_x C_y C_z	Force coefficients
L M N	$\frac{1}{2}\rho V_1^2 S l_1$	C_l C_m C_n	Moment coefficients
X_u Z_u Y_v X_w Z_w	$\rho V_1 S$	x_u z_u y_v x_w z_w	Force-velocity derivatives
Y_p X_q Z_q Y_r	$\rho V_1 S l_2$	y_p x_q z_q y_r	Force-angular velocity derivatives
M_u L_v N_v M_w	$\rho V_1 S l_2$	m_u l_v n_v m_w	Moment-velocity derivatives
L_p N_p M_q L_r N_r	$\rho V_1 S l_2^2$	l_p n_p m_q l_r n_r	Moment-angular velocity derivatives
Y_ξ X_η Z_η Y_ζ	$\rho V_1^2 S$	y_ξ x_η z_η y_ζ	Force-control movement derivatives
L_ξ N_ξ M_η L_ζ N_ζ	$\rho V_1^2 S l_2$	l_ξ n_ξ m_η l_ζ n_ζ	Moment-control movement derivatives
$M_{\dot{w}}$	$W l_2/g$ or $\rho S l_2^2$	$m_{\dot{w}}$ or $\overline{m}_{\dot{w}}$	Moment-downwash lag derivative
A B C E	$W l_2^2/g$	i_A i_B i_C i_E	Inertia coefficients

For the above table the following definitions of the lengths l_1 and l_2 apply:

	Longitudinal group	Lateral group
l_1	$\bar{c}$ = mean chord	b = span
l_2	l_t = tail arm	$b/2$ = semi-span

and V_1 is the flight velocity.

† Taken from the Aerodynamics Data Sheets of the Royal Aeronautical Society.

By tail arm is here meant the distance between the aerodynamic centre of the aircraft without tail and that of the tail itself.

For a fuller discussion reference should be made to the relevant *Aerodynamics Data Sheets* of the Royal Aeronautical Society. A revised and comprehensive system of notation which may be widely adopted has been recently put forward by Hopkin.[1]

11.8 Cascades

11.8.1 Introduction

When the flow in a duct is required to negotiate a rapid bend as, for example, a corner of a wind tunnel, large energy losses are liable to occur due to flow separation which usually develops from the inner face of the bend. To reduce the losses a cascade of aerofoils, or guide vanes as they are generally called, can be fitted and, if properly designed, they enable the flow to negotiate the bend without separation and ensure reasonable uniformity of flow downstream of the bend. The losses then are essentially due to the boundary layer drag of the vanes of the cascade and of the duct walls, augmented by secondary flows within the boundary layers induced by the transverse pressure gradients.

The theory of the flow past cascades of aerofoils has been developed as an extension of two-dimensional aerofoil theory, and it has applications not only to the design of guide vanes but also to the design of the blades of axial flow compressors and turbines. We shall confine ourselves here to demonstrating a few basic features of the flow that can be readily deduced from simple physical considerations. For a fuller account of the subject and bibliographies the reader is referred to Schlichting,[2] Robinson and Laurmann[3] and to Thwaites.[4]

11.8.2 Basic Relations for Cascade Flow

Take the y axis parallel to lines joining corresponding points of the aerofoils and the x axis normal to such lines, as illustrated in Fig. 11.8,1. Let q_1 be the velocity of the incident stream far upstream with components $-U_1$ and V_1, and let q_2 be the velocity far downstream with components $-U_2$ and V_2. Let p_1 and p_2 be the corresponding values of the static pressure. The distance between neighbouring aerofoils measured parallel to the y axis is called the *gap* and is denoted by h, and the chord of each aerofoil is denoted by c. The angle which the chord of an aerofoil makes with the x axis is referred to as the *stagger*, and is here denoted by σ.

Consider first the cascade in inviscid flow. It will be clear that if we consider the flow conditions along any line parallel to the y axis they will repeat themselves at intervals of length h. Therefore any two streamlines distant h apart when $x = 0$ will remain the same distance apart, measured parallel to the y axis, for all other values of x. The rate of mass flow between the two streamlines must be constant and far upstream and downstream this rate of mass flow is $\rho U_1 h$ and $\rho U_2 h$, respectively.

Therefore

$$U_1 = U_2 = U, \text{ say.} \tag{11.8,1}$$

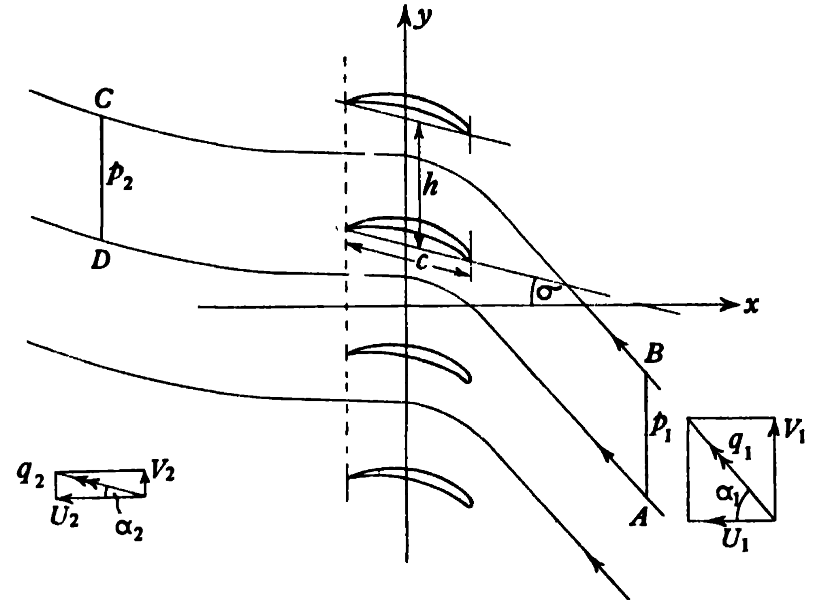

Fig. 11.8,1.

Bernoulli's equation for incompressible inviscid flow then leads to

$$\begin{aligned} p_2 - p_1 &= \frac{\rho}{2}[(V_1^2 + U_1^2) - (V_2^2 + U_2^2)] \\ &= \frac{\rho}{2}(V_1^2 - V_2^2). \end{aligned} \tag{11.8,2}$$

Consider the circuit denoted by $ABCD$ in Fig. 11.8,1, enclosing an aerofoil, where AD and BC are streamlines, AB and CD are parallel to the y axis and of length h and are sufficiently far upstream and downstream of the cascade for flow conditions there to be regarded as uniform. If we denote the force components per unit span acting on the aerofoil as X' and Y', then these can be equated to the corresponding resultants of the pressures acting on $ABCD$ and the corresponding net rates of flow of momentum.

Thus

$$\left.\begin{aligned} X' &= h(p_2 - p_1) \\ \text{and} \quad Y' &= \rho U h(V_1 - V_2) \\ &= \rho U^2 h(\tan\alpha_1 - \tan\alpha_2) \end{aligned}\right\} \tag{11.8,3}$$

where α_1 and α_2 are the angles that q_1 and q_2, respectively, make with the x axis reversed.

Since the velocity distribution along BC is identical with that along AD, it follows that the circulation round $ABCD$, which must be the same as that round the aerofoil, is

$$\Gamma = (V_1 - V_2)h. \tag{11.8,4}$$

Hence, from (11.8,3),

$$Y' = \rho U \Gamma \tag{11.8,5}$$

as for an isolated aerofoil.

Further, from (11.8,2) and (11.8,3)

$$X' = \frac{\rho}{2} h(V_1^2 - V_2^2)$$

$$= \frac{\rho}{2}(V_1 + V_2)\Gamma. \tag{11.8,6}$$

Write

$$V_m = (V_1 + V_2)/2, \tag{11.8,7}$$

and

$$\alpha_m = \tan^{-1} V_m/U. \tag{11.8,8}$$

The relationship between α_m, α_1 and α_2 is illustrated in Fig. 11.8,2.

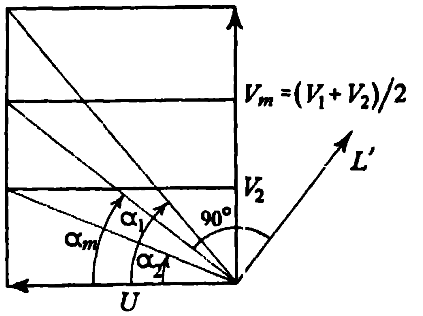

Fig. 11.8,2.

Then we see that the resultant force per unit span is L' where

$$L' = \rho\Gamma(U^2 + V_m^2)^{\frac{1}{2}}, \tag{11.8,9}$$

and its line of action is at right angles to the direction defined by the angle α_m. The lift coefficient is then

$$C_L = L'/[\tfrac{1}{2}\rho(U^2 + V_m^2)c] = \frac{2\Gamma}{c}\Big/(U^2 + V_m^2)^{\frac{1}{2}}$$

$$\frac{2h}{c}(\tan\alpha_1 - \tan\alpha_2)\cos\alpha_m. \tag{11.8,10}$$

We note that α_m and $(U^2 + V_m^2)^{\frac{1}{2}}$ are the direction and magnitude, respectively, of the vector mean of the stream velocities far upstream and downstream of the cascade.

The relation between the lift on the cascade aerofoil and the circulation round it (equation 11.8,9) is therefore the same as the Kutta-Joukowsky relation (equation 11.3,6) for an isolated aerofoil in a stream for which the velocity vector is this mean, and further in both cases the direction of the lift is normal to this velocity vector.

The above analysis applies to a cascade in an inviscid fluid. In a real fluid boundary layer drag of the cascade aerofoils (including possible separation and secondary flow effects) will result in a defect of flux of momentum downstream of the cascade, manifest mainly in the form of streamwise vortices and wakes streaming from the aerofoils. If we assume that far enough downstream these losses of momentum flux can be regarded as uniformly distributed over the flow, then instead of equation (11.8,2) we have

$$p_2 - p_1 = \frac{\rho}{2}(V_1^2 - V_2^2) - \Delta H, \qquad (11.8,11)$$

where ΔH is the mean loss of total head caused by the cascade.

The equation for X' in (11.8,6) then becomes

$$\begin{aligned} X' &= \frac{\rho}{2}h(V_1^2 - V_2^2) - h\Delta H \\ &= \rho V_m \Gamma - h\Delta H, \qquad (11.8,12) \end{aligned}$$

but the expressions for Y' (equations 11.8,3 and 11.8,5) remain unchanged.

If we now regard the lift and drag forces per unit span as the components normal to and along the direction defined by the angle α_m, then

$$\begin{aligned} L' &= Y' \cos \alpha_m + X' \sin \alpha_m \\ &= \rho \Gamma U \cos \alpha_m + (\rho \Gamma V_m - h\,\Delta H) \sin \alpha_m \\ &= \rho \Gamma (U^2 + V_m^2)^{\frac{1}{2}} - h\,\Delta H \sin \alpha_m, \end{aligned}$$

and

$$\begin{aligned} D' &= Y' \sin \alpha_m - X' \cos \alpha_m \\ &= h\,\Delta H \cos \alpha_m. \end{aligned}$$

If we define lift and drag coefficients as

$$C_L = L'/[\tfrac{1}{2}\rho(U^2 + V_m^2)c], \quad \text{and} \quad C_D = D'/[\tfrac{1}{2}\rho(U^2 + V_m^2)c]$$

then

$$C_D = 2h\,\Delta H \cos \alpha_m/[c\rho(U^2 + V_m^2)]$$

and

$$C_L = \frac{2h}{c}(\tan \alpha_1 - \tan \alpha_2) \cos \alpha_m - C_D \tan \alpha_m. \qquad (11.8,13)$$

11.8.3 Further Theoretical and Experimental Features of Cascade Flow

To proceed further we require to determine the outflow angle α_2 in terms of α_1 and the cascade geometry, as well as the pressure distribution on the cascade members. To this end the theory of potential flow past a cascade has been developed on lines closely similar to the theory of potential flow past isolated aerofoils.

Thus, it is possible by suitable conformal transformations to relate the flow past a cascade of aerofoils to that past a single contour with associated singularities and thence to the flow past a circular boundary also with associated singularities. Solutions can also be obtained by suitable transformations of the flow in the hodograph plane.† Again, it is possible to determine the cascade flow by representing the aerofoils as distributions of bound vorticity, as in thin aerofoil theory (see § 11.5,3), if required thickness effects can be represented by source distributions, as in § 11.5,6. The algebra involved is inevitably somewhat complicated and cannot be reproduced here; for details of some representative methods of analysis of cascade flow the reader is referred to Collar,[1] Merchant and Collar,[2] Lighthill,[3] Carter and Hughes,[4] Garrick,[5] Howell,[6] Goldstein and Jerison,[7] Schlichting[8] and Scholz.[9]

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† In the hodograph plane the cartesian components of velocity u, v are the coordinates. Hence the polar coordinates are the velocity magnitude and direction.

A result of the theory for a cascade of flat plates is

$$C_L = \frac{4h}{c}\frac{\sin(\alpha_m - \sigma)}{k\cos\sigma} \tag{11.8,14}$$

where σ is the stagger angle as defined in Fig. 11.8,1 and k is a parameter which is a function of the cascade geometry and is determined by the relation

$$\frac{c}{h} = \frac{1}{\pi}\left[\cos\sigma \ln\frac{k+1}{k-1} + 2\sin\sigma\tan^{-1}\left(\frac{\tan\sigma}{k}\right)\right]. \tag{11.8,15}$$

Write $k = 1 + \varepsilon$, say. Then, from (11.8,15) it follows that for small values of h/c

$$\varepsilon \doteqdot 2\exp\left[-\frac{\pi c}{h\cos\sigma} - 2\sigma\tan\sigma\right],$$

and hence ε is small and k is near unity.

The theory also leads to the result

$$\frac{\tan\alpha_1 - \tan\sigma}{\tan\alpha_2 - \tan\sigma} = \frac{k+1}{k-1} \tag{11.8,16}$$

and hence for small values of h/c the difference between α_2 and σ must be small. In other words the outflow angle α_2 is not very different from the angle of flow at the flat plate aerofoils and is not greatly influenced by the incident angle of flow α_1. This is a particular case of a more general result that is found for cascades of aerofoils with small gap/chord ratio, namely that over a fairly wide range of values of α_1 the outflow angle α_2 is found to be nearly constant and it approximates roughly to the angle of the camber line at the trailing edge of a cascade aerofoil, γ_{TE}. More detailed analysis of the theoretical and experimental differences between α_2 and γ_{TE} indicates that $(\alpha_2 - \gamma_{TE})/\gamma_{TE}$ varies between about $0{\cdot}1(h/c)^n$ and $0{\cdot}3(h/c)^n$, depending on the shape of the camber line, where n is close to unity for cascades through which the pressure falls ($p_1 > p_2$), as for turbine cascades, and n is nearer to $\frac{1}{2}$ for cascades through which the pressure rises ($p_1 < p_2$) as for compressor cascades. It will be readily appreciated that for the former type of cascade the boundary layers on the aerofoils develop in a more favourable (negative) pressure gradient and hence they do not thicken so rapidly and flow separation is less likely. For the latter type of cascade, on the other hand, the pressure gradient is mainly unfavourable, the boundary layers are consequently thicker and flow separation is much more likely. It is not therefore surprising to find that the 'turning' effectiveness of the former type of cascade is somewhat greater than that of the latter.

Apart from the imposed pressure gradient arising from the fact that the pressures upstream and downstream are not the same for a cascade, the pressure distributions over the aerofoils resemble qualitatively the distributions over isolated aerofoils. With increase of α_1 there is a tendency for a suction peak to develop and grow on the upper surface and eventually the subsequent adverse pressure gradient reaches a value for which flow separation develops and the cascade stalls. Maximum lift coefficients in terms of the downstream dynamic pressure are of the same order as those achieved with isolated aerofoils, viz. about 1·4. With reduction of α_1, on the other hand, a suction peak tends to develop on the lower surface near the leading edge, and for sufficiently small values of α_1 flow separation from the lower surface may result. Between these extreme conditions of upper and lower surface flow separation, the flow approximates reasonably closely to that predicted by inviscid flow theory. This range of α_1 for which no flow separation occurs depends not only on the cascade geometry but also on the Reynolds number. The lower the Reynolds number the more likely is laminar boundary layer separation, and the smaller is the working range of α_1; this is to be expected in the light of our discussion of flow separation from isolated aerofoils (see § 11.4). The effect of Reynolds number is a factor of some significance in the design of gas turbine engines for aircraft intended to fly at considerable heights, since then the Reynolds number of the blades of the compressors and turbines can be low enough for laminar boundary layer separation to be possible.

The boundary layer drag of a cascade can be calculated by much the same methods as those used to calculate the boundary layer drag of an isolated aerofoil (see § 6.21,2). In the working range of α_1 the boundary layer drag provides the major part of the total head losses, the remainder being provided by secondary flow effects. Schlichting and Scholz[1] have performed such calculations for a series of cascades covering a range of gap-chord ratios, but having the same theoretical angle of turn ($\alpha_1 - \alpha_2$), and with pressure rise (compressor cascade) and presssure fall (turbine cascade). The calculated loss was expressed as a coefficient in the form

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[1] H. Schlichting and N. Scholz, *Ing. Archiv*, **19**, 42–65 (1951).

$$\zeta = \Delta H/\tfrac{1}{2}\rho U^2$$

where U was the velocity component in the negative x direction of the far downstream and far upstream flow. The results are illustrated in Fig. 11.8,3.

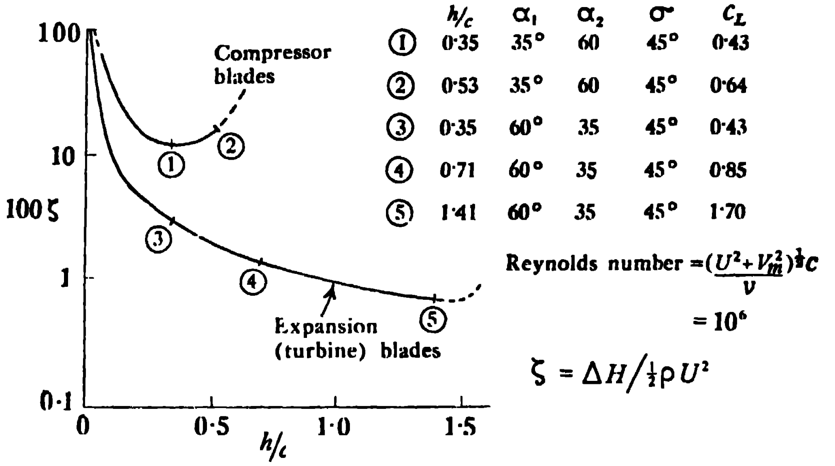


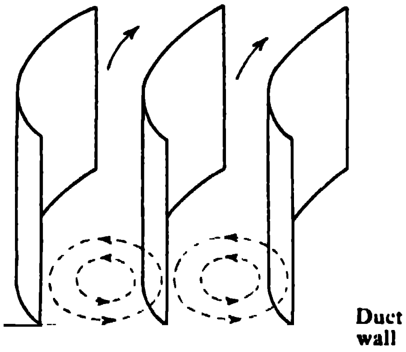
Fig. 11.8,3. Theoretical loss coefficient of a cascade as a function of gap-chord ratio, as determined by Schlichting and Scholz. Reproduced by kind permission of Prof. Schlichting from *Grenzschicht-Theorie*, Verlag G. Braun, Karlsruhe, 1951.

It will be seen that for the expansion (turbine) cascade the losses fall steadily with increase of h/c for a wide range of h/c due to the reduced frictional losses as the number of blades decreases over a given length of cascade. But as h/c increases the loading per blade must increase to provide the required turning effect and this implies increasingly adverse gradient over the upper surface, with associated increased frictional losses per blade. Therefore, a value of h/c is eventually reached beyond which the overall losses begin to increase again, and these losses may well be augmented at that stage by flow separation.

It will be noted that the losses with compressor blades are much greater than those with turbine blades and the range of h/c over which the losses are unlikely to be seriously increased by the adverse gradient effects, including flow separation, is much more limited.

With a fuller understanding of the design requirements to keep the direct boundary layer drag losses to a minimum has grown the need to know more about the effects of secondary flow, and this subject has received increasing attention in recent years. Secondary flow arises from the presence of the boundary layers on the walls of the duct normal to the span of the guide vanes, since due to these boundary layers the flow over the guide vanes is not completely two-dimensional. Because the flow in these boundary layers is slower than in the main flow outside them, the transverse pressure gradients required to keep the main flow turning tend to over-deflect the flow in the wall boundary layers. Hence a secondary or transverse flow develops in these boundary layers towards the upper surface of each blade and away from the lower surface of the adjacent blade. A compensatory transverse flow develops in the main stream outside the wall boundary layers away from the upper surface of each blade and towards the lower surface of the adjacent blade. Thus, between adjacent blades a transverse vortex pattern is set up near the duct wall, as illustrated in Fig. 11.8,4, the scale and intensity of which are determined by the geometry of the cascade and the thickness of the wall boundary layer.

Fig. 11.8,4. Secondary flow pattern between guide vanes in a duct near the duct wall.



Further, the fact that the flow conditions along the span of each blade are not constant, due to the presence of the wall boundary layers, implies that the circulation round each blade will vary along the span, particularly over the parts immersed in the wall boundary layers. In consequence, as for isolated aerofoils of finite span (see § 11.6), trailing vortices stream off the trailing edge of each blade aligned with the general stream direction there and of strength determined by the spanwise gradient of circulation along the blade. Hence, these trailing vortices are strongest near the blade ends. They are of opposite sign to the vortices between the blade ends referred to above.

The final pattern of vorticity involves an absorption of power in its continuous creation which is manifest as a total head loss additonal to that directly due to the boundary layers on the blades. In this sense it is analogous to the trailing vortex pattern of an isolated wing of finite span and the associated induced drag. Further, as can be inferred from Fig. 11.8,4, the secondary flow results in the concentration of the duct wall boundary layer into the corners formed by the wall and the upper surface of each blade, and such a concentration of 'tired air' may readily lead to an early separation of flow from these corners and an associated increase of total head loss. These facts underline the importance of the study of secondary flows. For further details the reader is referred to Squire and Winter,[1] Hawthorne[2] and Preston.[3]

EXERCISES. CHAPTER 11

1. Consider in an inviscid fluid a small tetrahedron formed by the planes defined by three mutually perpendicular axes, Ox, Oy and Oz, through a point O and a plane inclined to these axes at a small distance from O. Suppose the normal pressure on the plane is p. Show that
$$p_{xx} = p_{yy} = p_{zz} = -p$$
by considering the balance of forces on the fluid contained within the tetrahedron and the associated rates of change of momenta and then making the distance from O to the inclined plane shrink to zero. Deduce that in the flow of an inviscid fluid the normal stress on a surface element containing a given point is a function only of the position of the point and is independent of the orientation of the surface element. The pressure at the point is defined as minus this normal stress.
2. Consider in a real fluid a small box of sides of length δx, δy and δz parallel to the x, y and z axes, respectively, with a point O at one corner of the box. By considering the balance of moments and the associated rates of change of angular momenta about each of the sides through O and then making δx, δy and δz tend to zero show that
$$p_{yz} = p_{zy}, \quad p_{zx} = p_{xz}, \quad p_{xy} = p_{yx}.$$
3. If the tetrahedron of question 1 is considered in a viscous fluid and if ν denotes the direction of the normal to the inclined plane with direction cosines (l, m, n) show that the normal stress on that plane is given by
$$p_{\nu\nu} = p_{xx}l^2 + p_{yy}m^2 + p_{zz}n^2 + 2p_{yz}mn + 2p_{zx}nl + 2p_{xy}lm.$$
Hence show that the sum of the normal stresses on three mutually perpendicular planes through a point is a function only of the position of the point and is independent of the orientation of the planes. The pressure at the point is defined as minus one third of this sum, i.e.,
$$p = -\frac{1}{3}(p_{xx} + p_{yy} + p_{zz}).$$
4. Show that, however the x, y, z axes are orientated at a point O,
$$e_{xx} + e_{yy} + e_{zz} = \text{const.}$$
This quantity is called the expansion and it is zero for incompressible flow.
5. By considering the balance of forces and associated rates of change of momenta in the x direction for the fluid in the box of Exercise 2, or otherwise, show that
$$\frac{\rho Du}{Dt} = \frac{\partial p_{xx}}{\delta x} + \frac{\partial p_{yx}}{\delta y} + \frac{\partial p_{zx}}{\delta z},$$
where it is assumed that body forces can be neglected. Similar equations can be obtained for the y and z directions.
6. From the equations derived in Exercise 5 and the relations between the stress and rate of strain components given in equation 11.1.1 deduce that
$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u, \quad \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2 v, \quad \text{and} \quad \frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w,$$
where ∇^2 denotes the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.
These equations are called the *Navier-Stokes equations* and are fundamental to the general analysis of the flow of a viscous fluid.

7. Show that when the lift coefficient on a circular cylinder (based on diameter) is 4π the circulation is such that the two stagnation points are coincident and occur at $\theta = 270°$.

8. A vortex of strength Γ is held fixed in a uniform stream of velocity V parallel to an infinite wall distant d from the vortex. Show that the lift on the vortex is a maximum when $\Gamma = 2\pi dV$. Hence show that if the lift per unit length of vortex (L') is expressed in coefficient form as $C_L = L'/\frac{1}{2}\rho V^2 c$, where c is some length, then the maximum value of C_L is given by $2\pi d/c$.

9. In the light of the discussion about the relationship between the circulation and lift loading on a wing section and the boundary layer development on its surfaces provide simple qualitative explanations of the following facts:

   (i) With increase of Reynolds number, but with fixed transition points, the value of $dC_L/d\alpha$ increases.

   (ii) With forward movement of the transition points at constant Reynolds number the value of $dC_L/d\alpha$ decreases.

   (iii) With increase of Reynolds number, but with fixed transition points, the aerodynamic centre moves rearward towards the quarter-chord point.

10. Show that if referred to some point O the lift, drag and pitching moment coefficients of a wing section at incidence α are C_L, C_D and $C_M(O)$, then at that incidence the centre of pressure is a distance xc aft of O where

$$x = -C_M(O)/(C_L \cos\alpha + C_D \sin\alpha).$$

Equation 11.4,4 can be regarded as an approximate form of this relation for which the moment due to the drag is neglected and the incidence is assumed small. Note that when $C_L \cos\alpha + C_D \sin\alpha \rightarrow \pm 0$,

$$x \rightarrow \pm\infty \text{ (sign of } -C_M(O)).$$

11. The following measurements were made for a wing section in a wind tunnel:

α (degrees)	C_L	C_M (About $c/4$ point)	C_D
−4	−0·2	−0·038	0·0088
−2	0	−0·030	0·0080
0	0·2	−0·022	0·0088
2	0·4	−0·014	0·0112
4	0·6	−0·006	0·0152
6	0·8	0·002	0·0208
8	1·0	0·010	0·0280
10	1·2	0·018	0·0376

Find the position of the centre of pressure for each of the angles tabulated. Find the position of the aerodynamic centre for small angles of incidence. (Answer. The C.P. positions ahead of the quarter chord point in fractions of a wing chord are for the incidences in order as given: 0·189, 100, −0·11, −0·035, −0·01, 0·002, 0·01, 0·015. Note how close these results are to those given by the simple approximate formula of equation 11.4,4, except very close to the incidence for which $C_L \cos\alpha + C_D \sin\alpha = 0$. The aerodynamic centre position is at $0{\cdot}21c$ behind the leading edge.)

12. The value of C_{M0} for the wing of an aeroplane is given as −0·06, and the position of its aerodynamic centre is $0{\cdot}23\bar{c}$ behind some datum point where $\bar{c}$ is the mean chord and is equal to 3 m. If the C.G. of the aircraft is at $0{\cdot}28\bar{c}$ behind the datum point, determine the load on the tailplane required to trim the aeroplane (i.e., to balance the pitching moment of the wing) in level flight at an altitude of 3050 m, given that the wing area is 42 m²,

the forward speed is 120 m/s, the lift coefficient of the aircraft is 0·2, and the lift on the tailplane acts through a point distant 8 m horizontally behind the wing aerodynamic centre. The pitching moment on the tailplane as well as the effects of fuselage and nacelles on the overall pitching moment may be neglected. At 3050 m, the air density = 0·9050 kg/m^3.

(Answer. 5132 N acting as a download.)

13. Show directly from the analysis of section 11.5,2 that the aerodynamic centre for a flat plate of chord c at small incidences is at $c/4$ aft of the leading edge.

14. In Fig. 11.5,1 consider a point Q with coordinates $(2a, 0)$ in the field of flow past the circular cylinder. Suppose the circulation round the cylinder is such that the Kutta-Joukowsky condition is satisfied at the trailing edge of the flat-plate into which the circle is transformed. Find the velocity components at Q.

$\left(\text{Answer. } -\frac{3}{4}V\cos\alpha,\ \frac{9}{4}V\sin\alpha.\right)$

15. What are the coordinates and velocity components at the point Q' into which the point Q of Exercise 14 transforms?

$\left(\text{Answer. } \frac{5a}{2},\ 0;\ \ -V\cos\alpha,\ 3V\sin\alpha.\right)$

16. The profile of a thin aerofoil consists of a straight front portion AB and a straight rear portion BC where BC is inclined downwards to AB at a small angle η, and $BC/(AB + BC) = E$. Show that the angle of zero lift α_0 referred to AC as chord is given by

$$\pi\alpha_0 = \eta[(\theta_B - \sin\theta_B) - \pi(1 - E)]$$

where θ is defined by equation 11.5,13, and θ_B is the value of θ at B, i.e. $\cos\theta_B = 2E - 1$.

17. For the profile of Exercise 16 show that if α is measured from AB as chord line, the lift coefficient is

$$C_L = a_1\alpha + a_2\eta$$

where $a_1 = 2\pi$ and $a_2 = 2(\pi - \theta_B) + 2\sin\theta_B$.

Note that BC can be regarded as a skeleton flap. Its effectiveness is therefore measured by $\Delta C_L = a_2\eta$ and this is independent of α.

18. For the profile of Exercise 16, if BC is regarded as a flap, show that its deflection relative to AB causes a change in pitching moment about the leading edge given by

$$\Delta C_M = -\eta\left[\frac{\pi - \theta_B}{2} + \sin\theta_B - \frac{\sin\theta_B\cos\theta_B}{2}\right].$$

Hence show that if E is small the extra lift due to the flap acts approximately through the mid-chord point of the wing.

19. An untwisted wing of rectangular plan form has the section profile as described in Exercise 16, with E constant across the span. Show that the lift coefficient can again be put in the form

$$C_L = a_1\alpha + a_2\eta$$

where the ratio a_2/a_1 is independent of aspect ratio.

20. The camber line of a thin aerofoil is given by

$$y = \frac{-4kx(c + x)}{c},$$

in the notation of Fig. 11.5,3. Find the no-lift angle, the value of C_{M0}, the ideal angle of attack and the corresponding lift coefficient.

(Answer. $-2k$ (radians), $-\pi k$, 0, $4\pi k$.)

21. If the camber line of a thin aerofoil of chord c is given by

$$y = -kx\left[1 - a\left(\frac{x}{c}\right) + b\left(\frac{x}{c}\right)^2\right]$$

find a and b such that $C_{M0} = 0$. (Answer. $a = -15/7$, $b = 8/7$.) Note that in this case the camber line is reflexed towards the trailing edge.

22. Show that according to the first order theory of § 11.5,6 the perturbation velocity u at the surface of a section in the form of a thin ellipse at zero incidence is constant and given by

$$\frac{u}{V} = \frac{t}{c},$$

where t/c is the thickness-chord ratio and V is the undisturbed stream velocity.

23. A wing of elliptic plan form is flying steadily at 10,000 ft with a speed of 290 ft/s and supports a weight of 8,500 lbf. The wing area is 280 ft^2 and its aspect ratio is 7. Determine the values of (a) C_L, (b) the downwash velocity at the wing induced by the trailing vortices, (c) the circulation over the midspan section (d) the induced drag coefficient. At 10,000 ft. the air density is 0·001756 slug/ft^3.

(Answer. (a) 0·401, (b) 5·34 ft/s, (c) 473 ft^2/s, (d) 0·0073.)

24. The following measurements were made of the lift and drag coefficients (boundary layer plus induced) for a wing with elliptic loading and of aspect ratio 10. On the basis of lifting line theory (§ 11.6,2) calculate for the same values of the lift coefficients the corresponding values of the incidence and drag coefficients if the aspect ratio were 4.

α(deg.)	C_L	C_D	α(deg.)	C_L	C_D
−1	0	0·0080	10	0·916	0·0505
0	0·084	0·0082	12	1·080	0·0710
2	0·247	0·0100	14	1·230	0·0952
4	0·415	0·0139	16	1·367	0·1255
6	0·582	0·0216			
8	0·747	0·0344			

Answer. C_L	α(deg.)	C_D	C_L	α(deg.)	C_D
0	−1	0·0080	0·747	10·05	0·0611
0·084	0·23	0·0086	0·916	12·51	0·0908
0·247	2·68	0·0130	1·080	14·96	0·1269
0·415	5·14	0·0222	1·240	17·37	0·1670
0·582	7·60	0·0378	1·367	19·75	0·2150

25. From the following relation for the drag coefficient of a wing

$$C_D = C_{D0} + C_{Di} = C_{D0} + \frac{KC_L{}^2}{\pi A},$$

where K is a constant somewhat greater than unity (see equation 11.6,21), show that if the wing is in level flight supporting a given load W the drag of the wing is a minimum when its speed

$$V = \left[\frac{4KW^2}{\pi A \rho^2 S^2 C_{D0}}\right]^{1/4}.$$

Show that then

$$C_{D0} = C_{Di},\ C_L = (\pi A C_{D0}/K)^{1/2}$$

and C_L/C_D is a maximum $= \frac{1}{2}[\pi A/KC_{D0}]^{1/2}$.

26. The plan form of a slender plane wing is defined by

$$y_s^2/s^2 = \sin^2\left(\frac{\pi}{2}\frac{x}{c}\right),$$

where c is the chord length along its centre line and its span is $2s$. Show that at a small incidence α the pitching moment coefficient about the apex is

$$C_M = M/\tfrac{1}{2}\rho V^2 Sc = -(\pi A/4)\alpha$$

where S is the wing area and A is the aspect ratio. Where is the aerodynamic centre? (Answer. $c/2$ behind the apex and on the centre line.)

27. Assume that for a wing in a steady rate of roll p about the x axis at zero incidence the effect of the rolling motion is equivalent to that of an incidence distribution $\alpha = py/V$. With the assumptions of lifting line theory, show that in the notation of § 11.6,2 the rolling moment due to the rate of roll is

$$L = \rho V^2 s^3 \pi \gamma_2,$$

and that

$$\gamma_2 = -\frac{\mu_0 s}{2(2\mu_0 + 1)}\frac{p}{V},$$

if the wing is of elliptic plan form, where $\mu_0 = a_\infty c_0/8s$, c_0 being the centre line chord.

Hence show that the derivative

$$l_p = -\frac{a_\infty}{8}\bigg/\left(1 + \frac{2a_\infty}{\pi A}\right).$$

28. A cascade of aerofoils is such that in the notation of Fig. 11.8,1, $c = 4$ in., $h = 3$ in., $U_1 = 250$ ft/s, $V_1 = 0$. Neglecting losses due to viscosity effects find α_2, V_2 and the force components per foot span on each aerofoil, X' and Y', given that the pressure drop through the cascade is 17 lbf/ft^2,

$$\left(\begin{aligned}&\text{Answer. } \alpha_2 = -25^\circ\,30', V_2 = 119{\cdot}5 \text{ ft/s},\\ &X' = 4{\cdot}25 \text{ lbf/ft}, \; Y' = 17{\cdot}8 \text{ lbf/ft.}\end{aligned}\right)$$

29. Show that the flow past a cascade of aerofoils can be regarded as composed of that due to a uniform non-circulatory flow of components U and $(V_1 + V_2)/2$ and that due to a circulation about each aerofoil of strength $(V_1 - V_2)h$. Hence, by the principle of superposition, show that we can deduce that at any point on an aerofoil the velocity can be written in the form

$$q = aU + bV_1 + cV_2,$$

where a, b and c are functions only of the cascade geometry and of the position of the point considered. Deduce that if the Kutta-Joukowsky condition determines the circulation about the aerofoils then

$$\frac{\tan\alpha_1 - \tan\alpha_0}{\tan\alpha_2 - \tan\alpha_0} = \text{constant},$$

where α_0 defines the flow direction for which there is no deflection. The constant is a function only of the cascade geometry.

CHAPTER 12

PUMPS AND TURBINES

12.1 **Introduction**

A hydraulic machine is a device for the exchange of energy between a mechanical system and a fluid medium. Most hydraulic machines can be included in one or other of the types:

(a) Positive displacement machines.

(b) Rotodynamic machines.

The essential feature of positive displacement machines is a movable member fitted within a fixed housing, the fit being made close enough, perhaps with the help of rings, packings or labyrinths, to prevent appreciable leakage of fluid past the movable member. We have, in effect, a chamber of variable volume containing fluid under pressure. When this volume increases as the movable member is impelled by the fluid pressure, energy is transferred from the fluid to the mechanical system and, conversely, energy is transferred to the fluid when the volume decreases.† If the movable member has a reciprocating motion some arrangement of automatic or gear driven valves ensures that the energy flux is always in the same sense, i.e. towards the fluid in the case of a pump and away from it for a motor; gear pumps and the like employ rotating members and for them valves are not essential. Although there are certainly dynamic effects (which may be important) in the operation of a positive displacement machine, these are incidental and the machine depends essentially on hydrostatic principles. On the other hand, rotodynamic machines depend essentially on hydrodynamic forces and the exchange of energy occurs between a rotating rigid member (the rotor or runner) and the fluid, which is in motion. Apart from leakage, a positive displacement machine may have a non-zero difference of head across it when it is at rest but the difference of head across a rotodynamic machine is necessarily zero when the rotor is stationary. While positive displacement machines are still used quite extensively (especially in the form of piston or bucket pumps and gear pumps) there is an evident tendency towards the general use of rotodynamic machines. For this reason, and because there are relatively few useful applications of theory to positive displacement machines, the present chapter will be devoted exclusively to rotodynamic machines.

Screw propellers belong to the class of rotodynamic machines but are so different from pumps and turbines as to require separate treatment. Chapter 13 is devoted to propellers, fans and windmills while the present chapter is mainly concerned with hydraulic turbines and centrifugal pumps.

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† Positive displacement pumps may also be made of easily deformable material, such as rubber. The animal heart is an example, where the deformable material is muscle.

Turbines and rotodynamic pumps may be classified in various ways. One of these depends on the nature of the path of the fluid through the passages in the rotor. When the path is wholly or mainly in a plane perpendicular to the axis of rotation we have a radial flow machine, exemplified by centrifugal pumps (see § 12.11) and by Thomson turbines (see § 12.9), but when the flow is wholly or mainly parallel to the axis of rotation we have an axial flow machine. Finally we have mixed flow machines. Turbines may be classified as *impulse* or *reaction* machines in accordance with the absence or presence of pressure changes in the fluid as it passes through the rotor. In an impulse machine the change from static to dynamic head occurs wholly in a fixed nozzle or set of nozzles and the fluid issuing from these impinges on the moving blades or buckets of the runner. In a reaction machine at least a part of the change from static to dynamic head occurs in the passages of the rotor, which are filled by the fluid.

One of the principal aims of the designer of a pump or turbine is to secure the highest possible efficiency for the machine; in other terms, his aim is to reduce to a minimum the losses, which represent the degradation of mechanical energy into useless heat.† To achieve this most desirable end, the designer must begin by identifying the sources of loss and then go on to consider how each of them can be made as small as possible. On account of the importance of this topic a special section of this chapter is devoted to it (§ 12.6).

The efficiency of a hydraulic machine is not constant over the range of its conditions of operation. It is the aim of the designer to secure the highest possible efficiency in the condition which is of the greatest importance in service and this provides the 'design point'. There will then be some fall in efficiency in other conditions but, with skilful design, the fall will be slight within a considerable range of operating conditions.

In this chapter we consider only the steady states of hydraulic machines, while the fluid is treated as incompressible. Consistent units and 'complete' equations are used throughout unless the contrary is stated explicitly.

12.2 Types of Rotodynamic Machines

The aim in this section is to give merely a brief descriptive account of those rotodynamic hydraulic machines which are of particular importance at the present time. These are:

(a) Turbines (1) Pelton wheel
(2) Francis turbine
(3) Kaplan turbine

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† This aspect of the matter may be exemplified by the fact that the rise of temperature resulting from free fall over Niagara is about 0·1° C.

(b) Pumps (1) Centrifugal pump
(2) Half-axial or screw pump
(3) Axial flow propeller pump.

The general classification according to direction of flow and the classification of turbines as 'impulse' or 'reaction' types has already been explained in § 12.1.

(a) 1. *Pelton wheel.* This is the only hydraulic turbine of the impulse type now in general use. It is very well suited to high heads and, at its best, is very efficient. It is also a robust and simple machine.

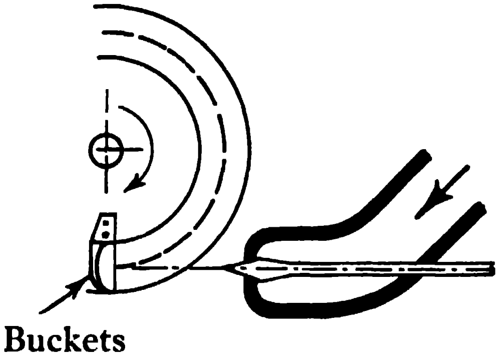

Fig. 12.2,1. Pelton wheel.

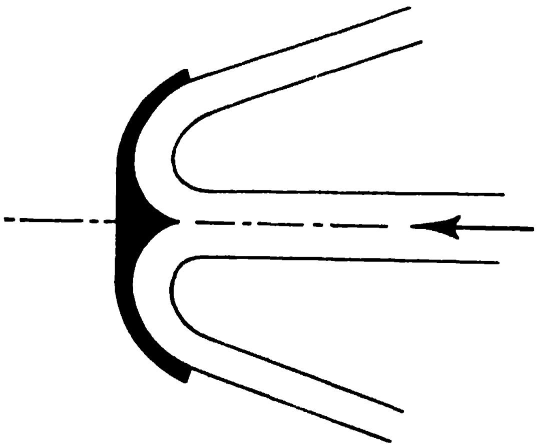

Fig. 12.2,2. Path of water relative to bucket of Pelton wheel.

The runner is a circular disk carrying a number (usually 20 or more) of cup-shaped buckets evenly spaced round its outer edge (see Figs. 12.2,1 and 12.2,6(a)). One or more jets of water issue from fixed nozzles situated in a housing round the runner and so arranged that the jet axes are tangent to a circle through the centres of the buckets. Each bucket has a ridge or 'splitter' down the centre, as shown in the figures, so the jet is divided symmetrically and issues equally on the two sides of the disk (see the diagram, Fig. 12.2,2). When running at highest efficiency (see § 12.7) the speed of the bucket centres is about half the speed of the jet and the absolute velocity of the water as it leaves the buckets is small, so little kinetic energy is wasted. The conversion of static into dynamic head takes place entirely in the fixed nozzles and the rate of discharge is controlled by spear valves. The Pelton wheel is suitable for heads ranging from 90 m upwards.

The shaft of a Pelton wheel is usually horizontal and then not more than two jets per wheel are used; they must not be so close that they interfere with one another. Occasionally a vertical axis is used and then there may be as many as six jets per wheel. Ideally there is no hydrodynamic end thrust but it is usual to fit a small thrust bearing on the shaft.

(a) 2. *Francis turbine.* This is an inward flow reaction turbine and the shaft is usually vertical (see Figs. 12.2,3 and 12.2,6(b)). The water enters horizontally all round the runner and is guided by vanes or 'wicket gates' which are adjustable and are ganged together by suitable gearing. In its passage through the runner the water reacts against the vanes of the runner and is at the same time deflected downwards. The exit angles of the runner vanes are so arranged that, at the 'design point', the water leaves the runner with a rather low axial velocity and very little whirl. This machine is capable of giving a high efficiency and it is very suitable for medium high heads (say 18 to 425 m). Like all inward flow turbines it is thoroughly stable.

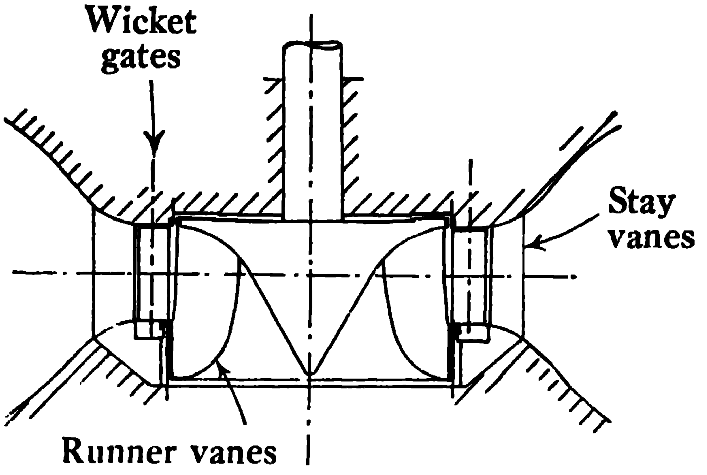

Fig. 12.2,3. Francis turbine.

(a) 3. *Kaplan turbine*. A diagram of this machine is shown in Fig. 12.2,4. The runner here is essentially a propeller working in reverse and the blades are mounted so that the blade angles can be adjusted together by means of suitable gearing while the machine is in operation. The guide vanes or wicket gates at entry are also adjustable (see Fig. 12.2,6(c)). This machine is very suitable for low heads, say up to 36 m. An electric generator coupled to a Kaplan type runner may be enclosed in a fairing within a faired tunnel and the assembly is then known as a *bulb turbine*; heads as low as 3·6 m may be utilised.[1]

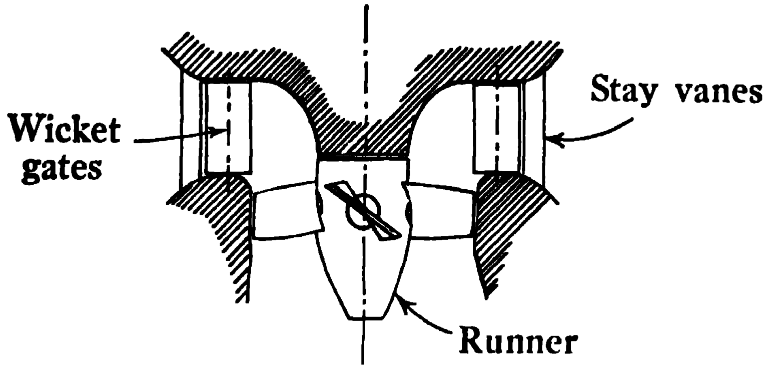

Fig. 12.2,4. Kaplan turbine.

It is often not convenient or practicable to arrange for the velocity of the water at exit from the runner of a Francis or Kaplan turbine to be very small. However, this does not necessarily imply a large wastage of energy provided that the machine is provided with an efficient *draught tube*. This is an expanding passage (diffuser or recuperator, see § 7.10) where the fluid velocity is gradually reduced and there is at least a partial conversion of dynamic to static head. In order that cavitation may be avoided (see § 12.12) the upper end of the draught tube, where it receives the water from the runner, must not be too high and may have to be below the free surface level in the tailrace.

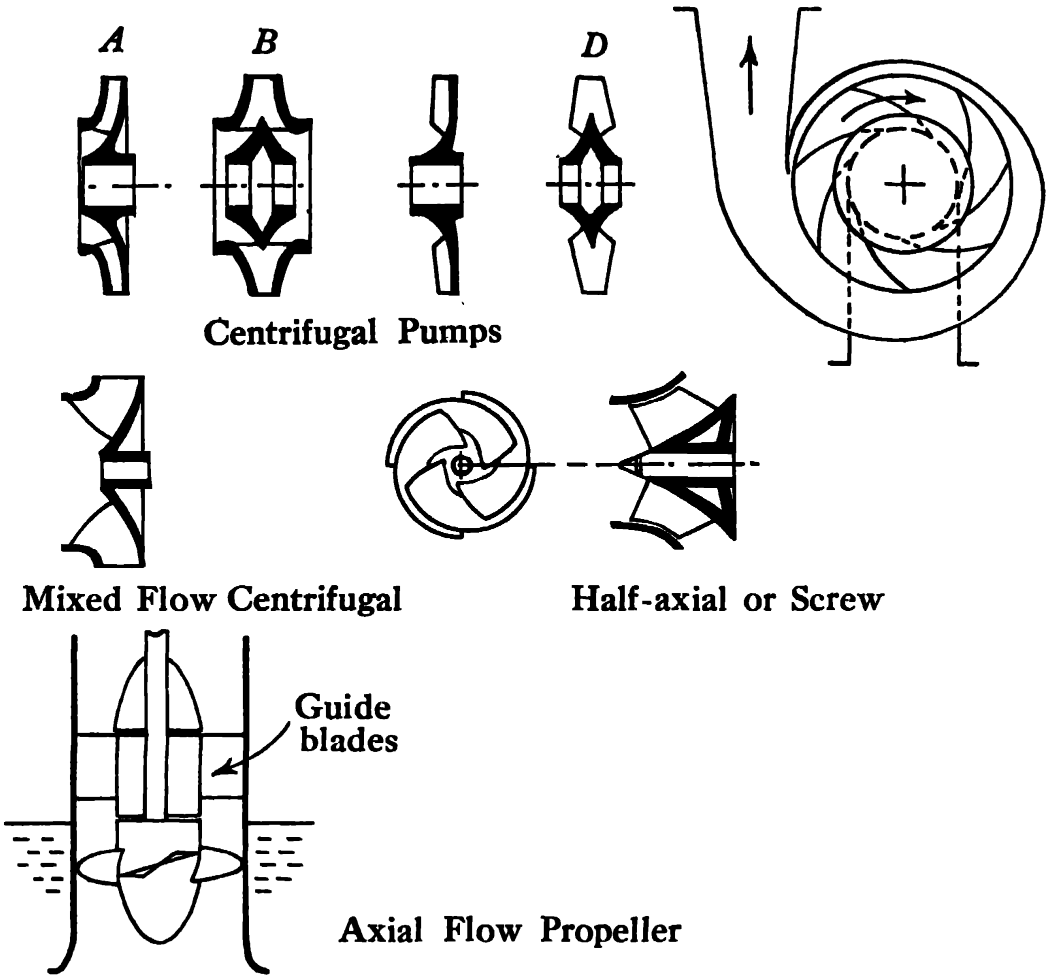

Fig. 12.2,5. Typical pump impeller types.

(b) 1. *Centrifugal pump*. The flow in the impeller of a centrifugal pump is mainly in a plane normal to the axis of rotation, which may be horizontal, vertical or inclined, but near the entry the flow may be wholly or nearly axial. Runners of several types are shown in Fig. 12.2,5. The blades of impeller *A* are shrouded and there is a single 'eye' on the left; impeller *B* has twin eyes and is symmetric, so the hydrodynamic thrusts are balanced. Impeller *C* is half-shrouded and unsymmetric, while *D* is unshrouded and symmetric. Unshrouded and part-shrouded impellers are used only where the attainment of high efficiency is unimportant.

The general arrangement of a typical centrifugal pump of high performance is also shown in end elevation in Fig. 12.2,5. The impeller discharges into the volute or recuperator, which is in effect a gradually expanding passage, in which some conversion of dynamic into static head occurs. The figure also shows a mixed flow centrifugal pump. Pumps of these types are suitable for heads of 6 m and upwards with the mixed flow type better suited to the lower heads.

(b) 2. *Half-axial or screw pump.* This is shown in Fig. 12.2,5 and it will be seen that the runner has a number of screw-shaped blades. The flow in the runner is partly axial and partly outward. This pump is suitable for heads from 3 to 15 m.

(b) 3. *Axial flow propeller pump.* Here the runner is a screw with a rather large boss or central fairing working within a cylindrical casing, as shown in Fig. 12.2,5. This is essentially a ducted water fan (see § 13.13) and it is suitable for heads up to about 6 m.

Photographs of the runners of the following machines† are shown in the Plate (Fig. 12.2,6) (a) Pelton wheel, (b) Francis turbine (2 runners), (c) Kaplan turbine, (d) Pump-turbine. Photograph (c) shows, in addition to the runner, the guide vanes and wicket gates. The pump-turbine (d) can work efficiently as a turbine or, in reverse, as a pump. This type of machine is much used in hydraulic power storage schemes in connection with public electricity supply systems; the machine runs as a turbine at times of peak load but pumps water into a storage reservoir at times of small power-demand.

The *area coefficient* of a runner at entry or exit is the ratio of the sum of the areas of the openings or orifices at entry or exit to the area of the peripheral face in which they lie.

12.3 Performance Coefficients

We shall consider here the application to rotodynamic machines of the principles of dynamical similarity and dimensional analysis which are discussed in a general way in Chapter 4 and we begin with an examination of the conditions for similarity of flow. The fluid is viscous but it will be regarded as incompressible. Suppose that we have two machines which are *exactly* similar geometrically; the similarity must extend to all significant parts of the entry and discharge systems, including, for example, free surface levels in the tailrace or sump. In accordance with §§ 4.2 and 4.5 the further conditions for similarity of flow are:

(a) The velocities of the mechanical moving parts must have the same ratios to the typical velocity of the fluid for both machines.

(b) The Reynolds number must have identical values for the two machines.

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† The authors are indebted to the Harland Engineering Co. Ltd. of Alloa, Scotland, for supplying the photographs.

PLATE 12.1

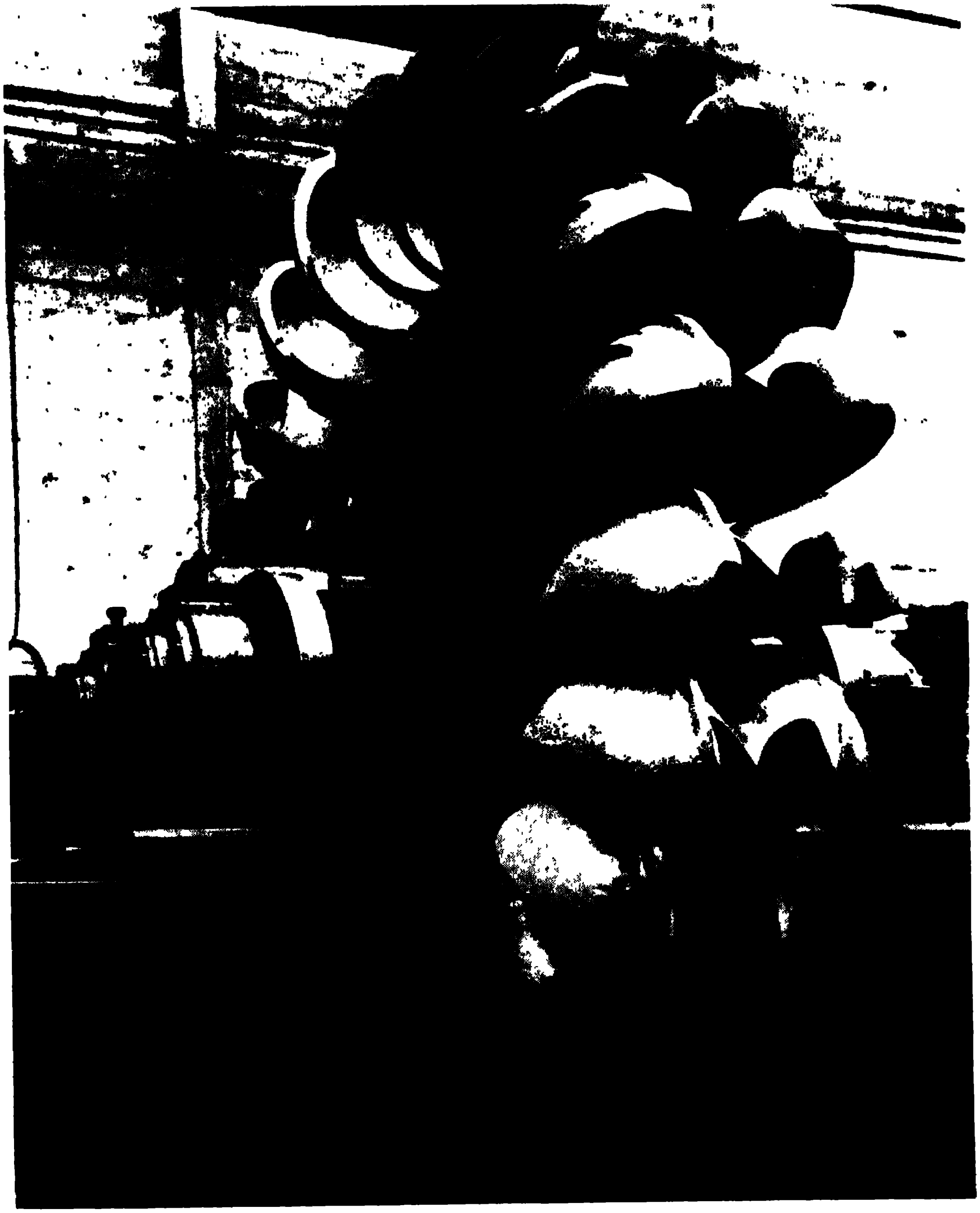

FIG. 12.2,6 (*a*). Typical Impulse Turbine Runner.

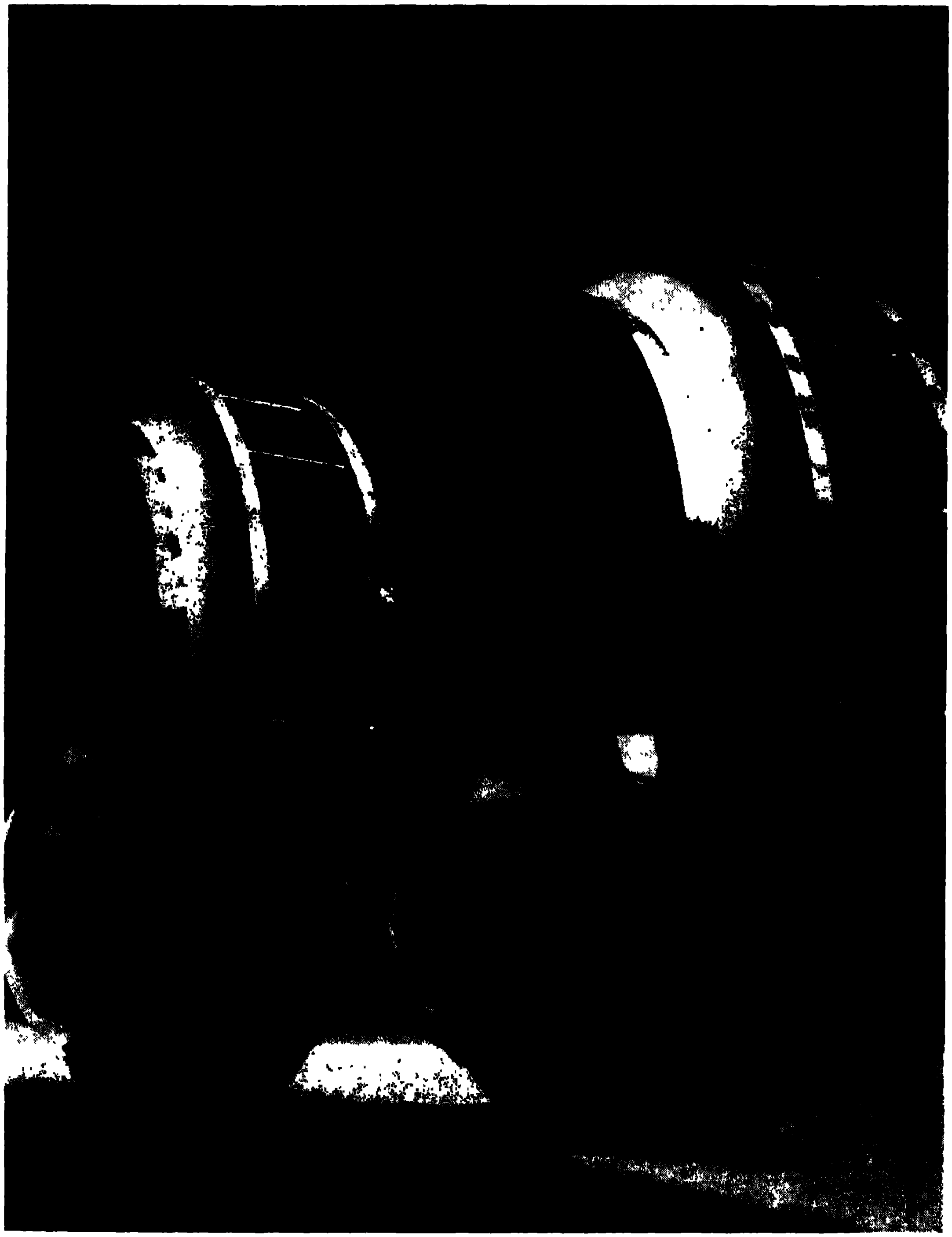

FIG. 12.2,6 (*b*). Francis Type Water Turbine Runners.

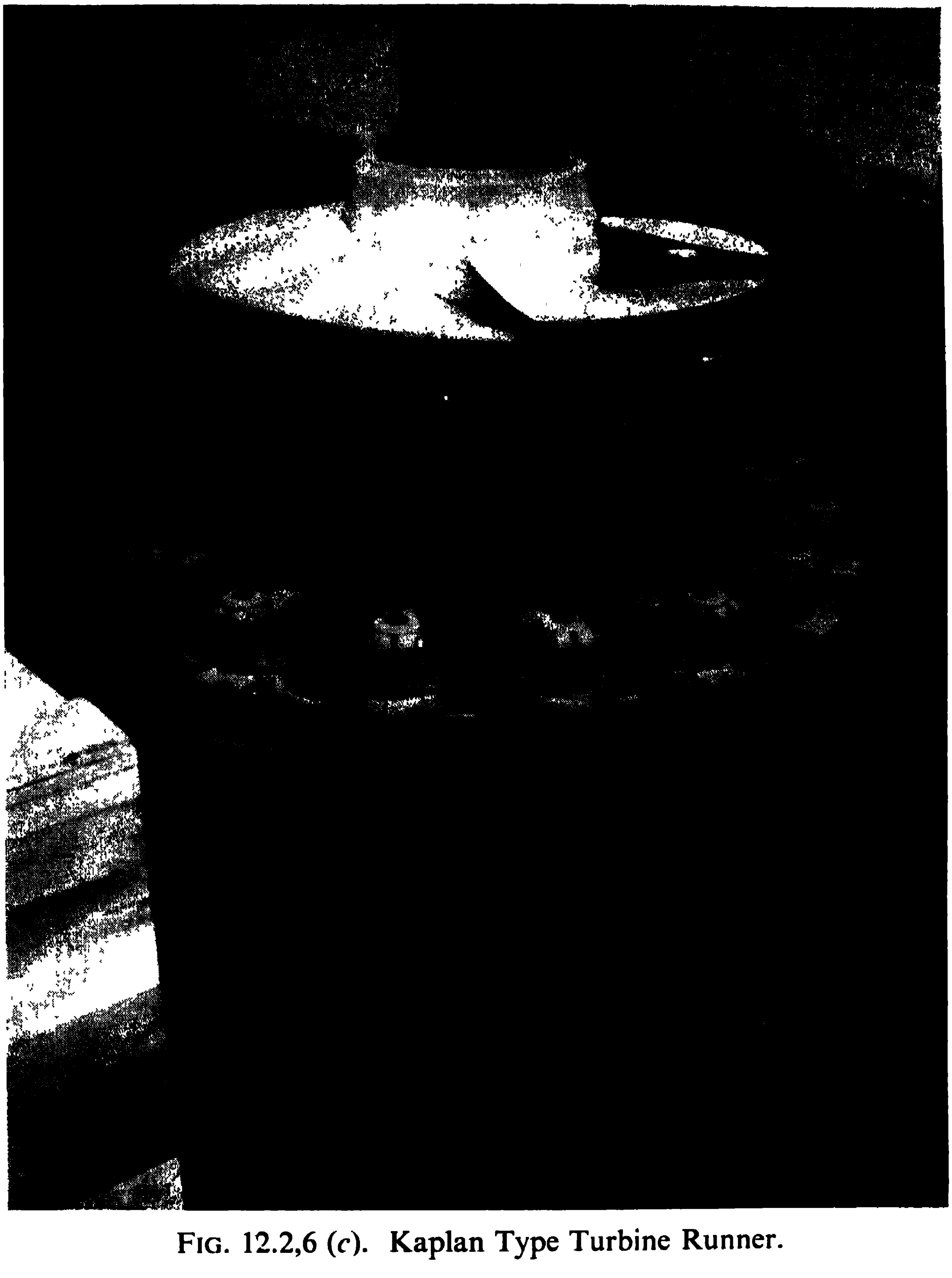

FIG. 12.2,6 (*c*). Kaplan Type Turbine Runner.

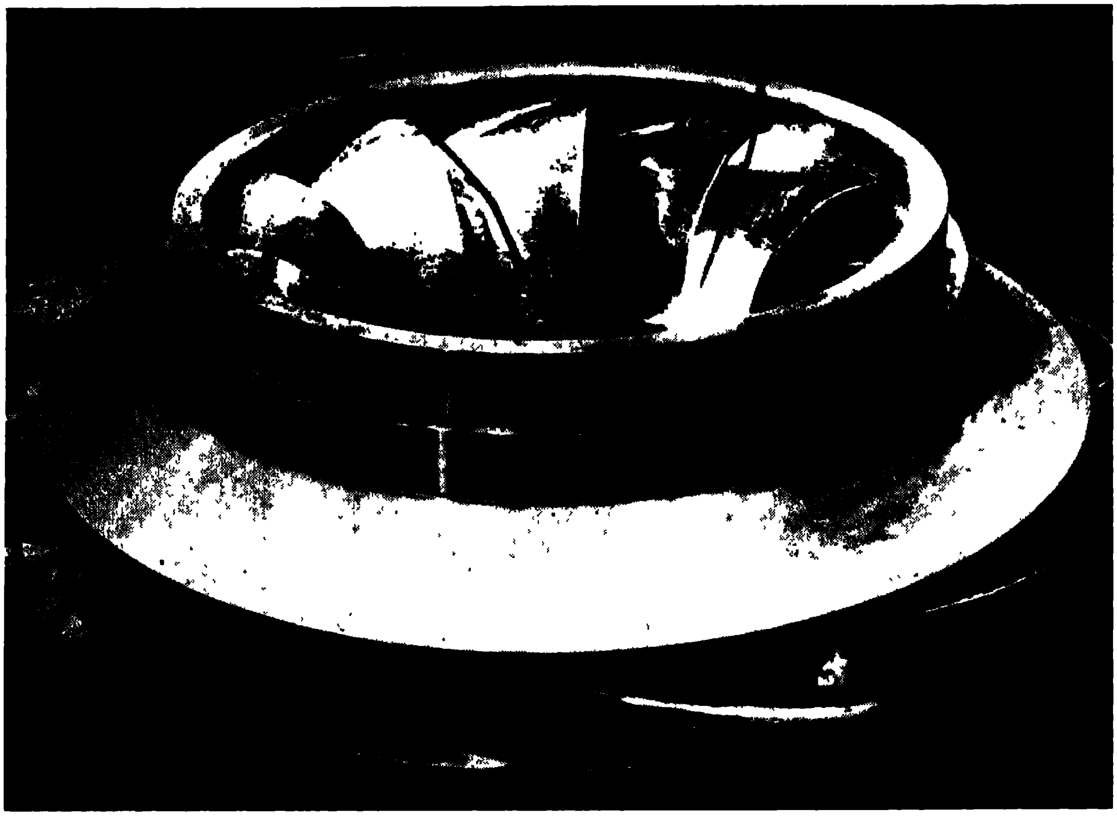

FIG. 12.2,6 (*d*). Pump Turbine Runner.

These conditions will next be discussed in detail and the following symbols will be used:—

D = external diameter of rotor
n = revolutions of the rotor in unit time
Q = volume of fluid passing through machine in unit time
M = torque
P = hydraulic power† $= 2\pi nM$
H = difference of total head across machine
ρ = density of fluid
μ = viscosity of fluid
R = Reynolds number
η = hydraulic efficiency (as a fraction).

For a machine with a single rotor (or any number of rotors connected by mechanical gearing) it is convenient to take nD as the typical velocity of the moving parts; it is the tip velocity of the rotor divided by π. Likewise it is convenient to take Q/D^2 as the typical velocity of the fluid, for all fluid velocities are proportional to this when the conditions of similarity are satisfied. The ratio of these is the non-dimensional parameter

$$\alpha = \frac{Q}{nD^3} \tag{12.3,1}$$

which will be called the *discharge number*.‡ Condition (a) can then be stated in the form that the discharge number has the same value for the two machines. In defining an appropriate Reynolds number for the present purpose we shall again take Q/D^2 as the typical velocity and we shall take D as the typical length. Accordingly the Reynolds number is

$$R = \frac{Q}{\nu D} = \frac{\rho Q}{\mu D}. \tag{12.3,2}$$

Thus there will be complete dynamical similarity provided that the machines are exactly similar geometrically and that both the discharge number and Reynolds number have the same values for the machines. It is rarely possible to secure identity of the Reynolds numbers for machines of widely different sizes but the variation of the efficiency and other non-dimensional performance coefficients with the Reynolds number ('scale effect') is not great over a moderate range of R. Hence, as a first approximation, scale effect may be neglected. However, identity of the discharge numbers is a vitally important condition for similarity.

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† For a turbine this is the power developed, but for a pump it is the power absorbed. The hydraulic power is that developed or absorbed by the rotor, so the net power of a turbine is obtained from this by multiplication by the mechanical efficiency. The power required to drive a pump is the hydraulic power divided by the mechanical efficiency. The power is here measured in consistent units.

‡ Called the 'characteristic discharge number' by Addison.

We shall now assume that the conditions for similarity are strictly satisfied. In such circumstances any pressure difference will be proportional to $\rho(nD)^2$ since the typical velocity of the fluid is proportional to nD. Hence H, the difference of head across the machine, will be proportional to n^2D^2/g and the non-dimensional quantity

$$\beta = \frac{gH}{n^2D^2} \tag{12.3,3}$$

will be constant; it is called the *head number* and it is the reciprocal of a species of Froude number (see § 4.4.) Likewise the power P will be proportional to the product of Q and the pressure difference across the machine. Thus P is proportional to $g\rho HQ$ and, since α is constant in similar conditions, it is proportional to $g\rho nHD^3$. The non-dimensional quantity

$$\gamma = \frac{P}{g\rho nHD^3} \tag{12.3,4}$$

is called the *power number*. Especially in relation to the design of turbines and in the selection of the most suitable type to meet given conditions, it is convenient to use a non-dimensional quantity depending on the power but not containing the diameter D. Such a quantity is

$$\gamma^2\beta^{-3} = \frac{n^4P^2}{\rho^2(gH)^5}$$

and its fourth root is Addison's *shape number for turbines*

$$n_{st} = \frac{n\sqrt{\dfrac{P}{\rho}}}{(gH)^{5/4}}. \tag{12.3,5}$$

This is non-dimensional since it is a function of the non-dimensional quantities β, γ and it is constant under conditions of strict similarity. In relation to the design of pumps we require a non-dimensional quantity which depends on the rate of discharge Q and which is independent of D. Now

$$\alpha^2\beta^{-3} = \frac{n^4Q^2}{(gH)^3}$$

and the fourth root of this is Addison's *shape number for pumps*†

$$n_{sp} = \frac{n\sqrt{Q}}{(gH)^{3/4}}. \tag{12.3,6}$$

Like the shape number for turbines, this is non-dimensional and its value is constant under conditions of strict similarity. The fluid in a pump or turbine is very often *water* and then the density ρ is practically constant; likewise g is practically constant. Hydraulic engineers in the past were in the habit of omitting ρ and g from the non-dimensional quantities and used what are called the *specific speeds* as follows:

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† To avoid fractions this is usually multiplied by 1000.

$$\text{For turbines} \quad \frac{N\sqrt{P}}{H^{5/4}}, \tag{12.3,7}$$

$$\text{For pumps} \quad \frac{N\sqrt{Q}}{H^{3/4}}. \tag{12.3,8}$$

These are not non-dimensional and their values therefore depend on the choice of units. In Britain P is measured in horse power, N in revolutions per minute and H in feet while Q may be in Imperial gallons per minute or cusecs. In the U.S.A. P, N and H are measured as in Britain but Q is measured in U.S. gallons per minute or in cusecs. On the Continent of Europe metric units are adopted.† This state of chaos is avoided by using the non-dimensional shape numbers.‡ Finally it may be noted that *both* shape numbers can, if desired, be applied to both turbines and pumps.

The overall efficiency of a turbine is the ratio of the power developed at the coupling to the energy supplied in unit time, where consistent units are used and the efficiency is expressed as a fraction, not a percentage. The overall efficiency η_0 is the product of the mechanical efficiency η_m and the hydraulic efficiency η_h, thus

$$\eta_0 = \eta_m \eta_h. \tag{12.3,9}$$

However, the mechanical efficiency (although undoubtedly very important) does not fall within the scope of hydraulics proper and we shall use the term efficiency to mean hydraulic efficiency and we shall usually omit the suffix h from η_h. Likewise the power P is the hydraulic power, so the net power of a turbine is $\eta_m P$. We have

$$P = 2\pi nM$$

while the energy supplied in unit time is $g\rho HQ$. Hence the efficiency is

$$\eta_t = \frac{P}{g\rho HQ} = \frac{2\pi nM}{g\rho HQ}. \tag{12.3,10}$$

For a pump the hydraulic efficiency is the ratio of the useful work done on the fluid in unit time to the power input to the rotor. Hence

$$\eta_p = \frac{g\rho HQ}{P} = \frac{g\rho HQ}{2\pi nM} \tag{12.3,11}$$

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† See footnote page 660.

‡ Let S be the specific speed and s the shape number (including the factor 1,000 for pumps).

Then $s = KS$ where, for pumps,

$K = 0{\cdot}064$ (n in revs. per minute, H in feet and Q in Imperial gallons per minute)
$K = 0{\cdot}058$ (as last, but Q in U.S. gallons per minute)

while for turbines

$K = 3{\cdot}66 \times 10^{-3}$ (specific speed based on horse power, revolutions per minute and head in feet).

Under conditions of strict similarity either of these efficiencies will be constant. For the same machine under differing conditions or for a family of machines which are strictly similar geometrically we shall have the efficiency a function of the discharge number α and the Reynolds number, or

$$\left.\begin{aligned}\eta &= f(\alpha, R)\\ &= f\left(\frac{Q}{nD^3}, \frac{\rho Q}{\mu D}\right).\end{aligned}\right\} \tag{12.3,12}$$

As already explained we may, as a first approximation, neglect scale effect for a moderate range of variation of R and we then have

$$\eta = f\left(\frac{Q}{nD^3}\right). \tag{12.3,13}$$

For a given machine which always operates with the same fluid and approximately at a constant temperature† the foregoing equation may be accepted as valid over the operational range of variation, always provided that the geometry of the machine is fixed. Many turbines are provided with adjustable blades (or other means of controlling the conditions of flow) and we shall suppose that the adjustment has just one degree of freedom whose non-dimensional measure is θ. Then we shall have in place of (12.3,13)

$$\eta = f\left(\frac{Q}{nD^3}, \theta\right). \tag{12.3,14}$$

The aim will be so to adjust θ that for every value of the discharge number the efficiency has the highest possible value and it may be possible to arrange for this to be done automatically.

Let us consider next the problem of determining the best type of turbine to meet given requirements and its typical linear dimension D. We shall suppose that the values of the following quantities are known: g, ρ, H, n and P. From these we calculate the shape number n_{st} by means of equation (12.3,5). As a result of accumulated experience it will be known what type of turbine has the best efficiency for this value of the shape number and we suppose that this one is chosen. For this particular type of turbine performance curves or tables will be available relating n_{st} to β. Hence knowledge of n_{st} enables us to find β and the diameter is then, by equation (12.3,3), given by

$$D = \frac{1}{n}\sqrt{\frac{gH}{\beta}}. \tag{12.3,15}$$

In practice there may not be complete freedom to choose the diameter and the question will then be one of selection from a range of possible 'stock' sizes. However, equation (12.3,15) always provides very useful guidance.

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† The reason for this proviso is that μ may change very greatly with temperature.

The theory underlying the foregoing is as follows. For machines of fixed shape or proportions we have similarity when α and R are fixed. This implies that any other non-dimensional performance coefficient is some function of α and R. In particular we may write

$$\beta = f_1(\alpha, R) \tag{12.3,16}$$

and

$$\gamma = f_2(\alpha, R) \tag{12.3,17}$$

where these relations may be found by experiment or calculation. Then, since

$$n_{st} = (\gamma^2\beta^{-3})^{1/4}$$

we can derive the relation between n_{st} and α, R which may be written

$$n_{st} = f_3(\alpha, R). \tag{12.3,18}$$

However, equation (12.3,16) establishes α as a function of β and R and (12.3,18) may accordingly be rewritten

$$n_{st} = f_4(\beta, R). \tag{12.3,19}$$

When the dependence on R is neglected n_{st} becomes a function of β alone, as assumed above.

Now let us consider the problem of determining the dimensions of a rotodynamic pump to meet given requirements. We suppose that the data are the values of g, ρ, H, n and Q. From these the value of the shape number n_{sp} is calculated by equation (12.3,6). Then from the known relation between β and n_{sp} the value of β is found and finally D is obtained from (12.3,15). The basic theory is essentially the same as for the turbine. It follows from this theory that the efficiency of a pump or turbine of given geometry is a function of the shape number and Reynolds number. Thus we may put

$$\eta = f(n_s, R) \tag{12.3,20}$$

or, when there is a geometrical variable θ (see above),

$$\eta = f(n_s, \theta, R) \tag{12.3,21}$$

which simplifies to

$$\eta = f(n_s, \theta) \tag{12.3,22}$$

when scale effect is neglected. Throughout this discussion cavitation has been assumed not to occur. This subject is considered in § 12.12.

A non-dimensional quantity much used in turbine design is ϕ, defined as the ratio of the peripheral speed of the runner to the fluid velocity corresponding to the difference of head across the machine. Thus, when consistent units are used, we have

$$\phi = \frac{\pi n D}{\sqrt{2gH}} \tag{12.3,23}$$

where D is now the diameter of the part where the peripheral speed is measured. By convention ϕ_1 refers to entry to the runner and ϕ_2 to outlet; very often the diameter at outlet is not constant and then the greatest value is used to calculate ϕ_2. It follows from equation (12.3,3) that

$$\phi_1 = \frac{\pi}{\sqrt{2\beta}}. \tag{12.3,24}$$

On account of this simple relationship ϕ_1 may be used in the same general manner as β in the similarity theory of the machines.

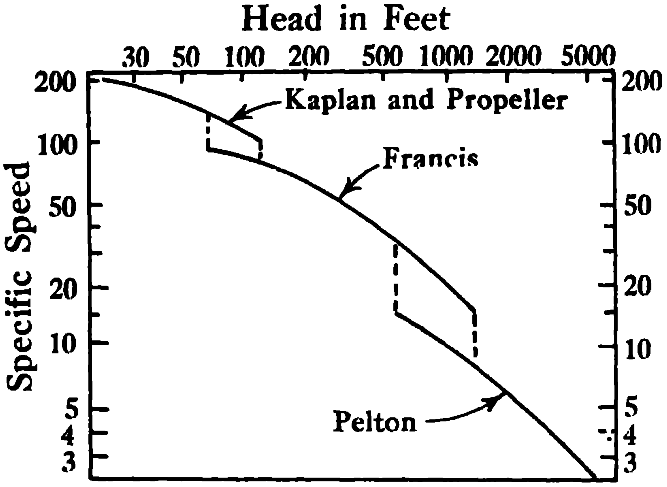

Fig. 12.3,1. Typical upper limits of specific speeds of turbines (based on H.P., R.P.M. and feet).†

The use of specific speeds in the selection of the most suitable type of turbine to meet given requirements is illustrated in Fig. 12.3,1 where the ordinate is specific speed (based on H.P., revolutions per minute and feet) to a logarithmic scale and the abscissa is head in feet. Maximum values of the specific speeds, in accordance with current practice, are shown for (a) Kaplan and propeller turbines, (b) Francis turbines and (c) Pelton wheels.

12.4 Dynamical Principles

In this section we consider the application of general dynamical principles to rotodynamic machines and, as explained in § 12.1, we shall restrict the discussion to steady states of the machines. The first general principle to be applied is the Principle of Angular Momentum (see § 3.2) and we shall use this to obtain the moment on the rotor about its axis of rotation. The principle states that the gain in angular momentum of the system consisting of the rotor and the fluid in unit time is equal to the moment applied to the rotor. To be definite think first of a pump. Then we have the relation:

Net driving torque applied to rotor = angular momentum (about axis of rotor) of fluid leaving rotor in unit time *less* angular momentum of fluid entering rotor in unit time. (12.4,1)

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† See footnote, page 660, for conversion to SI.

The *net* driving torque is the torque applied to the driving shaft less the opposing torque caused by mechanical friction and fluid friction (except that in the internal passages of the rotor). It is to be noted that the angular momentum of the rotor itself is constant since we have postulated that the machine is in a steady state. Next, for a turbine we have:

Gross torque developed by the rotor = angular momentum (about axis of rotor) of fluid entering rotor in unit time *less* angular momentum of fluid leaving rotor in unit time. (12.4,2)

The net or useful torque of the turbine is the gross torque less the opposing torque caused by mechanical and fluid friction. In general it is necessary to have complete knowledge of the distribution of velocity at entry to and exit from the rotor in order to apply the relations (12.4,1) and (12.4,2). However, very simple relations are obtained when the conditions at entry and at exit are uniform, as explained below.

Consider a radial flow machine and let us suppose for simplicity that the fluid is everywhere guided by very numerous and very thin vanes and let the vanes be arranged to give constant circumferential and radial components of velocity at entry and exit. The following symbols will be used:

w = circumferential component of absolute velocity of fluid (velocity of whirl),

r = radial distance from axis of rotor,

with the suffix 1 for entry and the suffix 2 for exit. The circumferential component of the momentum of an element of fluid of mass dm is $w\,dm$ and the moment of momentum is $rw\,dm$. The whole mass entering and leaving the rotor in unit time is ρQ and the angular momentum of the fluid entering the rotor in unit time is accordingly $\rho Q r_1 w_1$ while the angular momentum of the fluid leaving the rotor in unit time is $\rho Q r_2 w_2$. Hence we have for a pump

$$M_n = \rho Q(r_2 w_2 - r_1 w_1) \tag{12.4,3}$$

where M_n is the net driving torque. For a turbine the relation is

$$M_g = \rho Q(r_1 w_1 - r_2 w_2) \tag{12.4,4}$$

where M_g is the gross torque developed by the rotor. The equations (12.4,3) and (12.4,4) were, in principle, first given by Euler and are known as *Euler's relations*. There must be continuity of both the radial and circumferential components of velocity and this implies relations between the guide vane angles and rotor vane angles at entry and at exit. This aspect of the matter is considered in § 12.5. Whenever the velocities at entry and exit are not uniform† as postulated in the present simplified theory the expressions on the right hand sides of equations (12.4,3) and (12.4,4) will be replaced by integrals which are easily formulated in any given instance.

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† Also when the radii at entry and exit are not constant.

Let us next consider the power absorbed by the rotor of a pump. On account of (12.4,3) the net power absorbed (i.e. the power absorbed irrespective of mechanical and disk friction) is

$$P_n = \omega M_n = \rho Q(\omega r_2 w_2 - \omega r_1 w_1) = \rho Q(u_2 w_2 - u_1 w_1) \quad (12.4,5)$$

where u_1, u_2 are the linear velocities of the rotor at entry and exit, respectively. Then the 'runner head' or 'Euler head' is defined as

$$H_r = \frac{P_n}{g\rho Q} = \frac{u_2 w_2 - u_1 w_1}{g}. \quad (12.4,6)$$

For a turbine the gross power (i.e. the power developed by the runner without deduction for mechanical and disk friction) is, by equation (12.4,4),

$$P_g = \omega M_g = \rho Q(u_1 w_1 - u_2 w_2) \quad (12.4,7)$$

and the runner head is

$$H_r = \frac{P_g}{g\rho Q} = \frac{u_1 w_1 - u_2 w_2}{g}. \quad (12.4,8)$$

We shall now consider the variation of fluid pressure in the passages of the rotor and shall assume that these are filled with the fluid, which is regarded as incompressible. The body forces have the potential gz, where z is measured vertically upward from a horizontal datum plane while $\varpi = p/\rho$ since ρ is constant. The general theory of the flow of an inviscid fluid in a steadily rotating channel given in § 3.11 leads to equation (3.11,22) which becomes in the present application

$$\tfrac{1}{2}q'^2 + \frac{p}{\rho} + gz - \tfrac{1}{2}\omega^2 r^2 = C \quad (12.4,9)$$

where q' is the resultant velocity of the fluid *relative* to the rotor, ω is the constant angular velocity of the rotor and C is a constant which is to be determined by the conditions at entry. There is also the condition of continuity for the relative velocity of flow (see equations (3.11,9) and (3.11,10)) which takes exactly the same form as for flow relative to a fixed frame of reference. Equation (12.4,9) shows that there is a *centrifugal pressure*

$$p_c = \tfrac{1}{2}\rho\omega^2 r^2 \quad (12.4,10)$$

which is added to the pressure which would exist in the passage if the rotor were at rest and the flow through it unaltered. The corresponding centrifugal head is

$$H_c = \frac{\omega^2 r^2}{2g}. \quad (12.4,11)$$

As an example, let us suppose that the total normal sectional area of the passages is A_1 at entry and A_2 at exit. Then $q_1' = Q/A_1$ and $q_2' = Q/A_2$. Hence (12.4,9) yields, when divided by g,

$$\frac{Q^2}{2gA_2^2} + \frac{p_2}{g\rho} + z_2 - \frac{\omega^2 r_2^2}{2g} = \frac{Q^2}{2gA_1^2} + \frac{p_1}{g\rho} + z_1 - \frac{\omega^2 r_1^2}{2g}$$

or
$$\frac{p_2 - p_1}{g\rho} = \frac{\omega^2}{2g}(r_2^2 - r_1^2) - (z_2 - z_1) + \frac{Q^2}{2g}\left(\frac{1}{A_1^2} - \frac{1}{A_2^2}\right). \quad (12.4,12)$$

So far the fluid has been treated as inviscid but the last equation may be corrected to allow for frictional losses in the passages. The head lost may be estimated from the value of Q and the particulars of the passages just as if the rotor were stationary. This is not quite exact since the rotation may influence the boundary layer flow in the passages. It will usually be adequate to assume that the head lost is

$$H_l = k\left(\frac{q_2'^2}{2g}\right) \quad (12.4,13)$$

where k is constant.

The foregoing results show that it is convenient to have outward flow in a pump and inward flow in a turbine although these arrangements are not necessary. For example, the flow is outward in the 'Scotch turbine' or Barker's mill, which is however obsolete, while it would be possible to make a centripetal pump. A turbine with inward flow is to some extent self-governing since, if it overspeeds, the increase of centrifugal pressure tends to check the flow and so to reduce the torque.

In calculating loads on bearings it should be kept in mind that, in general, there will be hydrodynamic as well as hydrostatic loads, also weights. The hydrodynamic loads can be computed from the momentum changes. For example, water enters the rotor of a Francis turbine symmetrically in a plane perpendicular to the vertical axis of rotation and leaves the rotor downwards. Then the gain in downward momentum in unit time is $\rho v_a Q$, where v_a is the mean axial velocity at exit from the rotor. Accordingly, this is the value of the *upward* axial hydrodynamic thrust.

Example 1. *Show that equations* (12.4,8) *and* (12.4,12) *are not independent*

For definiteness consider an inward flow turbine and let θ be the angle between the blade and the tangent to the circumference of the runner. Then the velocity of whirl is

$$w = u + q' \cos\theta.$$

The square of the absolute velocity of the fluid is

$$v^2 = u^2 + q'^2 + 2uq' \cos\theta = q'^2 - u^2 + 2uw$$

or
$$uw = \tfrac{1}{2}(u^2 + v^2 - q'^2).$$

Now the excess of the head of the fluid just at entry over that at exit is

$$H_r = (z_1 - z_2) + \frac{1}{2g}(v_1^2 - v_2^2) + \frac{1}{g\rho}(p_1 - p_2)$$

and (in the absence of frictional losses) this is the work done by the fluid on the runner per unit weight. By (12.4,8) this is equal to

$$\frac{1}{g}(u_1 w_1 - u_2 w_2) = \frac{1}{2g}(u_1^2 + v_1^2 - q_1'^2 - u_2^2 - v_2^2 + q_2'^2).$$

When we equate the expressions for H_r and rearrange we get, since $u = \omega r$,

$$\frac{p_1 - p_2}{g\rho} = \frac{\omega^2}{2g}(r_1^2 - r_2^2) - (z_1 - z_2) - \frac{1}{2g}(q_1'^2 - q_2'^2)$$

which is equivalent to (12.4,12).

12.5 Velocity Diagrams

In order to achieve high efficiency in a rotodynamic machine it is necessary to avoid breakaway of flow from the blades since this leads to the development of vigorous turbulent flow with consequent dissipation of mechanical energy into useless heat. Breakaway of flow near the leading edges or blades is particularly objectionable and this will occur when the incident stream is too greatly inclined to the tangent to the centre line of the blade at the leading edge. The ideal condition of *smooth entry* is when the incident stream is exactly along this tangent† and this very often requires that the incident stream should be guided by a system of fixed blades or nozzles. The angles of such blades or nozzles can be found very easily by the use of velocity diagrams, as will now be explained.

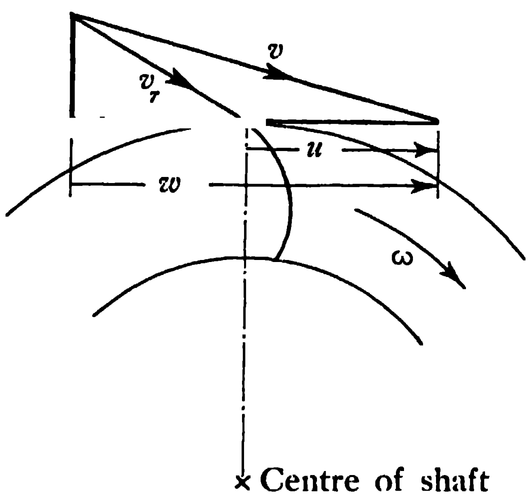

Fig. 12.5,1. Velocity triangle at entry to blade (inward radial flow).

Fig. 12.5,1 refers to a machine with inward radial flow and we shall consider the conditions at the outer end of the rotor blade. Let

v = absolute velocity of fluid at entry
w = component of the absolute velocity in the circumferential direction
v_r = velocity of fluid at entry relative to the rotor
u = circumferential velocity of the blade tip.

Then v is the vector sum of v_r and u as shown in the diagram. The tangent to the fixed guide vane at its exit must be parallel to v while the tangent to the rotor blade at entry must be parallel to v_r, if perfectly smooth flow is to be attained; the tangents referred to are those of the centre lines of the vanes or blades, which are supposed to be thin. The radial component of velocity is continuous at entry, i.e. v and v_r have the same component in the radial direction, as is evident from the diagram. When this radial component and u are given, together with one of the blade angles, the other is fixed by the geometry of the triangle of velocities. Usually it is not convenient to make the blade angles of the rotor variable but there is no difficulty in making the guide vanes adjustable. In this way it is possible to achieve perfectly smooth entry for a range of velocity ratios. A velocity diagram for the inner edge of the rotor blade can be constructed in exactly the same way as for the outer edge; if the radial flow is outward, the sense of the vector v_r must be altered accordingly. The radial component of velocity is denoted by f while the angle between the tangent to the vane and the tangent to the runner is θ.

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† More exactly, there is an *ideal entry incidence* which is usually small (compare § 11.5.3 and § 11.8.3).

Fig. 12.5,1 is a projection of the velocity vectors on a plane normal to the axis of rotation. However, if it be regarded as a projection on a plane parallel to the axis of rotation it is applicable to a machine with *axial flow*. In that case the circumferential velocity u will vary along the blade in proportion to the distance from the axis of rotation.

The simple velocity relations already discussed are strictly applicable only when there is completely uniform flow at entry and exit, such as could be obtained in an inviscid fluid by the use of very numerous and thin guide vanes. When the spacing of vanes and blades is bigger, as it must be in practice, the flow is not strictly uniform and the velocity relations are, in general, affected by circulation about the blades.

12.6 General Discussion of Losses

In § 12.1 we have already drawn attention to the importance of eliminating losses of available energy, which necessarily result in a fall in the efficiency of a pump or turbine. We consider it to be beyond our scope to discuss the mechanical efficiency of a machine in detail since this is a question of mechanical engineering. It is, however, opportune to remark that the hydraulic design should be such as to make easy the attainment of a high mechanical efficiency. For example, the design should be such that loads on bearings are as small as possible (see § 12.4). Hence the hydraulic and gravity loads should be in balance, so far as possible.

The principal causes of hydraulic losses are as follows:

(a) Fluid friction in nozzles, passages in the runner and at the surfaces of blades, buckets etc.

(b) Friction in any layer of fluid lying between moving and fixed surfaces (disk friction and the like).

(c) Leakage of fluid from regions of high head into regions of low head.
(d) Turbulence, especially that associated with 'detached' flow.
(e) Lack of uniformity of flow.
(f) Existence of secondary flows.

These are discussed separately below. The occurrence of cavitation will, in general, result in increased losses (see § 12.12).

As regards (a), some losses by fluid friction are inevitable and the problem is to keep them small. Now the frictional stresses rise rapidly with speed of flow so the regions of high speed flow should be kept as small as possible. For example, the velocity in the approaches to a nozzle should be much lower than in the nozzle itself. All surfaces should be well faired and smooth.

(b) This source of loss is absent in some machines, e.g. the Pelton wheel and Barker's mill. As a typical instance we may take a centrifugal pump where there is a layer of fluid between the outside of the impeller and the casing. The frictional loss cannot be avoided altogether, but it is reduced by keeping surface areas as small as possible and by enlarging the gap. However, any increase of the gap increases the loss by leakage (see item (c)), but this may be reduced by the use of labyrinths.

(c) Let the loss of fluid by leakage be equivalent to a rate of discharge Q_L. If the leakage occurs under the full difference of head the efficiency of the machine will be multiplied by the factor $(Q - Q_L)/Q$, which is less than unity; when the difference of head across the leak is smaller, the loss will be smaller in proportion. As already pointed out under (b), a reduction of leakage attained by reducing gaps will lead to an increase of the frictional loss. Losses may be reduced by the use of labyrinths.

(d) The kinetic energy associated with the irregular motions in a region of turbulent flow is dissipated into useless heat and therefore implies a loss of efficiency; moreover, the intensity of skin friction is much higher, other things being equal, when flow is turbulent than when it is laminar. When an important separation or 'breakaway' of flow occurs (see § 6.2) there is a large region of vigorous turbulence and a corresponding fall of efficiency. The prevention of separation by boundary layer control is discussed in § 6.24.

(e) The flow in the passages of a machine should be as nearly uniform as possible.† Forces and moments depend on the momentum of the stream and, with a given momentum, the kinetic energy is least when the flow is uniform. It is to be understood that absolute uniformity is usually unattainable (quite apart from the effects of surface friction) but gross departures from uniformity must be avoided. The variation of axial velocity in ducts should be kept small; constrictions and enlargements must be avoided (unless structurally necessary) since the changes from dynamic to static head and vice versa are never effected without loss. Loss by splashing and dispersal of droplets in Pelton turbines may be included under the present heading.

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† More generally, the flow should be as nearly irrotational as possible.

(f) Secondary flow is exemplified by that in a pipe bend (see § 7.8) and always occurs where the boundary layer at a fixed surface is exposed to a pressure gradient transverse to the stream (see also § 11.8.3). The secondary flow implies dissipation of energy and loss of efficiency. Secondary flows may be reduced by the subdivision of passages but this may increase the frictional loss, so a compromise must be accepted.

Losses may be grouped together in various ways, according to the viewpoint adopted, and it may be convenient to group some of the hydraulic with the mechanical losses. For example, disk friction and frictional resistances to relative motion arising in labyrinths may conveniently be treated as additions to mechanical friction. Loss by leakage is often regarded as a separate item and the remaining hydraulic losses may be divided into those occurring within the runner and those external to it.

12.7 The Pelton Wheel

The Pelton wheel is the only hydraulic turbine of the impulse type which is in current use and it is a highly efficient machine but only suitable for high heads. A general description of this turbine has been given in § 12.2 and a simplified theory will now be developed in which the motion of the bucket is treated as rectilinear. Fig. 12.7,1 shows a section of the bucket by a plane parallel to the axis of rotation and containing the axis of the incident jet, also the velocity vectors of the incident and emergent streams. The following symbols will be used:

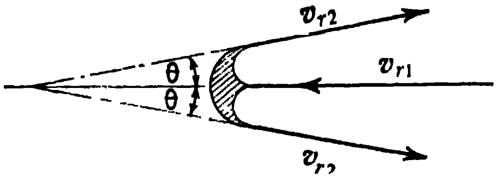

Fig. 12.7,1. Idealised diagram of bucket and velocity vectors.

v = absolute velocity of stream issuing from nozzle
A = total effective area of nozzles
r = radius from axis of rotation to axis of incident jets
ω = angular velocity of wheel
v_{r1} = velocity of incident jets relative to bucket
v_{r2} = velocity of emergent jets relative to bucket
θ = acute angle between emergent and incident jets
$u = r\omega$ = velocity of centre of bucket.

Then we have

$$v_{r1} = v - u \tag{12.7,1}$$

and we shall assume that as a result of frictional resistance

$$v_{r2} = k_1 v_{r1} \tag{12.7,2}$$

where k_1 is less than unity; since the jet is exposed to the atmosphere the pressure of the fluid is assumed to be constant and, in the absence of friction, the velocity would be constant. The change of velocity in the direction of the jet axis is therefore

$$v_{r1} + v_{r2}\cos\theta = v_{r1}(1 + k_1\cos\theta). \tag{12.7,3}$$

The foregoing argument is based on an idealised picture of the phenomena and in practice things are not so simple. For example, there is considerable scatter of the fluid emerging from the buckets. We shall assume that the effective change of velocity is

$$\Delta v = v_{r1}(1 + c) \tag{12.7,4}$$

where c is less than unity but should be as near to unity as possible. The force exerted on the bucket by one jet is

$$T = \rho Q_1 \Delta v$$

where Q_1 is the volume discharged from one nozzle in unit time. This gives rise to a moment about the axis of rotation equal to $r\rho Q_1\Delta v$ and the total torque is

$$M = r\rho Q\Delta v \tag{12.7,5}$$

where Q is the total rate of discharge from the nozzles. We shall assume that the loss of head by friction in a nozzle is $k_2(v^2/2g)$. Hence

$$\frac{v^2}{2g} = H - k_2\left(\frac{v^2}{2g}\right)$$

or

$$v = \sqrt{\frac{2gH}{1 + k_2}} \tag{12.7,6}$$

where H is the total available head at the entrance to the nozzle, and

$$Q = Av. \tag{12.7,7}$$

Equation (12.7,5) becomes, in view of (12.7,1), (12.7,4) and (12.7,7)

$$M = r\rho Av(v - u)(1 + c) \tag{12.7,8}$$

and the power developed is

$$P = M\omega = \rho Auv(v - u)(1 + c). \tag{12.7,9}$$

Suppose that ρ, A, v and c are all constant. Then P will be a maximum when $u(v - u)$ is a maximum, i.e. when $u = v/2$. Hence the maximum power is†

$$\begin{aligned} P_{\max} &= \tfrac{1}{4}\rho Av^3(1 + c) \\ &= \tfrac{1}{4}(1 + c)\rho A\left(\frac{2gH}{1 + k_2}\right)^{3/2} \end{aligned} \tag{12.7,10}$$

---

† In practice there is an additional loss caused by windage on the wheel and then the efficiency is greatest when u/v is a little less than 0·5.

by equation (12.7,6). Since the rate of supply of energy is independent of the speed of the wheel, it follows that the efficiency is a maximum when the power developed is a maximum. Now the rate of supply of energy is

$$\rho gHQ = vg\rho HA$$

and therefore

$$\eta_{\max} = \frac{P_{\max}}{vg\rho HA} = \frac{1}{4}\frac{v^2(1+c)}{gH} = \frac{1+c}{2(1+k_2)} \qquad (12.7,11)$$

by (12.7,6). In the ideal case where $c = 1$ and $k_2 = 0$, $\eta_{\max}$ is unity.

A very important parameter in the design of a Pelton wheel is the ratio (jet diameter) : (bucket diameter) If this is too small the power and efficiency will both be low but if it is too large the water will not be properly deflected backwards by the bucket, with large resulting losses and low efficiency. According to experience the optimum value of the ratio is about 1/5. There must be sufficient clearance between the buckets and the casing to allow the discharged fluid to escape freely without being largely splashed back on the runner. The buckets of a Pelton wheel are notched, as shown in Fig. 12.2,6(a), in order to prevent the approaching buckets from interfering with the proper impingement of the jet on the bucket which is normal to it. The spacing or pitch of the buckets will be roughly proportional to their diameter and is commonly about 1·5 times the diameter.

A typical curve of efficiency against load for a Pelton turbine, as measured on a small experimental machine, is given in Fig. 12.9,1. It will be noted that the curve has a very flat top, so the efficiency is high over a wide range of conditions.

12.8 Barker's Mill or Scotch Turbine

Barker's Mill is now obsolete but merits some discussion because it is the simplest of all reaction turbines and illustrates the basic principles of such turbines very well. This water mill was invented about the end of the 17th century but the dynamical principle of its operation, namely jet reaction, was used in the *aeolipile* invented by Hero (Heron) of Alexandria, whose date is uncertain but is somewhere near the beginning of the Christian era. The aeolipile consists of a hollow metal sphere mounted on trunnions so as to rotate freely about a diametral axis; one of the trunnions is hollow and admits steam generated by boiling water in a cauldron. The steam escapes from the sphere through two short tubes situated at opposite ends of a diameter lying in the diametral plane normal to the axis of rotation. The tubes are bent round so that the steam jets are tangential to the sphere in the diametral plane while the jets are oppositely directed so that the moments of their reactions about the axis are added. Although the aeolipile was no more than a 'philosophical toy' it was both the first steam engine and the first turbine of any kind.

A diagram of Barker's mill in the form developed by James Whitelaw[1] is shown in Fig. 12.8,1. The runner has two equal and symmetrically situated hollow curved arms and the jets issue from the open ends of these while water is supplied under pressure through a watertight gland at the centre of the runner, whose axis of rotation is vertical. We shall use the following symbols in developing the theory:

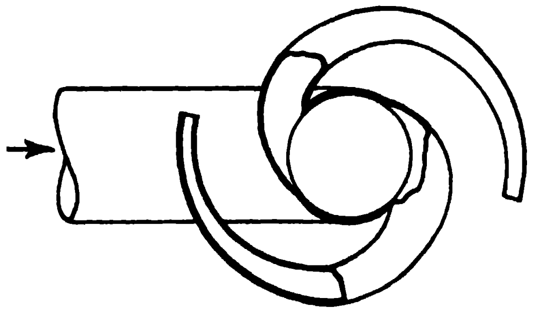

Fig. 12.8,1. Whitelaw's turbine.

$v_r = q'$ = velocity of fluid relative to runner at exit
w = absolute velocity of fluid at exit (this is in the circumferential direction)
r = radius from axis of rotation to centre of orifice at exit
H = excess of head at centre of runner over atmospheric head at same level
ω = angular velocity of runner
k = a frictional loss coefficient.

Other symbols have their usual meanings. To find the torque we use equation (12.4,4) and we may take the radius at inlet to be zero. Hence

$$M = wr\rho Q = \rho r(v_r - \omega r)Q \qquad (12.8,1)$$

since

$$w = v_r - \omega r. \qquad (12.8,2)$$

Therefore the power developed is

$$P = \omega M = \rho\omega r(v_r - \omega r)Q, \qquad (12.8,3)$$

a relation which is valid irrespective of frictional losses in the passages of the runner. But the energy supplied in unit time is $g\rho HQ$ and the efficiency is accordingly

$$\eta = \frac{P}{g\rho HQ} = \frac{\omega r(v_r - \omega r)}{gH}. \qquad (12.8,4)$$

We can calculate the efficiency in another way by finding the amount E_w of energy wasted in unit time. First, the absolute velocity of the fluid just after leaving the runner is $(v_r - \omega r)$ and the kinetic energy of the fluid issuing in unit time is $\frac{1}{2}\rho Q(v_r - \omega r)^2$. Suppose that the resistance to flow in the passages of the runner gives rise to a loss of head amounting to $k(v_r^2/2g)$.

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[1] *Description of Whitelaw and Stirrat's Patent Water-Mill* (London, Mechanics' Magazine office, 1843).

This causes a loss of energy in unit time equal to

$$Q \times g\rho \times k\left(\frac{v_r^2}{2g}\right) = \tfrac{1}{2}k\rho Q v_r^2$$

and therefore

$$E_w = \tfrac{1}{2}\rho Q[(v_r - \omega r)^2 + kv_r^2]. \tag{12,8.5}$$

Consequently the efficiency is

$$\eta = \frac{g\rho HQ - E_w}{g\rho HQ} = 1 - \frac{(v_r - r\omega)^2 + kv_r^2}{2gH}. \tag{12.8,6}$$

The two expressions for the efficiency must be equal and we find after a little reduction that this requires

$$(1 + k)v_r^2 = 2gH + r^2\omega^2. \tag{12.8,7}$$

This is exactly the equation which results from consideration of the change of head in the passages of the runner when allowance is made for the frictional loss. Since the flow within the runner is horizontal and the difference of head between the centre of the runner and atmosphere is H, equation (12.4,5) yields when corrected for the loss of head by friction

$$\frac{q'^2}{2g} - \frac{\omega^2 r^2}{2g} = H - k\left(\frac{q'^2}{2g}\right)$$

which reduces to (12.7,7) since $q' = v_r$. The expression for the efficiency can be put in a very simple form in the special case where k is zero (no loss by fluid friction in the passages of the runner). The total energy supplied in unit time

$$= (\text{useful work done} + \text{energy wasted}) \text{ in unit time}$$

$$= \rho Q(v_r - \omega r)[\omega r + \tfrac{1}{2}(v_r - \omega r)] = \tfrac{1}{2}\rho Q(v_r^2 - \omega^2 r^2)$$

so the efficiency is

$$\eta = \frac{\rho Q(v_r - \omega r)\omega r}{\tfrac{1}{2}\rho Q(v_r^2 - \omega^2 r^2)} = \frac{2\omega r}{v_r + \omega r}. \tag{12.8,8}$$

This is unity when $v_r = \omega r$ but then by (12.18,3) the power is zero. The more general expression (12.8,6) in this case gives zero efficiency in view of equation (12.8,7).

It will be convenient to express the efficiency, torque and power in terms of the non-dimensional quantity

$$\lambda = \frac{H_c}{H} = \frac{r^2\omega^2}{2gH} \tag{12.8,9}$$

which is equal to the reciprocal of β, as defined in (12.3,3), multiplied by a constant. We thus have

$$r\omega = \sqrt{(2\lambda gH)} \tag{12.8,10}$$

while (12.8,7) yields

$$v_r = \sqrt{\left(\frac{2gH(1+\lambda)}{1+k}\right)}. \tag{12.8,11}$$

Equation (12.8,4) is valid for all values of k and now becomes on reduction

$$\eta = 2\left[\sqrt{\left(\frac{\lambda(1+\lambda)}{1+k}\right)} - \lambda\right]. \tag{12.8,12}$$

We find in the usual way that η is a maximum when

$$\lambda = \frac{1}{2}\left[\sqrt{\left(\frac{1+k}{k}\right)} - 1\right] \tag{12.8,13}$$

and

$$\eta_{\max} = 1 - \sqrt{\left(\frac{k}{1+k}\right)}. \tag{12.8,14}$$

This falls from 0·9 when $k = 0{\cdot}01$ to 0·7 when $k = 0{\cdot}1$ and we see that the attainment of a high efficiency depends vitally on keeping the frictional loss of head to a minimum. Next, we have

$$Q = v_r A$$

where A is the total effective area of the jet orifices and by equation (12.8,1) the torque is

$$M = 2\rho gHAr\left[\frac{1+\lambda}{1+k} - \sqrt{\left(\frac{\lambda(1+\lambda)}{1+k}\right)}\right]. \tag{12.8,15}$$

This shows that M vanishes when $\lambda = 1/k$ and it follows from (12.8,7) that $v_r = \omega r$. The torque has its maximum value

$$M_{\max} = \frac{2\rho gHAr}{1+k} \tag{12.8,16}$$

when $\lambda = 0$, i.e., when the runner is at rest, and the torque always decreases as λ increases. From (12.8,3) the power is

$$P = \rho A(2gH)^{3/2}\left[\frac{(1+\lambda)\sqrt{\lambda}}{1+k} - \lambda\sqrt{\left(\frac{1+\lambda}{1+k}\right)}\right]. \tag{12.8,17}$$

This vanishes when $\lambda = 0$ and when $\lambda = 1/k$. The maximum value occurs when $dp/d\lambda$ is zero and it is found that the condition for this is

$$9k\lambda^3 - (3 - 12k)\lambda^2 - (3 - 4k)\lambda - 1 = 0. \tag{12.8,18}$$

This cubic equation cannot, in general, be solved exactly but a good approximation to the relevant root is

$$\lambda = \frac{1 - k + 2k^2 - k^3}{3k}. \tag{12.8,19}$$

The rough rule is that the power is greatest when the speed of rotation is $1/\sqrt{3}$ times the speed corresponding to zero torque. The speed for maximum efficiency is less than that for maximum power approximately in the ratio $\sqrt{(3)}/2$. It is noteworthy that these rates of rotation increase rapidly as the friction coefficient k is reduced. In practice the power might be limited by the occurrence of cavitation, which is not taken into account in the present theory. Another comment is that, while this turbine is not unstable, its stability is very much less than that of an inward flow machine (see also § 12.4).

Barker's mill is a pure reaction turbine, i.e., the change from static to dynamic head takes place entirely in the runner. In modern turbines, however, only a part of this conversion occurs in the runner and the remainder occurs in fixed passages or nozzles.

12.9 Inward Flow Turbines

At present very many reaction turbines are of the inward flow type but the flow is not usually entirely in a plane perpendicular to the axis of rotation; the flow at exit from the runner is, in modern machines, as nearly as possible axial. The tendency towards instability of the outward flow reaction turbine (see § 12.8) was early noted and several inventors produced inward flow machines, notably Poncelet, James Thomson and Francis. The Francis turbine is now very extensively used for moderate heads and it alone is discussed here. Strictly, it is a mixed flow turbine as the discharge is axial.

A general description of the Francis turbine has been given in § 12.2 where it was explained that the water approaches the runner through a set of guide vanes or wicket gates, which are adjustable to suit the running conditions. Part of the total head of the water is converted into dynamic head in the guide passages, so the Francis turbine is not, like Barker's mill, a pure reaction machine. The fraction of the total head converted to dynamic head in the guide passages is of the order of one half. As the water flows inward through the passages in the runner it is opposed by the centrifugal pressure and this accounts for much of the difference of head across the runner (see § 12.4).

In order that the kinetic energy in the water discharged from the runner shall be a minimum its velocity must be everywhere in the axial direction and uniform. Then the velocity of whirl w_2 at exit from the runner is everywhere zero and, by equation (12.4,8), the runner head is

$$H_r = \frac{u_1 w_1}{g}. \tag{12.9,1}$$

Even when the ideal conditions at exit are not exactly met this gives a good approximation to the runner head. As explained in § 12.2, a large part of the kinetic energy in the water leaving the runner may be recovered in the draught tube.

A curve showing the efficiency of a small experimental Francis turbine at various fractions of full load is given in Fig. 12.9,1 where a curve for a Pelton wheel is also included. Rather higher efficiencies are attainable with large Francis turbines.

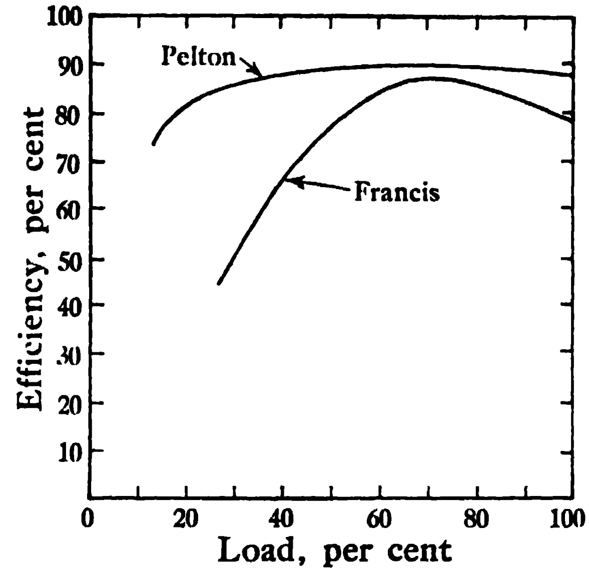

Fig. 12.9,1. Efficiency curves for Francis and Pelton turbines.

12.10 Kaplan Turbine

A general description of this turbine has been given in § 12.2 while a photograph of a runner, showing also the wicket gates, appears in Fig. 12.2,6(c). This machine is essentially a ducted water fan working in reverse and much of the general theory of ducted fans is applicable, with some obvious modifications (see § 13.13). It is characteristic of a fan or propeller

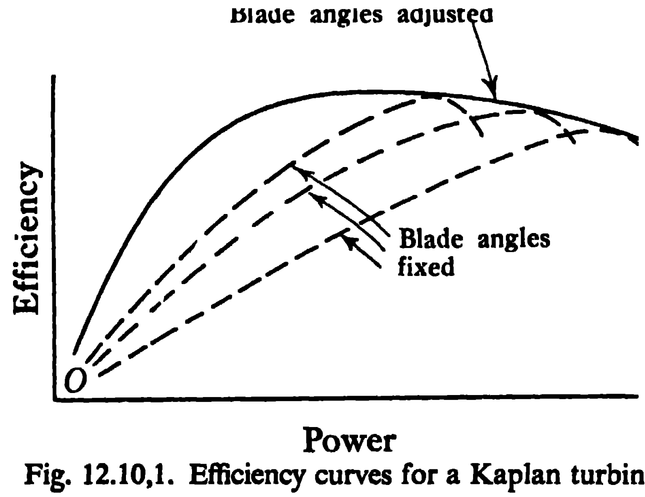

Fig. 12.10,1. Efficiency curves for a Kaplan turbine.

having fixed pitch or blade angles, whether delivering or absorbing power, that the efficiency falls rather rapidly when the 'advance ratio' (see § 13.3) exceeds the value for maximum efficiency. Similarly the efficiency of a Kaplan turbine, with fixed blade settings, falls rather rapidly when the discharge or quantity number (see equation (12.3,1)) exceeds the value for maximum efficiency. However, the envelope curve for efficiency, which is touched by the efficiency curves for the individual blade settings, is flat topped as shown in Fig. 12.10,1. Thus, provided that the blades are always properly adjusted, the Kaplan turbine is capable of giving a high efficiency for a wide range of conditions.

12.11 **Centrifugal Pumps**

General descriptions of types of centrifugal pumps have been given in § 12.2. Diagrams of these are shown in Fig. 12.2,5 and there is a photograph of a pump-turbine runner in Fig. 12.2,6(d).

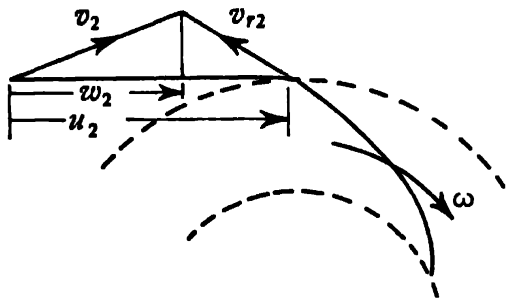

Fig. 12.11,1. Velocity triangle at exit from centrifugal pump impeller.

The general principle of the pump is that fluid enters the impeller at the eye in a direction which is axial or nearly so, passes out through the blade passages and is discharged at the periphery of the impeller with a velocity relative to it which has an *outward* and a *backward* circumferential component; the absolute circumferential component in the forward direction is then the tip speed of the impeller less the circumferential component of the relative velocity. A typical velocity diagram is shown in Fig. 12.11,1. After leaving the impeller the velocity of the fluid is reduced (with a concurrent conversion of some dynamic head into static head) either in guide passages or in free flow in the volute or recuperator (see Fig. 12.2,5). In the passage through the impeller the static head of the fluid is increased by the centrifugal head, quite independently of any changes associated with variations of the relative velocity of flow (see equation 12.4,12). If it be assumed that the velocity of whirl of the fluid is zero at the entry to the impeller the runner head, as given by equation (12.4,6), is

$$H_r = \frac{u_2 w_2}{g}. \qquad (12.11,1)$$

As in the corresponding equation for the Francis turbine, there is little error in taking this to be generally true. In the optimum condition of running the fluid leaves the impeller with a considerable velocity of whirl and there is some recovery of static head in the volute. The *manometric head* H_m is

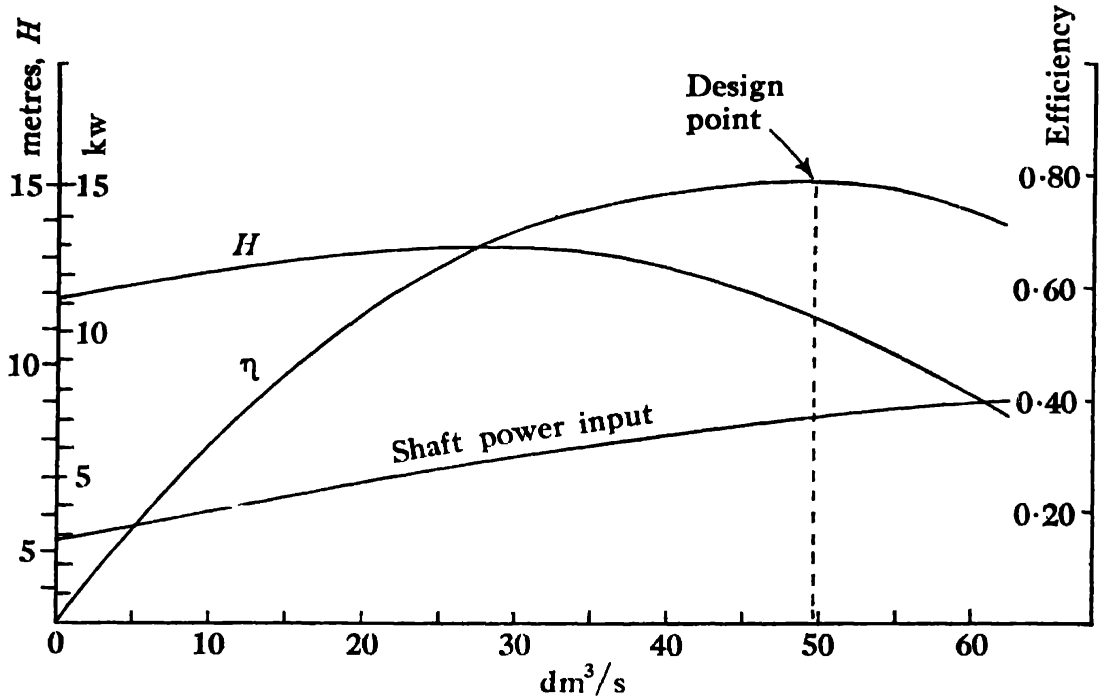

Fig. 12.11,2. Performance curves for 216 mm double entry centrifugal pump, at constant speed $N = 1350$ r.p.m.

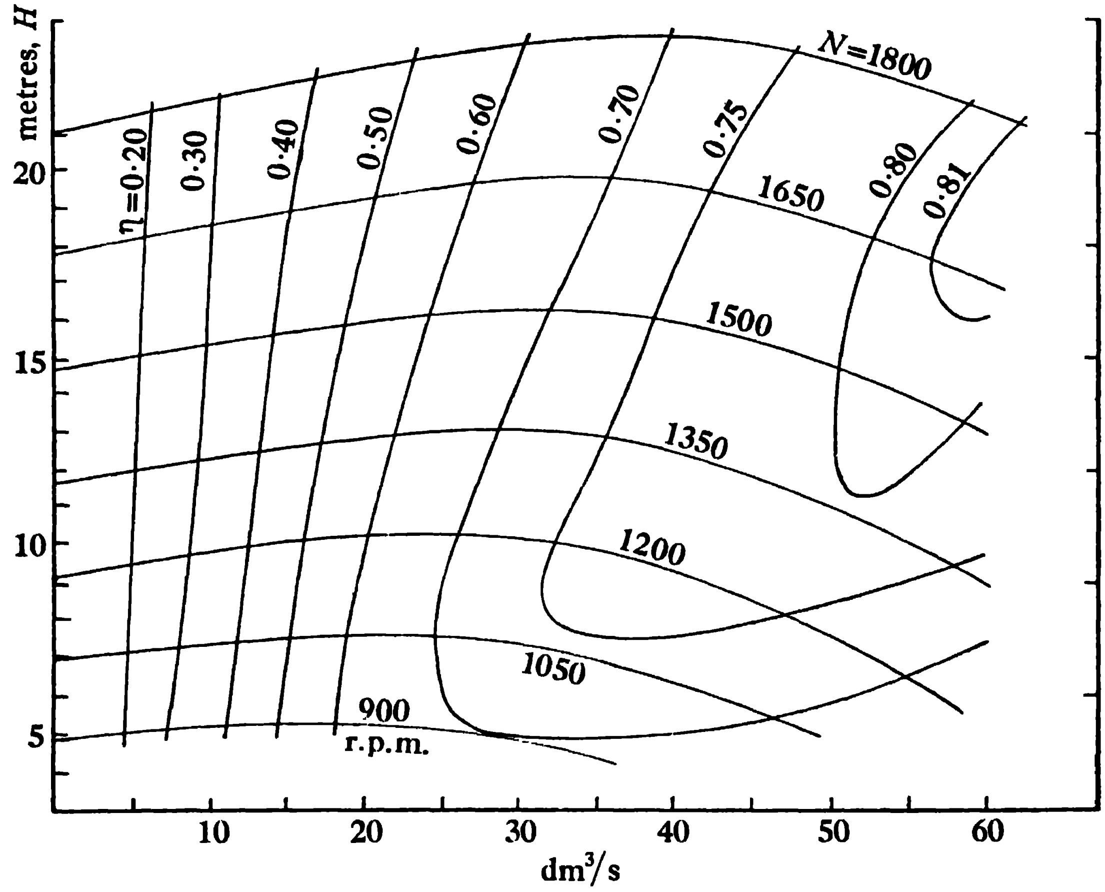

Fig. 12.11,3. Centrifugal pump characteristics (varying speed).

defined to be the gain in pressure head across the pump and the *manometric efficiency* is

$$\eta_m = \frac{gH_m}{u_2 w_2}. \qquad (12.11,2)$$

The curves in Fig. 12.11,2 show typical constant speed characteristics of a centrifugal pump plotted against rate of discharge; the ordinates show head, efficiency and power absorbed. At the 'design point' the efficiency is a maximum. When the speed is varied there will be a family of curves for any quantity corresponding to the speeds selected. Fig. 12.11,3 shows curves of head plotted on a base of Q for various speeds of rotation and with contours of constant efficiency superposed.

12.12 Cavitation

A general discussion of cavitation has been given in § 3.12 and the reader may be reminded that the word covers the occurrence and development of bubbles (regions devoid of fluid) within the fluid, especially in places where the pressure is low because the velocity is high. The occurrence of gross cavitation (large regions devoid of fluid) is incompatible with the proper working of a hydraulic machine and is not found in practice under ordinary conditions of running unless there has been a serious error in design or installation. However, cavitation on the small scale, more particularly with the transient development of very small bubbles, may have only a very slight effect on the performance of the machine and yet be most objectionable because it leads to serious damage to metal surfaces by pitting and accelerated corrosion (see § 3.12). Hence even incipient cavitation is to be avoided.

Cavitation occurs when the *absolute* pressure falls below a critical value, which is influenced by the presence and nature of nuclei for bubble formation; the corresponding gauge pressure is usually negative (vacuum). The absolute pressure (or head) at any point is the absolute total pressure less the sum of the position and dynamic pressures; hence cavitation may be avoided by

(a) making the position head low or

(b) making the dynamic head low.

Condition (a) requires that in the critical regions for cavitation the vertical height above datum level shall be small and in practice this often requires that these critical regions shall lie well below the free surface level in the sump or inlet branch for a pump or in the tailrace for a turbine. Condition (b) requires that the highest velocity of flow, even where this occurs only in a very restricted region, shall not be too high. These two aspects of the matter are considered separately below.

The Thoma cavitation factor is the non-dimensional number

$$\sigma = \frac{H_a - Z_0}{H} \qquad (12.12,1)$$

where H_a = pressure head at free surface in contact with atmosphere (or corresponding to atmospheric pressure)

Z_0 = height of outlet of turbine runner *above* free surface in tailrace

H = difference of total head across machine.

The cavitation factor for any given machine and installation has a critical value σ_c when cavitation is about to begin and cavitation occurs when σ is *less* than σ_c. The critical value depends on the specific speed of the turbine (see § 12.3) and may be estimated from the values given in Table 12.12,1.

TABLE 12.12,1.

Minimum Values of Thoma Cavitation Factors for Turbines
(After Addison†)

Specific speed, based on revs. per min., horse power and ft.‡	23	56	90	135	180
Minimum value of σ	0·04	0·15	0·35	0·8	1·3

† H. Addison, *A Treatise on Applied Hydraulics*, 3rd Edition, p. 345 (Chapman and Hall).

‡ To obtain specific speed, equation 12.3,7, based on rev/min kW and metres, multiply these tabulated values by 3·814.

It must be emphasised that the tabulated figures are no more than a useful general guide since the incidence of cavitation depends greatly on the details of the design.

Although by no means obvious at first sight, there is a region of very high velocity near the leading edge of a blade or hydrofoil, unless the effective angle of incidence of the stream lies within a very restricted range, and in the region of high velocity there is a 'peak suction'. Thus it follows that, except for optimum conditions of incidence or entry, there will be high peak suctions near the leading edges of blades or hydrofoils and these will be likely to give rise to local cavitation. The suction peaks can be reduced by careful design of the blade profiles but they are at their worst with thin sharp nosed blades (except when the entry conditions are precisely correct). Regions of high suction, with attendant cavitation, also sometimes occur near the trailing edges of turbine blades.

12.13 Methods for the Measurement of Efficiency

The efficiency of a rotodynamic machine can only be measured under steady conditions of running and, for large machines, a considerable period for settling down to a really steady state is needed. For a pump, the efficiency is the ratio of the useful work done on the fluid to the energy supplied at the coupling during the same time interval while for a turbine it is the ratio of the work developed at the coupling to the hydraulic energy supplied in the same interval. In both cases it is thus necessary to measure a quantity of hydraulic energy and the corresponding amount of mechanical work. Usually the quantities are reckoned per unit of time and are therefore measured as power.

The mechanical power developed by a turbine at the coupling may be measured directly by a brake in the ordinary way when the power is small but this is impracticable for machines of high power. Turbines of large output are nearly always coupled to electric generators and the power developed at the coupling is obtained from the electric power output (converted to mechanical units) by division by the overall efficiency of the generator. The chief uncertainty here concerns this efficiency. The power output can be measured directly, if any means exist for the simultaneous measurement of torque and rate of rotation but torque meters of large capacity are very rarely available. With some obvious changes, the foregoing remarks apply to the power input to a pump.

Whether a pump or turbine is under test, it is necessary to determine the hydraulic energy gained (or expended) in unit time. This is equal to $g\rho QH$ where Q is the quantity of fluid passing in unit time and H is the difference of total head across the machine. Methods for the measurement of Q have been discussed in § 5.10 while measurements of pressure and head are treated in § 5.11.

The above methods for efficiency measurement are time-consuming and use of the *thermodynamic method*[1] is becoming more common. Friction raises the temperature of the fluid and if pressure and temperature difference between inlet and outlet are measured, the inner or hydraulic efficiency can be calculated. Knowledge of change of enthalpy of the fluid with pressure and temperature is necessary. Specific enthalpy i is defined as internal energy e plus pressure energy p/ρ (flow work) per unit mass; if total energy h is defined as i plus kinetic energy plus potential energy per unit mass (see equation 9.3,13), then

$$h = e + p/\rho + v^2/2 + gz = i + v^2/2 + gz. \qquad (12.13,1)$$

It is assumed that v and z are known.

Specific enthalpy i may be expressed as a function of any pair of independent variables. With p, T and p, s for example, $i = f(p, T)$ and $i = f(p, s)$ and so

$$di = \left(\frac{\partial i}{\partial p}\right)_T dp + \left(\frac{\partial i}{\partial T}\right)_p dT \qquad (12.13,2)$$

and

$$di = \left(\frac{\partial i}{\partial p}\right)_s dp + \left(\frac{\partial i}{\partial s}\right)_p ds, \qquad (12.13,3)$$

where s is the specific entropy.

Inner efficiency of a turbine is given by the ratio between W, the exchange of energy by work and the ideal energy exchange (entropy s constant); thus between inlet 1 and outlet 2,

$$\eta_{tu} = \frac{W}{(h_2 - h_1)_s} = \frac{(h_2 - h_1)}{(h_2 - h_1)_s}. \qquad (12.13,4)$$

With equations 12.13,1, 2 and 3, equation 12.13,4 gives

$$\eta_{tu} = \frac{\int_1^2 \left(\frac{\partial i}{\partial p}\right)_T dp + \int_1^2 \left(\frac{\partial i}{\partial T}\right)_p dT + \frac{v_2^2}{2} - \frac{v_1^2}{2} + gz_2 - gz_1}{\int_1^2 \left(\frac{\partial i}{\partial p}\right)_s dp + \frac{v_2^2}{2} - \frac{v_1^2}{2} + gz_2 - gz_1}. \qquad (12.13,5)$$

Similarly for pumps, η_{pu} is the reciprocal of (12.13,5). The quantity $\left(\frac{\partial i}{\partial T}\right)_p$ is c_p (see equation 9.2,11). It can also be shown that

$$\left(\frac{\partial i}{\partial p}\right)_s = 1/\rho \quad \text{and} \quad \left(\frac{\partial i}{\partial p}\right)_T = \left[1/\rho - T\left(\frac{\partial 1/\rho}{\partial T}\right)_p\right].$$

Average values of these properties between chosen values of pressure have been tabulated. It is possible to evaluate η_{tu} or η_{pu} either by computer[2] or by direct reference to enthalpy tables while using 12.13,1 with 12.13,4 or 5. For turbines it is expedient to use, in 12.13,4, the published average values of the properties. Accurate measurement of temperature difference has usually been made by resistance thermometer placed either in a pocket fixed in the main flow or in a specially made calorimeter through which a small sampling flow is extracted. Details of arrangements and of the precautions which must be used are given in the references. Thermodynamic methods are not suitable when the difference of head across the machine is less than 30 m.

EXERCISES. CHAPTER 12.

1. Show that the specific speed of a single jet Pelton turbine is $N_s = 494\,\phi\, d/D\, C_v^{1/2}\eta^{1/2}$, where ϕ is the ratio of the peripheral velocity of the buckets to the jet velocity, d/D is the ratio jet/wheel diameter, C_v is the coefficient of velocity in the jet and η is the efficiency. (Express the power in terms of ρg, d, C_v, H and η, and express N in terms of ϕ, D and H, where H is the total head in the penstock behind the jet.)
2. In the above example show that, if n jets of the same size are fitted, the specific speed is increased $\sqrt{n}$ times.
3. A Pelton wheel operates under a head of 274 m. The jet diameter is 100 mm the coefficient of velocity for the nozzle is 0·97, and the ratio bucket speed to jet speed is 0·47. The angle of deflection of the water in passing over the buckets is 160° and the relative velocity is reduced by 10% in the process. Calculate the diameter of the runner, the hydraulic efficiency, the power

developed at the buckets and the shaft power if the speed is 600 r.p.m. and the mechanical efficiency is 0·9.

(Answer 1·066 m; 0·92; 1,297 kW; 1,167 kW.)

4. A Pelton wheel is supplied with water from a reservoir 274 m above the nozzle, through a long pipe. The combined efficiency of the pipe and nozzle is 0·85 and the friction losses in the pipe amount to 19·8 m. If the bucket speed is 0·46 of the jet speed and if the water is deflected through 160° by the buckets and has its velocity reduced by 10% in the process, estimate the hydraulic efficiency of (a) the runner alone; (b) the turbine. If the mechanical efficiency of the turbine is 0·95, estimate (c) the overall efficiency of the turbine alone, based on head at inlet to the nozzle, and (d) the overall efficiency of pipe and turbine. (Answer 0·92; 0·84; 0·80; 0·74.)

5. Test results from a water turbine operating under a head of 21 m are as follows:

Unit speed	34·4	37·1	38·9	40·7	42·5	44·4	46·2
Unit power	50·3	51·1	51·3	51·3	50·6	49·5	48·0
Unit discharge	6·45	6·33	6·25	6·17	6·08	5·98	5·86

A geometrically similar turbine under a head of 10·5 m is to run at the same speed. Draw characteristic curves for the proposed turbine and use them to determine the discharge and power output at maximum efficiency. Ignore scale effects arising from any difference in Reynolds numbers.

Note. Test results of power P and discharge Q at speed N under head H are frequently plotted for unit head ($H = 1$); unit power, unit speed and unit quantity are then respectively $P_1 = P/H^{3/2}$, $N_1 = N/H^{1/2}$ and $Q_1 = Q/H^{1/2}$. (Answer 191 r.p.m.; 865 kW; 9·90 m^3/s.)

6. In the mixed flow reaction turbine detailed below, calculate the entrance head above the tailrace level, and the pressure head at the entrance to the draft tube:—guide blade angle 20°; head loss in guides 2·74 m; mean inlet diameter 0·85 m; velocity of flow at inlet and outlet, 11·22 and 9·14 m/s respectively; loss in runner 0·0678 of the head; entrance to draft tube 2·44 m above tailrace; loss in draft tube 0·014 of the head; velocity in tailrace 3·0 m/s; speed 582 r.p.m.; axial discharge from runner.

If the area at inlet to the runner is 0·5 m^2, and mechanical losses amount to 3·6 m of head, estimate the shaft horse power.

(Answer 92·2 m, atm — 4·96 m, 4,280 kW.)

7. A vertical shaft axial flow reaction turbine works under a head of 9·15 m. Water leaves the runner axially at a pressure of $-20{\cdot}71$ kN/m^2. The external and internal diameters of guides and runner are 1·83 m and 1·22 m respectively. The axial depth of the runner is 0·25 m and the area coefficient at inlet and outlet is 0·89 for both guide and runner passages. Find a suitable guide vane outlet angle, the speed of the turbine, the discharge, power, hydraulic efficiency and pressure head at inlet to runner.

Assume:—Runner blade angle, 90° at inlet; 24° at outlet. Loss in guide passages, 5% of the velocity head at exit from the guides. Loss in runner, 10% of the total head. Loss in draft tube friction and in tail water energy, 40% of the velocity head at outlet from the runner.

(Answer 24°; 109 r.p.m.; 5·03 m^3/s; 379 kW; 0·84; 2.40 m)

8. A model is to be built for studying the performance of a turbine having a runner diameter of 1 m and maximum output 2,240 kW under a head of 50 m. What is the appropriate diameter of the runner of the model if the

corresponding power is 10 kW and the available head is 8 m? Show that the corresponding model speeds will be about 50% greater than those of the turbine. (Answer 0·265 m.)

9. Estimate the horse power developed and the hydraulic efficiency of the following vertical shaft Francis turbine:—speed 180 r.p.m.; guides, elevation above tailrace 2·9 m, discharge angle 20°, velocity coefficient 0·97; runner, inlet diameter 3·0 m, mean outlet diameter 2·1 m, vane inlet angle 85°, vane outlet angle 25°, inlet flow area 2·92 m^2, outlet area 3·02 m^2, friction loss 19% of relative outlet velocity head; draught tube, inlet 2·4 m above tailrace, outlet area 11 m^2, loss 50% of the kinetic head at inlet. (Answer 28·2 MW; 0·88.)

10. In a projected low head hydro-electric scheme 283 m^3/s of water are available under a head of 3·66 m. Alternative schemes to use Francis turbines having a specific speed of 400 r.p.m. or propeller turbines with a specific speed of 686 r.p.m. are investigated. The normal running speed is to be 50 r.p.m. in both schemes. Compare the two proposals in so far as the numbers of machines are concerned and estimate the power to be developed by each machine. The units in either installation are to be of equal power and the efficiency of each type may be assumed to be 0·9. (Answer 6 Francis or 2 propeller turbines.)

11. A centrifugal pump, situated 4·0 m above sump level, lifts water to a tank 36·0 m above sump level. The suction and delivery pipes are 150 mm diameter and the heads lost are respectively 2 m and 7 m. The impeller is 400 mm in diameter and 32 mm wide at the exit; the blade outlet angle is 39° 50′ and $N = 1200$ r.p.m. If the manometric efficiency is 0·82 and the gross efficiency is 0·71, what power would be required to drive the pump and what discharge would be expected? What pressure head would be indicated on gauges attached at the suction and delivery flanges of the pump? Neglect the effect of blade thickness and assume radial flow at inlet. (Answer 65·5 kW, 101·5 dm^3/s, −7·69 m, 39·00 m.)

12. A centrifugal air blower works under the following conditions: $N = 3{,}000$ r.p.m.; mass flow of air = 340 kg/min, compressed from 100 to 108·6 kN/m^2 abs.; manometric efficiency = 0·85; air enters radially; $f_2 = f_1 = 45$ m/s; inlet diameter = 0·6 outlet diameter; blade angle at outlet = 45°; mean specific volume of air = 817 dm^3/kg. Calculate the diameter of the impeller, the blade angle at inlet and the axial widths at inlet and outlet. (Answer 738 mm; 33°; 74 mm; 44 mm.)

13. A centrifugal pump delivers 1·6 m^3/s of water when $N = 300$ r.p.m. Water enters the impeller radially at 2·4 m/s and the radial component of the velocity is constant at this value. Calculate the effective head H, the shaft power required, and the hydraulic efficiency gH/w_2u_2. Assume that 45 per cent of the velocity head at outlet from the impeller is recovered as pressure head in the diffuser and that hydraulic losses in the runner are equivalent to 9 per cent of w_2u_2/g. Assume that mechanical losses are 15 per cent of w_2u_2/g, that $\theta_2 = 26°$, that velocity in delivery pipe is 2·4 m/s, and that impeller diameter is 1·0 m. (Answer 12·6 m, 312 kW, 0·73.)

14. A 3-stage air compressor is to be designed to deliver air to a mine ventilating system. The head required at the inlet is 272 mm of water. A preliminary design for the compressor was made and a model of this was constructed to a scale of 1/8. The model was 533 mm diameter and when run at a speed of 1,200 r.p.m. it delivered 2·0 m^3/s of air and required 2·09 kW to drive it.

The efficiency was 0·75. Predict the performance of the full scale compressor and estimate the power it would require.

(Answer 277 r.p.m., 236 m^3/s and 838 kW.)

15. Water approaches a moving cascade of blades with an absolute velocity of 70 m/s at an angle of 60° to the line of movement of the blades. The blades are moving in the plane of the cascade, perpendicular to the span of the blades. Water leaves the blades with an absolute velocity perpendicular to the direction of movement of the blades. If the relative velocity over the blades is reduced by 5 % during passage, calculate the forward blade velocity and the blade inlet and outlet angles for this velocity.

(Answer 11·75 m/s, 69° and 79°.)

16. A centrifugal blower delivers 11 m^3 of air per second against a head of 50 mm of water, when running at 512 r.p.m. Estimate the power required if the efficiency is 0·75. Estimate the discharge of a blower of similar design, 1·5 times the diameter when discharging against a head of 60 mm of water. State the operating speed and calculate the probable power necessary. $\rho_{\text{air}} = 1{\cdot}2$kg/m^3. (Answer 7·17 kW, 27·1 m^3/s, 374 r.p.m., 21·2 kW.)

17. The centre line of the shaft of a centrifugal pump is at the water level in the sump. If $\theta_2 = 30°$, $D_2 = 450$ mm, $f_2 = 3$ m/s and $N = 1{,}000$ r.p.m., calculate the total head expected. Take the pressure head at impeller outlet to be 18 m, volute losses to be $0{\cdot}5\, v_2^2/2g$ and assume radial entrance to impeller. (Answer 26·8 m.)

18. A centrifugal pump has an impeller of external and internal diameters 420 mm and 210 mm respectively. It is proportioned to give constant radial velocity and at outlet the vanes make an angle of 25° with the tangent to the periphery. The static lift (from sump to tank) is 18 m, friction losses in the inlet and delivery piping are respectively 0·8 and 4·0 m, the velocity in the inlet pipe is 5 m/s and in the delivery pipe is 3 m/s. Manometric efficiency $= 0{\cdot}70$, $N = 1{,}150$ r.p.m., $Q = 0{\cdot}15$ m^3/s and it may be assumed that water enters the impeller radially. Determine impeller width at discharge, vane inlet angle, pressure head regained after outlet from the impeller if head loss in impeller is 6·0 m, and show that the pump will start delivering at the given speed. (Answer 20 mm; 23° 40′; 5·1 m; $(u_2^2 - u_1^2)/2g > 18$ m)

19. When running at 1,600 r.p.m. the characteristics of a centrifugal pump are as follows:

H m	38·4	39·6	39·9	39·0	36·9	33·2
η	0	0·64	0·74	0·78	0·80	0·79
Q m^3/s	0	2·8	5·6	8·5	11·3	14·2.

Plot the curves of head and efficiency expected for 1,300 r.p.m. If the impeller diameter is 0·3 m, give the hydraulic efficiency gH/u_2^2 for zero delivery. (Answer 0·60)

20. A centrifugal pump develops an effective head of 15 m; its diameter is 0·25 m and the delivery is 0·06 m^3/s. The impeller blades are directed backwards, making an angle of 30° with the tangent at outlet, and the effective width at outlet is 25 mm. Diffuser vanes are fitted, recovering 2·4 m of head. It may be assumed that hydraulic losses in the impeller amount to 1·8 m, that the radial velocity remains constant and that there is negligible velocity of whirl at inlet. Calculate the speed, the hydraulic efficiency, the best angle for the entrance to the diffuser blades and the head loss in the diffuser blades. Assume that the velocity in the suction and delivery pipes is 3·7 m/s.

(Answer 1,343 r.p.m.; 0·68; 13° 50′; 5·14 m,)

CHAPTER 13

PROPELLERS, FANS AND WINDMILLS

13.1 Introduction

The screw propeller is a device for obtaining a propulsive thrust from a rotary motion; the propeller itself is mounted on the vehicle (usually a ship or aircraft) and rotates in the fluid medium surrounding the vehicle. Suppose that the propeller rotates steadily making n revolutions in unit time while the constant speed of advance of the vehicle relative to the undisturbed fluid is V. Let the propulsive thrust provided by the screw in the direction of motion be T while the torque required to maintain the rotation of the screw is Q. Then the useful work done in unit time is TV while the energy supplied is $2\pi nQ$. Consequently the efficiency of the propeller (as a fraction) is

$$\eta = \frac{TV}{2\pi nQ} \tag{13.1,1}$$

where all the quantities are measured in consistent units. It will be noted that we have been careful to specify that the speed V is measured relative to the fluid where it is undisturbed by the propeller itself or by the vehicle. A ship's propeller is usually situated near the stern and therefore lies in the wake created by the hull. The rate of advance V_A of the propeller through the wake differs from V, in general. To avoid the complications introduced by the presence of a hull or other body it is usual and convenient to begin the theory of the propeller with the case where it is *isolated*; we may suppose the propeller to be at the forward end of a shaft which is so long that the presence of the body containing the driving mechanism has a negligible influence on the flow at the screw. The fluid is otherwise unbounded.

A fan is a screw propeller which is arranged to provide a current of fluid (usually air) in a duct or chamber and the centre of the fan is fixed relative to the walls. In many cases (as, for example, in return flow wind tunnels (see § 5.5)) the channel is closed and the fluid circulates continuously.

Lastly, a windmill is a screw propeller working in reverse so as to take energy from a natural current of air. The basic theory of an isolated windmill does not differ from that of an isolated propulsive screw except that the signs of both the torque and thrust are reversed.

13.2 Geometry of the Screw Propeller

The geometry of the screw propeller is based on the surface called the *helicoid*. In Fig. 13.2,1 the point A moves uniformly along the fixed axis

OZ while the plane containing OZ and the straight line AB rotates uniformly about OZ and the angle β is constant. Then the surface swept by the line AB is a helicoid; it is a *right helicoid* when AB is perpendicular to OZ. The line OZ is the axis of the helicoid. Any point on AB at a fixed distance from A describes a curve called a *helix* and this lies entirely on the helicoid. When the plane OAB makes one complete revolution about OZ let the point A advance the distance P_i along OZ. Then P_i is defined to be the *pitch* of the helicoid and of all the helices which lie upon it. A more general helicoidal surface is obtained when the straight generator AB is replaced by an arbitrary plane curve lying in the plane OAB and moving with it. Such a surface is entirely composed of helices of pitch P_i. A helicoid is called *right handed* when the helices lying in the surface are similar to a right handed corkscrew; otherwise it is left handed.

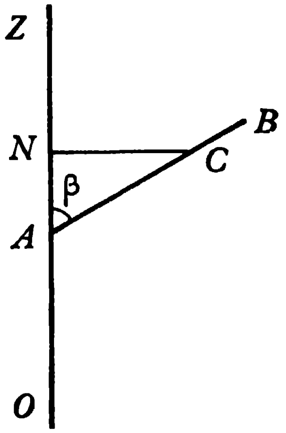

Fig. 13.2,1. Generation of the helicoid.

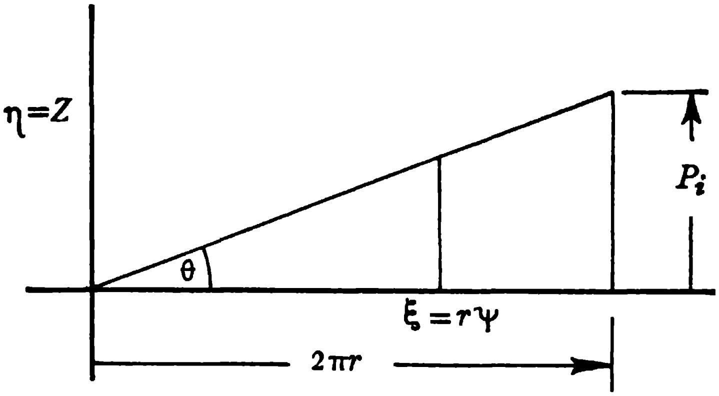

Fig. 13.2,2. Developed helix.

Consider the helix described by a point C on AB at the perpendicular distance $NC = r$ from OZ. This helix lies on the surface of a circular cylinder of radius r with OZ as axis. If this cylinder be supposed slit along one of its generators it can be flattened into a plane and the helix will then appear as a straight line (see Fig. 13.2,2). Thus, let distances measured circumferentially round the cylinder be denoted by ξ while axial distances are denoted by $\eta = z$. Then

$$\xi = r\psi \tag{13.2,1}$$

$$\eta = \frac{P_i \psi}{2\pi} \tag{13.2,2}$$

where ψ is the angle rotated by NC about OZ. Hence

$$\eta = \left(\frac{P_i}{2\pi r}\right)\xi \tag{13.2,3}$$

and the angle θ at which the helix cuts the normal circular sections of the cylinder is therefore given by

$$\tan\theta = \frac{P_i}{2\pi r}\,. \tag{13.2,4}$$

The helicoid is generated by the motion of a straight line and it is thus an example of a *ruled surface*. However, the generators do not intersect and it follows that the surface cannot be developed into a plane, i.e. *the helicoid is not a developable surface*. However, any small region of the surface may be supposed developed into a plane with only very slight distortion.

It follows from the definition that if the helicoid be slid without rotation through any distance z along OZ and then rotated about OZ through the angle $2\pi z/P_i$ radians in the appropriate sense, it will coincide with its initial situation. Hence the helicoid will remain constantly in complete contact with an equal fixed helicoid when any constant axial velocity V is combined with the constant angular velocity

$$\omega = \frac{2\pi V}{P_i} \tag{13.2,5}$$

about the axis of the surface. This is a screw motion.

A screw propeller consists of a number B of *equal* blades which are *evenly spaced* on a circular disk (see Fig. 13.2,3) and the normal to this disk through its centre is the axis of the propeller. The blades are mounted at their roots on the propeller boss or hub which is carried by the driving shaft. In the simplest type of propeller one face (called the driving face) of each blade is a true helicoidal surface generated by a straight line as shown in Fig. 13.2,1. When the angle β is other than a right angle the blade is said to be *raked*. The diameter D of the propeller is defined as the diameter of the *propeller disk* which is the circle through the tips of the blades. The pitch P_i of the propeller is that of the driving faces of the blades and the pitch-diameter ratio (often called simply the pitch ratio) is

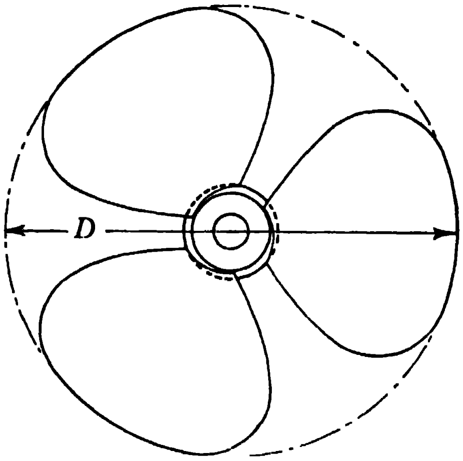

Fig. 13.2,3. Screw propeller.

$$\varpi = \frac{P_i}{D}. \tag{13.2,6}$$

For geometrically similar propellers this ratio is constant.

Imagine the propeller to be cut by a circular cylindrical surface of radius r whose axis is that of the propeller. This cylinder will cut each blade in the same closed section which may be developed into a plane as already explained; for the simplest type of propeller the section of the driving face will develop into a straight line. The length of the section is the *blade width* or *blade chord* c at the radius r. The *solidity* σ of the propeller at the radius r is defined to be the ratio of the sum of the blade chords at this radius to the circumference of a circle of radius r. The *surface area* of the screw is defined as the sum of the true areas of the driving faces between boss and tip. The form of the back of the blade will be determined by the distribution of thickness along the chord of the section. For marine propellers the developed form of the blade section is often a segment of a circle. However, for airscrews (and for some marine screws) the blade section takes the form of an aerofoil profile. This profile is defined by the ordinates of its upper and lower surfaces which thus determine the shapes of the back and driving face respectively. The ordinates are measured from the straight 'chord line' and the blade angle θ (see Fig. 13.2,2) is equal to the inclination of the chord line to the propeller disk at the radius considered. The driving face of a propeller of this kind may not be a true helicoidal surface, but the geometric pitch P_i at the radius r is obtained from the blade angle by use of equation (13.2,4). Sometimes the pitch varies along the blade from root to tip. Reference will be made in § 13.9 to the 'experimental mean pitch' of a propeller.

Example 1. *Equations of a helix*

Let OZ be the axis of the helix, r the radius and P_i the pitch. Then we can express the coordinates of a point on the helix by the parametric equations

$$x = r\cos\psi, \qquad y = r\sin\psi, \qquad z = \frac{P_i\psi}{2\pi}.$$

When ψ is eliminated we get the pair of equations

$$x^2 + y^2 = r^2$$

$$z = \frac{P_i}{2\pi}\tan^{-1}\left(\frac{y}{x}\right).$$

Example 2. *Equations of right and raked helicoids*

A point on the surface can be identified by the parameters r, ψ. For the raked helicoid (see Fig. 13.2,1)

$$x = r\cos\psi, \qquad y = r\sin\psi, \qquad z = r\cot\beta + \frac{P_i\psi}{2\pi}.$$

Hence

$$z = \sqrt{(x^2 + y^2)}\cot\beta + \frac{P_i}{2\pi}\tan^{-1}\left(\frac{y}{x}\right).$$

For the right helicoid $\beta = \pi/2$ and $\cot\beta = 0$. Hence the equation is

$$z = \frac{P_i}{2\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

Example 3. Equation of general helicoidal surface

Let the generating curve be

$$z = f(r).$$

Then $\quad x = r \cos \psi, \quad y = r \sin \psi, \quad z = f(r) + \dfrac{P_i \psi}{2\pi}$

Hence the equation to the surface is

$$z = f[\sqrt{(x^2 + y^2)}] + \frac{P_i}{2\pi} \tan^{-1}\left(\frac{y}{x}\right).$$

Example 4. Area of helicoidal surface

Let dS be an element of surface area and dS' the area of its projection on the plane OXY. Then we have the general formula

$$\frac{dS}{dS'} = \sqrt{\left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]}.$$

Take the equation to the general helicoidal surface as in Example 3 and for convenience put $p = P_i/2\pi$. Then we obtain

$$\frac{\partial z}{\partial x} = \frac{xrf'(r) - py}{r^2}, \qquad \frac{\partial z}{\partial y} = \frac{yrf'(r) + px}{r^2}$$

and
$$\frac{dS}{dS'} = \frac{1}{r}\sqrt{[r^2(1 + f'(r)^2) + p^2]}.$$

It is convenient to take dS' as a polar element defined by dr and $d\psi$ so that

$$dS' = r\, dr\, d\psi.$$

Then
$$dS = \sqrt{[r^2(1 + f'(r)^2) + p^2]}\, dr\, d\psi$$

where the expression under the radical is a function of r alone.

For the raked helicoid $f'(r) = \cot \beta$ and we get

$$dS = \sqrt{[r^2 \operatorname{cosec}^2 \beta + p^2]}\, dr\, d\psi.$$

Now let ds be an element of arc of the helix $r =$ constant on the surface and dw an element of length of the straight generator AB (Fig. 13.2,2). Then (see Fig. 13.2,2)

$$ds = r \sec \theta\, d\psi = \sqrt{(r^2 + p^2)}\, d\psi$$

and
$$dw = dr \operatorname{cosec} \beta.$$

Consequently we obtain

$$dS = \sqrt{\left(\frac{r^2 + p^2 \sin^2 \beta}{r^2 + p^2}\right)}\, ds\, dw.$$

For the right helicoid $\sin \beta = 1$ and $dS = ds\, dw = ds\, dr$. Unless β differs largely from a right angle the last equation can be used with only trifling error.

Let c be the blade chord measured on the helix $r =$ constant. Then for a right helicoid we have

$$\text{area of one blade} = \int_{R_0}^{R} c\,dr$$

where R, R_0 are the radii at the tip and root respectively. But $Bc = 2\pi r\sigma$ where σ is the solidity. Hence

$$\text{total blade area} = 2\pi \int_{R_0}^{R} \sigma r\,dr.$$

13.3 Slip and Advance Ratio

Suppose that the screw advances through the fluid with a velocity V parallel to the axis of rotation while rotating at the rate of n revolutions in unit time. Now, if the screw were engaged with a fixed 'nut' of corresponding form it would advance a distance equal to the pitch P_i in each revolution and the rate of advance would be

$$V' = nP_i.$$

The excess of V' over V is called the *slip* and the ratio of the slip to V' is called the *slip ratio* s. Thus the slip ratio is given by

$$s = \frac{V' - V}{V'} = 1 - \frac{V}{nP_i} \tag{13.3,1}$$

and this is often quoted as a percentage. By equation (13.2,6)

$$s = 1 - \frac{J}{\varpi} \tag{13.3,2}$$

where

$$J = \frac{V}{nD} \tag{13.3,3}$$

and is called the *advance ratio*. It will be noted that s and J are non-dimensional quantities. For similar screw propellers ϖ is constant and equation (13.3,2) shows that the slip ratio is then determined by J alone.

It is usual in aeronautics to plot the non-dimensional performance coefficients of propellers (see § 13.4) on a base of J. When the Reynolds and Mach numbers are fixed the performance coefficients of a family of similar propellers become functions of J alone.

13.4 Propeller Coefficients

The propeller coefficients which will be explained are non-dimensional and can easily be derived by the usual procedure of dimensional analysis (see Chapter 4). Here we shall consider a family of isolated propellers (see § 13.1) which are exactly similar geometrically and shall assume that the conditions for the similarity of flow are all satisfied. This implies that the

Reynolds number (VD/ν) and tip Mach numbers are kept constant. However, for marine propellers the Mach number is always so small that it can be regarded as constantly zero and need not be further considered. The dependence of the non-dimensional performance coefficients on the Reynolds number is called *scale effect.*

Provided that J is constant, *corresponding* velocities and velocity components for all propellers of the similar family will be proportional to V, for corresponding velocities due to rotation are proportional to $nD = V/J$ by equation (13.3,3). Hence corresponding pressures will be proportional to ρV^2 while corresponding forces will be porportional to ρV^2D^2. In particular, the thrust T will be proportional to ρV^2D^2 and the non-dimensional thrust coefficient accordingly is

$$T_c = \frac{T}{\rho V^2 D^2}. \tag{13.4,1}$$

But $V = JnD$ so the last equation yields

$$J^2T_c = \frac{T}{\rho n^2 D^4} = C_T = k_T \tag{13.4,2}$$

which is another non-dimensional thrust coefficient. When all the conditions for similarity are satisfied (including constancy of J) corresponding moments will be proportional to ρV^2D^3. Hence we derive the non-dimensional torque coefficient

$$Q_c = \frac{Q}{\rho V^2 D^3} \tag{13.4,3}$$

while

$$k_Q = J^2Q_c = \frac{Q}{\rho n^2 D^5} \tag{13.4,4}$$

is also non-dimensional. As an alternative to these torque coefficients we may adopt the power coefficient

$$C_P = \frac{P}{\rho n^3 D^5} \tag{13.4,5}$$

where P is the power absorbed (measured in consistent units). Since

$$P = 2\pi nQ \tag{13.4,6}$$

it follows from (13.4,4) and (13.4,5) that

$$C_P = 2\pi k_Q. \tag{13.4,7}$$

Also the efficiency of the screw is

$$\begin{aligned} \eta &= \frac{TV}{P} = \frac{\rho n^2 D^4 C_T V}{\rho n^3 D^5 C_P} = \frac{C_T}{C_P}\left(\frac{V}{nD}\right) \\ &= \frac{JC_T}{C_P} = \frac{J}{2\pi}\left(\frac{k_T}{k_Q}\right). \end{aligned} \tag{13.4,8}$$

Thus the efficiency can be calculated at once when the values of C_T and C_P corresponding to the selected value of the advance ratio J are known.

The power coefficient C_P and J both depend on the diameter D. However

$$\frac{J^5}{C_P} = \frac{\rho V^5}{n^2 P} \tag{13.4,9}$$

is independent of D and it is non-dimensional, being a combination of C_P and J. The fifth root of this quantity is used in aeronautics under the name of the *speed-power coefficient*

$$C_s = \sqrt[5]{\left(\frac{\rho V^5}{n^2 P}\right)}. \tag{13.4,10}$$

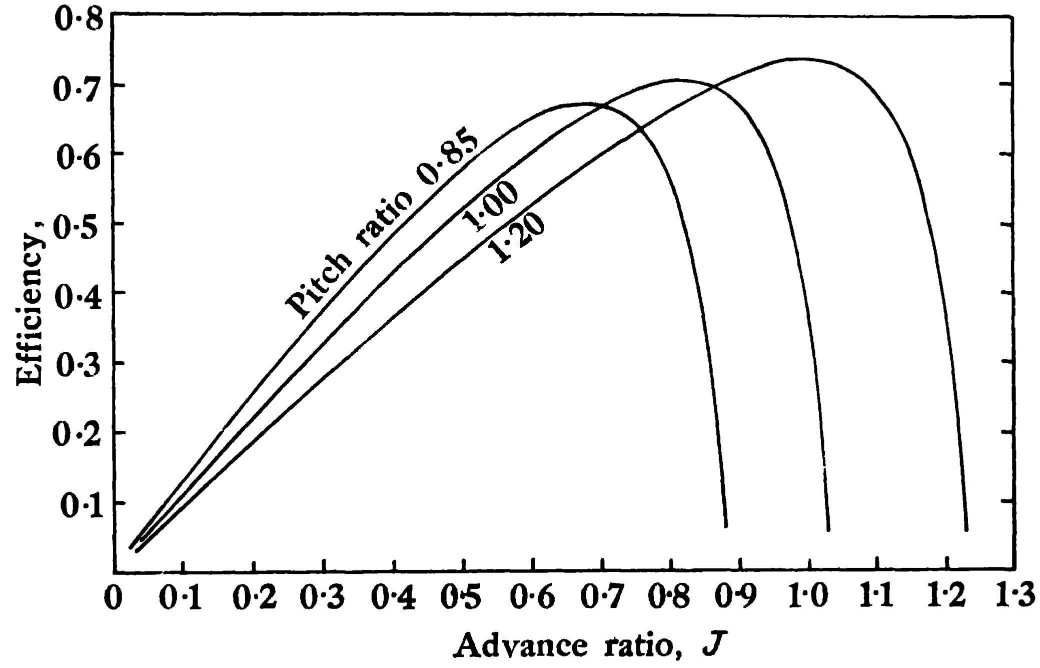

Fig. 13.4,1. Typical efficiency curves for a family of screw propellers.

When the relation between C_P and J is known the relation between C_s and J can be found by using (13.4,9). Now C_s can be calculated from the data of a design and when this is known the value of J becomes determined. (We assume here that the pitch ratio and other shape parameters have been selected.) Then the required diameter D can be found from J and the other data, thus

$$D = \frac{V}{nJ}. \tag{13.4,11}$$

Evidently the speed-power coefficient is of great utility in propeller design (see further § 13.10).

Fig. 13.4,1 shows some typical curves of propeller efficiency plotted on a base of advance ratio for several values of the pitch-diameter ratio; the other geometrical particulars are the same for all the propellers. It will be noted that the efficiency rises gradually with J until the maximum is reached and thereafter falls rather rapidly. The value of J for maximum efficiency increases as the pitch-diameter ratio rises.

13.5 Theory of the Actuator Disk

A screw propeller develops a propulsive thrust by imparting rearward velocity to the fluid which passes through the propeller disk; the thrust is, in fact, equal to the rearward momentum communicated to the fluid in unit time. The fluid coming from the propeller is called the *slipstream* and it possesses kinetic energy since it has a rearward velocity. Hence the efficiency of the propeller is inevitably less than 100 per cent, for the lost energy in the slipstream is provided by the propeller. For a real propeller operating in a viscous fluid there are other sources of energy wastage which reduce the efficiency still further. Since the torque is balanced by the angular momentum communicated to the fluid in unit time, the fluid in the slipstream must have a rotary motion, implying a further wastage of energy in the kinetic form.† On account of viscosity the blades of the propeller are subject to an additional resistance (profile drag) while local velocity perturbations near the blade tips give rise to further losses.

The complete investigation of propeller efficiency involves a complicated analysis but it is instructive to begin the theory with a conceptual scheme of the utmost simplicity; the efficiency so obtained will be an upper bound to the attainable efficiency and is related to actual propeller efficiencies in the same kind of way as the efficiency in a Carnot cycle is related to the actual efficiency of a heat engine. The first step in the simplification of the problem consists in replacing the actual propeller with its small number of blades by another of the same diameter having a very large number of very narrow equal blades, the solidity (see § 13.2) at any radius being the same as for the actual propeller. When this multibladed propeller develops a positive thrust the pressure in the fluid behind the propeller disk will be higher than the pressure just in front of the disk and we may regard the propeller as a mechanism which produces this pressure increment. Such a mechanism has been called an *actuator* and the theory of the ideal actuator is based on the following assumptions:

(1) The pressure increment or thrust per unit area is constant over the disk.
(2) The rotational component of the velocity in the slipstream is zero. Thus the actuator is an ideal mechanism which imparts momentum to the fluid in the axial direction only.
(3) The fluid is inviscid and incompressible.
(4) There is continuity of velocity through the disk.

In developing the theory it is convenient to suppose the actuator to be brought to rest by superposing on the whole system of actuator and fluid a steady and uniform velocity equal and opposite to the velocity of advance. We then have a stationary actuator situated in a stream of fluid which is uniform except where it is disturbed by the actuator. The pressures and forces are uninfluenced by the superposed velocity in accordance with the Principle of Relativity.

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† This loss can be eliminated by the use of a pair of coaxial propellers rotating in opposite directions (Contra-rotating pair).

A diagram of the actuator and the flow is given in Fig. 13.5,1 which also shows the velocities and pressures. The streamlines shown pass through the edge of the disk and everything is completely symmetric about the axis of the disk. The flow far upstream from the disk is uniform, the velocity being V in the axial direction and the pressure has the uniform value p_0 (we suppose for simplicity that body forces are absent). Far downstream from the disk the velocity in the slipstream is again axial and uniform and has the value $V + v_1$. Since the streamlines here are straight there can be no pressure gradient normal to them, so the pressure is again p_0. Just at the actuator the axial component of velocity, which is continuous at the disk, is $V + v$, where v is to be determined; there is also a small lateral or radial component of velocity u_r which depends on the radial distance from the axis. The pressure just in front of the disk is p and that just behind is $(p + p')$, where p' is the constant thrust per unit area. First, apply Bernoulli's theorem to a streamline in the flow approaching the disk

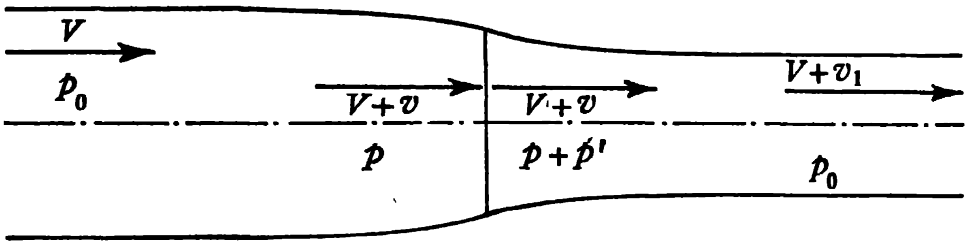

Fig. 13.5,1. Diagram of actuator disk.

$$p + \tfrac{1}{2}\rho[(V + v)^2 + u_r^2] = p_0 + \tfrac{1}{2}\rho V^2. \qquad (13.5,1)$$

Second, apply the theorem to the streamline as it leaves the disk

$$p + p' + \tfrac{1}{2}\rho[(V + v)^2 + u_r^2] = p_0 + \tfrac{1}{2}\rho(V + v_1)^2. \qquad (13.5,2)$$

Hence, by difference,

$$p' = \rho v_1(V + \tfrac{1}{2}v_1), \qquad (13.5,3)$$

and the thrust is

$$T = p'A = \rho v_1(V + \tfrac{1}{2}v_1)A \qquad (13.5,4)$$

where A is the area of the actuator disk. Now the axial thrust on the tubular surface generated by the streamlines passing through the edge of the disk is zero.† Hence the thrust is equal to the difference between the axial momentum passing per unit time in the far distant slipstream and that in the approaching stream. Since the motion is steady, the mass passing in unit time is everywhere the same as at the disk, i.e. $\rho(V + v)A$. Therefore

$$T = \rho v_1(V + v)A \qquad (13.5,5)$$

---

† This follows from the fact that the drag (in a perfect fluid) of a body whose external form is that of the tubular surface is zero. When such a body moves uniformly through the fluid the kinetic energy of the fluid is constant, so the drag must be zero.

since the total gain of axial velocity is v_1. On comparison of (13.5,4) with (13.5,5) we see that

$$v = \tfrac{1}{2}v_1. \tag{13.5,6}$$

Thus the induced axial velocity of inflow at the actuator is just half the total gain of axial velocity. We have here established a fact of fundamental importance in the theory of the propeller, namely, that the velocity at the propeller itself differs from the velocity of advance. It will be shown later that for a real propeller, which imparts rotary as well as axial momentum to the fluid, there is also an induced rotary or circumferential velocity at the propeller.

By equations (13.5,5) and (13.5,6) the thrust is given by

$$T = 2\rho v(V + v)A \tag{13.5,7}$$

$$= 2a(1 + a)\rho V^2 A \tag{13.5,8}$$

where

$$v = aV \tag{13.5,9}$$

and a is called the *axial inflow factor* and is non-dimensional. When we substitute for A in terms of the diameter D of the disk we obtain

$$T_c = \frac{T}{\rho V^2 D^2} = \frac{\pi}{2}a(1 + a). \tag{13.5,10}$$

The last equation is a quadratic in a and can be solved to yield

$$a = \frac{1}{2}\left[\sqrt{\left(1 + \frac{8T_c}{\pi}\right)} - 1\right] \tag{13.5,11}$$

where the radical must be taken with the positive sign since a and T_c vanish together. Next, we must find the efficiency η of the actuator. The kinetic energy passing in the slipstream and wasted in unit time is

$$E_w = \tfrac{1}{2}v_1^2 \times \rho(V + v)A$$

$$= \tfrac{1}{2}\rho v_1^2(V + \tfrac{1}{2}v_1)A$$

while the useful work done in unit time is

$$E_u = TV = \rho v_1(V + \tfrac{1}{2}v_1)VA.$$

Hence the efficiency is

$$\eta = \frac{E_u}{E_u + E_w} = \frac{V}{V + \frac{1}{2}v_1} = \frac{V}{V + v} = \frac{1}{1 + a}. \tag{13.5,12}$$

It will be seen that the efficiency of the actuator approaches unity as a tends to zero, but the thrust then also tends to zero. For a real propeller, subject to viscous drag, the efficiency tends to zero as a tends to zero and the highest efficiency occurs when a is small but not zero.

Example. Relation between power input to the actuator and efficiency

We have

$$\eta P = TV = \tfrac{1}{2}\pi a(1 + a)\rho V^3 D^2$$

while

$$1 + a = \frac{1}{\eta} \quad \text{and} \quad a = \frac{1-\eta}{\eta}.$$

Hence

$$\frac{P}{\rho V^3 D^2} = \frac{\pi}{2}\frac{1-\eta}{\eta^3}.$$

For example, when $\eta = 0{\cdot}9$ the non-dimensional power coefficient has the value 0·215.

13.6 Blade Element Theory of the Propeller

In the blade element theory of the propeller we consider in detail the forces on a small section of blade lying between the radii r and $r + dr$ and then derive the resultant thrust and torque for the propeller by integration. In reckoning the velocity of the element relative to the fluid we take account of the velocity of advance V of the propeller as a whole and the rotary component $r\Omega$ of the velocity of the element, where

$$\Omega = 2\pi n \tag{13.6,1}$$

is the angular velocity of the propeller.† We saw in § 13.5 that there is an axial inflow at the propeller disk, so the resultant axial velocity is $V(1 + a)$, where a is the axial inflow factor. Similarly, there is a rotary inflow which, however, is negative as will be demonstrated later. Hence the effective rotary or circumferential component of the relative velocity of the element is $(1 - a')r\Omega$, where a' is the rotational inflow factor. In the meantime we assume the existence of these inflow factors and consider their evaluation later. The blade element is treated as a section of an aerofoil‡ of infinite span whose velocity relative to the fluid agrees with that of the element in magnitude and direction. By resolution of forces, the contributions of the element to the thrust and torque are obtained from the lift and drag.

A diagram showing the developed blade section at radius r and the velocities is given in Fig. 13.6,1. The chord line is inclined at the blade angle θ to the propeller disk and the angle of incidence, measured from the chord line, is

$$\alpha = \theta - \phi \tag{13.6,2}$$

where

$$\tan\phi = \frac{(1 + a)V}{(1 - a')r\Omega} \tag{13.6,3}$$

$$= \frac{(1 + a)}{\pi(1 - a')}\left(\frac{R}{r}\right)J \tag{13.6,4}$$

and R is the tip radius while J is the advance ratio given by equation (13.3,3).

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† In the present theory V and Ω are constant.

‡ This is legitimate even when the blade section is not what is commonly called an aerofoil, provided that appropriate values of the lift and drag coefficients are used.

It will be seen that ϕ (and then α) can be calculated when J and the values of the inflow factors are given. Then the lift and drag coefficients (C_L, C_D) appropriate to the incidence α can be obtained from graphs of blade section characteristics in two-dimensional flow (infinite aspect ratio). Since the area of the blade element is $c\,dr$, we have

$$\text{lift on element} = \tfrac{1}{2}\rho W^2 c C_L\,dr$$

$$\text{drag on element} = \tfrac{1}{2}\rho W^2 c C_D\,dr$$

where W is the resultant relative velocity given by

$$W^2 = V^2(1+a)^2 + r^2\Omega^2(1-a')^2. \tag{13.6,5}$$

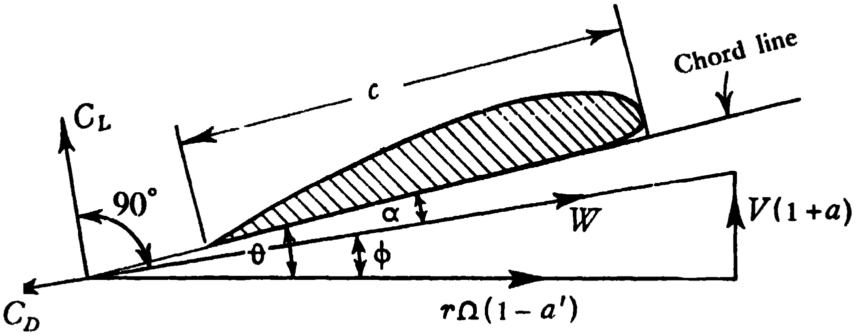

Fig. 13.6,1. Developed blade section, with components of relative velocity.

Let the thrust and torque on the element be dT' and dQ' respectively. Then it is clear from the diagram that

$$\frac{dT'}{dr} = \tfrac{1}{2}\rho W^2 c(C_L\cos\phi - C_D\sin\phi) \tag{13.6,6}$$

$$\frac{dQ'}{dr} = \tfrac{1}{2}r\rho W^2 c(C_L\sin\phi + C_D\cos\phi). \tag{13.6,7}$$

The total thrust and torque can then be obtained by multiplication by B and integration with respect to r over the range R_0 to R. The efficiency of the blade element is

$$\begin{aligned}\eta' &= \frac{V\,dT'}{2\pi n\,dQ'} = \frac{V}{\Omega}\frac{dT'}{dQ'}\\ &= \left(\frac{V}{r\Omega}\right)\frac{C_L\cos\phi - C_D\sin\phi}{C_L\sin\phi + C_D\cos\phi}\\ &= \left(\frac{1-a'}{1+a}\right)\tan\phi\left(\frac{C_L\cos\phi - C_D\sin\phi}{C_L\sin\phi + C_D\cos\phi}\right)\\ &= \left(\frac{1-a'}{1+a}\right)\frac{\tan\phi}{\tan(\phi+\gamma)}\end{aligned} \tag{13.6,8}$$

where
$$\tan\gamma = \frac{C_D}{C_L}, \tag{13.6,9}$$

so γ is the 'glide angle' of the blade profile when the angle of incidence is α.

Example. Non-dimensional expressions for thrust and torque gradings.

Let T be the total thrust. Then the *thrust grading* is

$$\frac{dT}{dr} = B\frac{dT'}{dr} = \tfrac{1}{2}\rho W^2 Bc(C_L \cos\phi - C_D \sin\phi).$$

But $Bc = 2\pi r\sigma$, where σ is the solidity,

and $$W^2 = r^2\Omega^2(1-a')^2 \sec^2\phi = 4\pi^2 r^2 n^2 (1-a')^2 \sec^2\phi.$$

Hence $$\frac{dT}{dr} = 4\pi^3\sigma\rho r^3 n^2(1-a')^2(C_L\cos\phi - C_D\sin\phi)\sec^2\phi.$$

Now k_T is given by equation (13.4,2) and the non-dimensional *thrust grading* is

$$\frac{dk_T}{d\left(\dfrac{r}{R}\right)} = R\frac{dk_T}{dr} = \frac{R}{\rho n^2 D^4}\frac{dT}{dr} = \frac{1}{16\rho n^2 R^3}\frac{dT}{dr}$$

$$= \frac{\pi^3}{4}\sigma\left(\frac{r}{R}\right)^3(1-a')^2(C_L\cos\phi - C_D\sin\phi)\sec^2\phi.$$

Similarly the non-dimensional *torque grading* is

$$\frac{dk_Q}{d\left(\dfrac{r}{R}\right)} = \frac{\pi^3}{8}\sigma\left(\frac{r}{R}\right)^4(1-a')^2(C_L\sin\phi + C_D\cos\phi)\sec^2\phi$$

where k_Q is given by equation (13.4,4).

13.7 Vortex Theory of the Propeller

The vortex theory of the screw propeller provides one method of obtaining the values of the axial and rotational inflow factors which have already been introduced in the blade element theory (see § 13.6). We may, in anticipation, summarise the results of this theory as follows:

(1) The induced flow at any radius r depends only on the thrust and torque gradings at this radius. In other words, the blade elements at differing radii are hydrodynamically independent.

(2) The induced axial inflow at a blade element is just half the axial velocity in the distant part of the slipstream which comes from the elements at that radius. This is in agreement with the findings of the simple actuator theory (see § 13.5).

(3) The induced circumferential or rotational velocity of inflow at a blade element is just half the circumferential velocity in the distant part of the slipstream which comes from the elements at that radius.

The axial and circumferential components of the velocity in the distant slipstream can be found from considerations of momentum and then the components of the velocity of inflow at the element are just half of these (see below).†

The vortex theory is not a completely general or exact theory and is based on the following simplifying assumptions:

(1) The propeller is treated as possessing a very large number of very narrow, equal and evenly spaced blades, the total blade chord (or solidity) at any radius being the same as for the actual propeller.

(2) The vortices which trail from a blade element are treated as having the form of an exact helix whose pitch is equal to the advance of the screw per revolution.

The assumptions of the vortex theory become more and more nearly in accordance with fact as the pitch ratio and solidity fall while at the same time the number of blades increases and the value of the non-dimensional thrust coefficient falls. When this coefficient is large the contraction of the diameter of the slipstream downstream from the propeller disk (as a fraction of the disk radius) increases and it becomes increasingly erroneous to treat the vortices as helices of constant radius. Since each blade of the propeller is treated as having, at every radius, a very small chord, it is assumed that the forces on an element of one blade and of radial length dr are the same as if this element were situated in a uniform two-dimensional stream and that the lift force per unit length depends on the local circulation and effective velocity as in the Kutta-Joukowsky relation (see § 11.3). The effective velocity at the element is the resultant of the velocity of the element relative to the undisturbed fluid and of the velocity induced by the vortices which trail from the whole blade system; this induced velocity is, for convenience, resolved into axial and circumferential components which are the velocities of inflow, sometimes called the interference velocities.

In approaching the vortex theory of the propeller it is helpful to revert for a moment to the 'lifting-line theory' of the straight finite and symmetric aerofoil with it span normal to the stream (see § 11.6.2). The basic fact is that the local value of the lift per unit span is given by

$$L' = \rho V \Gamma \tag{13.7,1}$$

where Γ is the local value of the circulation round the wing or the total local strength of the bound vortices. It is customary to think of Γ as being built up by the superposition of 'horse-shoe vortices' each of which has a finite spanwise portion of length $2y$ and a pair of infinite straight legs extending downstream from each end of this; these infinite legs are the 'trailing vortices.' However, we may alternatively consider the vortex system to be built-up as follows: At the section of the wing situated at the distance y

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† A simplified approximate theory, based on the fact that the induced velocity is nearly perpendicular to the blade velocity, is given in the Example at the end of this section.

from the plane of symmetry take an element dy. Then we may take a horse-shoe vortex of strength Γ (equal to the local value of the circulation at the wing section considered) whose plan is the element dy with a pair of infinite straight legs extending downstream from the ends of the element. At an adjacent element the circulation will be $\Gamma + d\Gamma/dy\, dy$ and the net strength of the vortex trailing from the outboard end of the first element considered, as obtained by superposition, will be

$$\Gamma - \left(\Gamma + \frac{d\Gamma}{dy}dy\right) = -\frac{d\Gamma}{dy}dy. \qquad (13.7,2)$$

This is shown in Fig. 13.7,1 where, for clearness, the adjacent elements of length dy are shown slightly separated. Now, in the case of the propeller, let the circulation round one blade at the radial distance r from the axis be Γ. Then we may regard the basic vortex, which corresponds to the elementary horse-shoe vortex of the straight aerofoil, to consist of a radial element dr from which helical vortices trail downstream, these helices having a pitch equal to the advance of the propeller per revolution, while the strength of the vortex is Γ. The direction of spin is always the same if we follow the vortex round, but this implies that the spins are opposite handed in the inner and outer helical trailing vortices. The complete vortex system for all the blade elements at the radius r and of radial length dr consists of B such vortices spaced evenly round the circle of radius r in the propeller disk. Hence these trailing vortices form two solenoids, an outer one of radius $r + dr$ and an inner one of radius r (see Fig. 13.7,2). It will be shown that when the number B is very large, the velocity induced by the solenoids and the radial elements joining them at their upstream ends is zero except in the region between the solenoids.

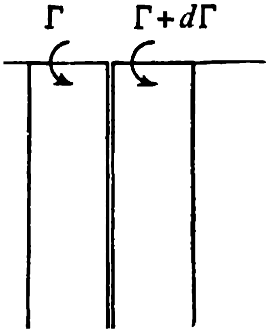

Fig. 13.7,1. Vortices trailing from a wing.

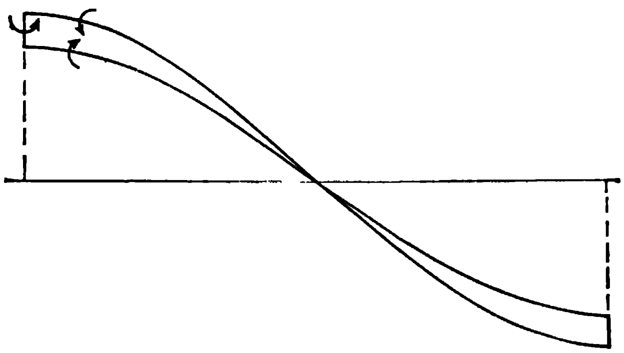

Fig. 13.7,2. Helical vortices springing from blade element.

Let us begin by considering a single solenoid of vortices, which are all of the same strength, since we assume that all the blade elements at a given radius are in all respects identical. We shall show that the solenoid is equivalent to a uniform distribution of ring vortices together with a uniform distribution of straight vortices, both lying on the surface of the circular cylinder upon which the solenoid lies. It is known (see § 2.13) that we may replace any very short straight element of a vortex by a bent vortex element having the same end points. In Fig. 13.7,3 parts of three adjacent vortices of the solenoid are shown by the dotted lines and each of these is supposed to be replaced by a stepped line, the faces of the steps being in the axial and circumferential directions. It will be seen that the circumferential steps combine into complete circular rings while the axial steps combine to form straight lines parallel to the axis of the cylinder. We shall consider separately the flows induced by these sets of vortices and first let us take the rings which we suppose very closely spaced with a total vortex strength per unit axial length equal to γ (a constant). At a great distance from the open end of the solenoid (i.e. far downstream from the propeller) the induced velocity inside the cylinder will be uniform and equal to v_1, say, while the induced velocity outside the solenoid is zero. This can be proved by integration of the velocities induced by the vortex elements (see § 2.13) but much more simply as follows. Within the solenoid and far from its open end the induced velocity is the same as if the solenoid extended to infinity in both directions. For such an infinite solenoid there is nothing to distinguish one normal section from another; consequently the three components u, v, w of the induced velocity must be independent of the axial coordinate x. Now the induced flow is irrotational so we have (see § 2.11)

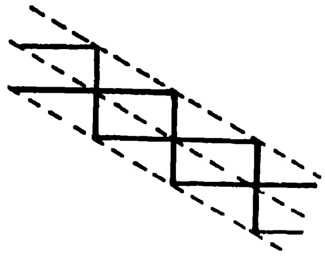

Fig. 13.7,3. Helical vortices replaced by axial and circumferential steps.

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \qquad \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \qquad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.$$

But we have just seen that $\partial v/\partial x$ and $\partial w/\partial x$ are zero, so the last two equations yield

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} = 0$$

and $\partial u/\partial x$ is also zero. Hence u is constant inside the solenoid. The velocities v and w are independent of x and, since $\partial u/\partial x$ is zero, constitute a two-dimensional flow in planes normal to the axis. But there is complete symmetry about the axis and the only two-dimensional irrotational 'flow' with axial symmetry is a state of rest,† i.e. $v = w = 0$. The same argument holds for the region outside the solenoid but here u is zero at an infinite radial distance from the axis, so u is everywhere zero and the fluid is at rest outside the solenoid. It only remains to find the constant value of u inside the solenoid. Consider the circulation in the rectangular circuit $ABCD$ of length dx (see Fig. 13.7,4) which embraces vortices of total strength $\gamma\, dx$.

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† If this were not so there would be a line source along the axis of the solenoid.

The flow from A to B is $u\,dx$ and it is zero for the other sides. Hence the circulation is $u\,dx$ and this must be equal to the total strength of the vortices embraced. Finally[†]

$$u = \gamma. \tag{13.7,3}$$

Next consider the state of affairs exactly at the plane of the end of a solenoid extending to infinity in one direction. This may be considered as one half of a solenoid extending to infinity in both directions and it can be seen that the axial components of the induced velocities at the plane of junction are the same for the two halves. Consequently the axial induced velocity over the circular end of the solenoid is $\frac{1}{2}\gamma$ while the axial induced velocity at greater radial distances from the axis is zero. Finally, when we combine the effects of semi-infinite solenoids of strength γ with radius $r + dr$ and of strength $-\gamma$ with radius r we get the following result by addition of the induced velocities:

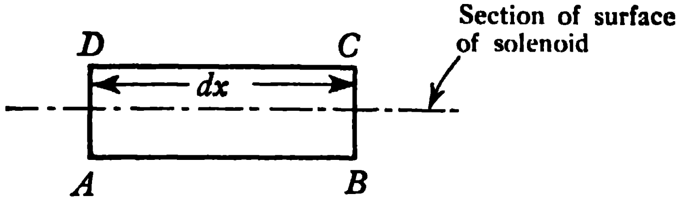

Fig. 13.7,4. Circulation in small rectangular circuit.

(a) In the region far distant from the open end the induced velocity is zero outside the outer solenoid and inside the inner solenoid while there is a uniform axial velocity $u = \gamma$ between the solenoids.

(b) On the plane of the end of the solenoids the axial induced velocity is zero except between the solenoids where it has the value $\frac{1}{2}\gamma$.

These results show that, so far as the influence of the ring vortices is concerned, the blade elements are independent.

We next consider the axial vortices and here we take the pair lying on the cylinders of radii $r + dr$ and r and on a plane through the axis, together with the radial element dr at the propeller disk, the whole forming an elementary horse-shoe vortex. The complete vortex system then consists of a set of such horse-shoe vortices, all of the same strength and evenly spaced around the axis. It follows from the symmetry of the arrangement that the induced velocity is everywhere circumferential, i.e. normal to the axis and to the perpendicular to the axis from the point considered. Also by symmetry the velocity is independent of angular position about the axis. Hence, if the induced velocity at distance r' from the axis is u_t there will be circulation in the circle of radius r' given by

$$\Gamma = 2\pi r' u_t. \tag{13.7,4}$$

Now the circulation in the circle is equal to the total strength of the vortices threading it (see § 2.11). When $r' < r$ the circulation will be zero since no vortices thread the circle and when $r' > r + dr$ the circulation will also be zero since the vortices on the cylinders of radii r and $r + dr$ give cancelling contributions to the vortex strength. Hence u_t is zero except in the annulus between the radii r and $r + dr$ and here its value is

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[†] If we take a uniform axial flow u inside a circular cylinder with the fluid at rest outside it is obvious that we have ring vortices whose strength per unit axial distance is u.

$$u_t = \frac{\Gamma_1}{2\pi r} \tag{13.7,5}$$

where Γ_1 is the total strength of the axial vortices on the cylinder of radius r. These results finally demonstrate that the blade elements are independent.

Immediately up-stream of the propeller disk u_t is zero, for no vortices thread a circle with its centre on the axis.† Now this velocity may be supposed analysed into two parts, namely, u_{t1} due to the radial vortices at the blade elements and u_{t2} due to the two systems of semi-infinite vortices on the cylinders of radii r and $r + dr$. Hence we have

$$u_{t1} = -u_{t2}$$

upstream of the propeller disk. Just downstream of the disk u_{t1} is reversed in sign but unchanged in magnitude while u_{t2} is unchanged.
Hence

$$u_t = 2u_{t2}. \tag{13.7,6}$$

Far downstream from the disk u_{t2} has just double the value at the disk (compare the discussion of the ring vortices above) while u_{t1} tends to zero. Hence the total value of u_t remains fixed (as we already know). Now the induced circumferential velocity at the blade elements is u_{t2}, being the mean of the values just before and just behind these elements. Accordingly the circumferential velocity of inflow is just half the circumferential velocity downstream.

We shall now find the axial and rotational inflow factors by combining the foregoing results of the vortex theory with relations deduced from momentum principles and we shall use the notation of § 13.6. Consider the annulus of the propeller disk of internal radius r and width dr. The mass of fluid passing in unit time is $2\pi r \rho V(1 + a)\, dr$ and the rearward velocity acquired is twice the velocity of inflow, i.e. $2aV$. Hence the thrust on the annulus, which is equal to the momentum communicated to the fluid in unit time, is

$$dT = 4\pi a(1 + a)\rho V^2 r\, dr. \tag{13.7,7}$$

But by Example 1 of § 13.6 we have also

$$dT = \tfrac{1}{2}\rho W^2 B c \lambda_1\, dr \tag{13.7,8}$$

where

$$\lambda_1 = C_L \cos\phi - C_D \sin\phi. \tag{13.7,9}$$

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† This also follows from Kelvin's theorem (see § 3.7) since there is no circulation far up-stream.

Also $$Bc = 2\pi\sigma r$$
and (see Fig. 13.6,1)
$$W = V(1 + a)\operatorname{cosec}\phi.$$
Hence (13.7,8) becomes
$$dT = \pi\sigma\rho V^2(1 + a)^2\lambda_1 \operatorname{cosec}^2\phi\, r\, dr \tag{13.7,10}$$
and by comparison with (13.7,7) we derive
$$\frac{a}{1 + a} = \tfrac{1}{4}\sigma\lambda_1 \operatorname{cosec}^2\phi = \frac{\sigma\lambda_1}{2(1 - \cos 2\phi)}. \tag{13.7,11}$$

The circumferential velocity of inflow is $r\Omega a'$ and the circumferential velocity in the wake is therefore $2r\Omega a'$ while the moment of this velocity about the axis is $2r^2\Omega a'$. When we multiply this by the mass passing the annulus in unit time we get the gain of angular momentum per unit time which is equal to the torque dQ on the annulus. Hence
$$dQ = 4\pi\rho V\Omega a'(1 + a)r^3\, dr. \tag{13.7,12}$$
But consideration of the forces on the blade elements yields
$$dQ = \tfrac{1}{2}\rho W^2 Bc\lambda_2 r\, dr$$
where $$\lambda_2 = C_L \sin\phi + C_D \cos\phi \tag{13.7,13}$$
while $$W = r\Omega(1 - a')\sec\phi$$
and, on combination with the expression for W already quoted,
$$W^2 = r\Omega V(1 + a)(1 - a')\sec\phi \operatorname{cosec}\phi.$$
Consequently
$$dQ = \pi\sigma\rho V\Omega(1 + a)(1 - a')\lambda_2 \sec\phi \operatorname{cosec}\phi\, r^3\, dr. \tag{13.7,14}$$
By use of (13.7,12) we now obtain
$$\frac{a'}{1 - a'} = \tfrac{1}{4}\sigma\lambda_2 \sec\phi \operatorname{cosec}\phi = \frac{\sigma\lambda_2}{2\sin 2\phi}. \tag{13.7,15}$$

The values of the inflow factors are determined by equations (13.7,11) and (13.7,15) but it is to be remembered that ϕ also depends on these factors (see equation 13.6,3). This question is further considered below.

In order to predict the performance of a given propeller on the basis of the vortex theory we may follow a semi-inverse method which is due to Glauert. The data will be the values of the solidity σ and blade angle θ at various radii, together with values of the lift and drag coefficients for the blade sections at these radii, each for a range of the angle of incidence α. Take the blade elements at radius r and *assume* an angle of incidence α. Then $\phi = \theta - \alpha$ is known while λ_1 and λ_2 are obtained from the characteristics C_L and C_D of the blade section at radius r by use of equations (13.7,9) and (13.7,13); C_L and C_D are appropriate to incidence α and to two-dimensional flow or infinite aspect ratio. Now the values of a and a' can be found from (13.7,11) and (13.7,15) respectively and then the value of J is given by (see equation 13.6,4)

$$J = \pi\left(\frac{r}{R}\right)\frac{(1-a')}{(1+a)}\tan\phi, \tag{13.7,16}$$

while the gradings of the thrust and torque coefficients can be got from the formulae of Example 1 § 13.6. This procedure is followed for a number of radii and for a range of angles of incidence. From the tabulated or graphed results of these calculations it will be possible to estimate the values of the thrust and torque gradings at all the selected radii corresponding to any given value of J and so to derive the values of k_T and k_Q by integration with respect to the non-dimensional radius r/R. Finally the graphs of k_T and k_Q on a base of J can be drawn.

Example. The inflow velocity is approximately perpendicular to the blade velocity

The velocity of the blade element at radius r relative to the undisturbed fluid has the components V and $r\Omega$ in the axial and circumferential directions respectively. Now the induced velocity in the axial direction is aV and that in the circumferential direction is $-a'r\Omega$ (for this component *reduces* the velocity of the blade relative to the fluid). The condition that the induced velocity may be perpendicular to the blade velocity is accordingly

$$\frac{a'r\Omega}{aV} = \frac{V}{r\Omega}, \qquad \text{or} \qquad \frac{a'}{a} = \left(\frac{V}{r\Omega}\right)^2.$$

But we find from equations (13.7,11) and (13.7,15) that

$$\begin{aligned}\frac{a'}{a} &= \left(\frac{1-a'}{1+a}\right)\left(\frac{\lambda_2}{\lambda_1}\right)\tan\phi\\ &= \left(\frac{V}{r\Omega}\right)\left(\frac{\lambda_2}{\lambda_1}\right) \text{ from (13.6,3).}\end{aligned}$$

Now C_D is usually much smaller than C_L and if we take C_D to be zero we get

$$\frac{\lambda_2}{\lambda_1} = \tan\phi \text{ and } \frac{a'}{a} = \left(\frac{V}{r\Omega}\right)\tan\phi.$$

Both a and a' are normally much smaller than unity and if we neglect them we obtain from (13.6,3)

$$\tan\phi = \tan\phi_0 = \frac{V}{r\Omega}$$

and then

$$\frac{a'}{a} = \left(\frac{V}{r\Omega}\right)^2$$

which shows that the interference velocity is perpendicular to the blade velocity. To summarise, the inflow velocity at a blade element is normal to the velocity of the element relative to the undisturbed fluid when C_D is zero and the interference factors a and a' are very small. In the same circumstances the effective velocity of the element is therefore the same as if a and a' were zero, i.e.

$$W^2 = V^2 + r^2\Omega^2 = W_0^2.$$

The interference factors are, in fact, small when k_T and k_Q are small. Let w be the resultant induced velocity. Then with the same simplifications as before

$$w = aV\sec\phi = \tfrac{1}{4}\sigma\lambda_1 V\sec\phi\operatorname{cosec}^2\phi$$

$$= \tfrac{1}{4}\sigma C_L V\operatorname{cosec}^2\phi = \tfrac{1}{4}\sigma C_L W\operatorname{cosec}\phi.$$

Since this is normal to the blade velocity and small we have approximately

$$\phi = \phi_0 + \beta$$

where $\tan\phi_0 = \dfrac{V}{r\Omega}$ and $\beta = \dfrac{w}{W} = \tfrac{1}{4}\sigma C_L\operatorname{cosec}\phi$

$$= \tfrac{1}{4}\sigma C_L\operatorname{cosec}\phi_0 \text{ approximately.}$$

Then $$\alpha = \theta - \phi = \theta - \phi_0 - \beta$$

while $$C_L = a_1\alpha$$

where a_1 is the value of $dC_L/d\alpha$ for infinite aspect ratio and α is measured from the no-lift incidence. Accordingly

$$\alpha = \theta - \phi_0 - \tfrac{1}{4}\sigma a_1\alpha\operatorname{cosec}\phi_0$$

or $$\alpha = \frac{\theta - \phi_0}{1 + \tfrac{1}{4}\sigma a_1\operatorname{cosec}\phi_0}.$$

As an approximation we may now use this value of α to find λ_1 and λ_2 with C_D restored and then find the thrust and torque gradings.

13.8 Blade Tip Effects and Broad-Bladed Propellers

In the vortex theory, as explained in § 13.7, the propeller is treated as having a very large number of very narrow blades and the radial component of the velocity of the fluid is regarded as zero. For an ordinary propeller having a small number of blades the radial flow is not negligibly small and is, indeed, important near the tips of the blades. The existence of the radial velocity implies that the other components of velocity are not exactly as given by the vortex theory. Consequently the values of the thrust, torque and efficiency are not exactly in accord with the vortex theory. The existence of the radial velocity implies increased kinetic energy losses in the slipstream and therefore a reduction in the efficiency, but this is often not serious. It is beyond the scope of an elementary treatise to discuss blade tip effects in detail and we shall confine ourselves to a general explanation of the phenomena.

The system of vortices which trail from one blade of a propeller forms a vortex sheet which is approximately of helicoidal form (see § 13.2) but, in general, the pitch is not constant along a radius. However, any departure from constancy of pitch implies that the kinetic energy in the wake is greater than the minimum which is possible with a given value of the thrust. Hence, when the blade is such that the efficiency is a maximum, the vortices trailing from a blade all have the same pitch, as first pointed out by Betz. If we neglect the contraction of the slipstream downstream from the propeller disk, each vortex sheet is then a true helicoidal surface, and there are B such surfaces evenly spaced round the axis. This system of helicoids moves downstream relative to the undisturbed fluid as if the helicoids were rigid and we can regard their velocity as being axial, for any rotation of such surfaces can be compensated by a suitable axial displacement (see § 13.2). The radial flow is induced by the axial motion of the helicoids and an exact solution (for a perfect fluid) to the equations of motion has been obtained by Goldstein.[1] The radial velocity is greatest near the tip radius and falls at both greater and lesser radii. Consequently the thrust and torque gradings as given by the vortex theory are most seriously in error in the region of the blade tips. Lock[2] has developed methods for dealing with propellers having a finite number of blades which are not restricted to the case where the distribution of circulation along the blades is the optimum. The design of broad-bladed marine propellers is treated in a paper by Eckhardt and Morgan,[3] where references to other papers are also given, while the theory is given by Ginzel.[4]

13.9 The Experimental Mean Pitch of a Propeller

If the driving faces of the blades of a propeller were true helicoidal surfaces of pitch P_i, the blades infinitely thin and the fluid of vanishingly small viscosity the thrust and torque would both be zero when the advance per revolution was equal to P_i or the slip zero. For a real propeller, whose blades must have a finite thickness, working in a viscous fluid, the thrust will vanish when the advance per revolution has a certain value P_e which differs from P_i and is normally somewhat larger but in this condition the torque, though small, will not be zero. Then P_e is called 'the experimental mean pitch' of the propeller and in many ways the propeller behaves as if its blades were infinitely thin helicoidal surfaces of pitch P_e. Strictly speaking the experimental mean pitch depends on the values of the Reynolds and Mach numbers and on the state of the surfaces of the blades but normally such effects are very small.

13.10 Propeller Design

The data of a propeller design are usually the rate of rotation and shaft power (which also determine the torque), the rate of advance through the fluid and its density. The problem is then to design the propeller so that the thrust (and therefore also the efficiency) is a maximum. Sometimes the admissible design is limited by certain requirements; a very common example of this is an upper limit on the diameter of the propeller. In this book we shall confine our attention to the hydrodynamic design of the propeller but it must be kept in mind that strength requirements will limit the admissible thinness of the blades. There are three principal methods or bases for the hydrodynamic design of screw propellers:

(a) The use of non-dimensional coefficients obtained from tests on model or full-scale propellers.

(b) Pure theory (such as the vortex theory) with possibly some empirical adjustments.

(c) The method of the representative blade section or standard radius. This is really a special case of (b) and is explained below.

Even in methods (b) and (c) some purely empirical information, such as the values of the drag coefficients of blade sections, is used. These three methods will now be explained in some detail.

Let us first consider design based on the use of non-dimensional coefficients and for simplicity let us assume for the moment that all the geometric particulars of the propeller have been chosen with the exception of the diameter D and pitch P_i.† The problem is to determine these so that the efficiency shall be a maximum, given the data:

Shaft power P
Revolutions in unit time n
Speed of advance of the centre of the propeller through the fluid V
Density of the fluid ρ.

These are expressed in consistent units; for example, when the units of length, time and force are the foot, second and pound-weight respectively, P will be measured in ft. lb. per sec., n in revolutions per sec., V in ft. per sec. and ρ in slugs per ft.^{3} It will be supposed that tables or graphs are available showing the relation between

$$J = \frac{V}{nD} \tag{13.10,1}$$

and the speed-power coefficient

$$C_S = \sqrt[5]{\left(\frac{\rho V^5}{n^2 P}\right)} \tag{13.10,2}$$

for a range of values of the pitch ratio (see equations (13.2,6), (13.3,3) and

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† The quantities which are assumed to have been already selected are shape parameters and thus non-dimensional.

(13.4,10)); these relations are obtained from model or full-scale tests and are assumed to be valid for the value of the Reynolds number of the propeller being designed. For any given pitch ratio the propellers considered are, accordingly, all geometrically similar and, as explained in § 13.4, the values of C_P, C_T, C_S and η are thus all functions of J alone. Hence we can, for example, plot C_S on a base of J for a range of values of the pitch ratio ϖ or, alternatively, we may prepare graphs of J on a base of C_S for a number of evenly spaced values of ϖ and we can superpose on these the contours of constant efficiency η. For definiteness let us suppose first that the diameter of the propeller is unrestricted. Then we can calculate C_S from equation (13.10,2) and the data of the design. From the chart we can now pick out the value of ϖ for which η is a maximum and the corresponding value of J. Then the diameter is obtained from

$$D = \frac{V}{nJ} \tag{13.10,3}$$

while the pitch is

$$P_i = \varpi D. \tag{13.10,4}$$

If, as happens not infrequently, the optimum diameter is larger than is, for some reason, admissible let D_m be the maximum permitted diameter. Then with $D = D_m$ we find J and calculate C_S as before. A point on the chart is now located and the value of the pitch ratio can be read off. The propeller designed in this way will absorb the specified power at the specified rate of rotation and when advancing at the specified speed. The amount by which its efficiency falls below the maximum possible with the given value of C_S will depend on the extent by which D_m falls below the diameter appropriate for maximum efficiency. In all cases the thrust is given by

$$T = \frac{\eta P}{V}. \tag{13.10,5}$$

We must next consider how the other geometric characteristics of the propeller are to be chosen so that the efficiency may be a maximum. In general the design should be guided by the known results obtained from propellers of various geometric forms and always for the value of C_S appropriate to the design. Most help will be obtained from the results of tests in which one geometric parameter at a time is varied systematically. One of the most important of these parameters is the 'solidity' (see § 13.2) which determines the total blade chord at any radius. The choice of the number of blades will depend on the value of the solidity for maximum efficiency. When this is small it will probably be best to choose a two-bladed screw but when it is large 3, 4 or more blades will be required. The choice is also governed by a variety of practical considerations in any given case. For blades of a given shape (e.g. of elliptic form when 'developed') the blade width may conveniently be specified by the value of the *mean width ratio* which is defined as the ratio of the mean chord or width of one blade to the propeller diameter. The blade thickness will be governed by strength requirements and the profile should be one having good hydrodynamic characteristics, i.e. a high lift/drag ratio. Sometimes it is important to secure a high thrust under conditions of high slip (or low J) and then the characteristics of the profile at large angles of incidence must be favourable.

Pure theory cannot be applied directly to solve the design problem but it would be possible to construct the design charts (in which J is plotted on a base of C_S) from theoretical calculations, say based on the vortex theory. Theory can also be applied to check a design which might, for example, be based on the interpolation of experimental results. The method of obtaining the performance of a given propeller on the basis of the vortex theory has been explained at the end of § 13.7.

We shall now explain the method of the 'representative section' or 'standard radius'. We take the particulars of the blades at a radius equal to a standard fraction of the tip radius (usually in practice 0·7 of the tip radius) and work out the values of the thrust and torque gradings at this radius on the basis of the vortex theory; this can be done without knowledge of the blade characteristics at other radii on account of the mutual independence of elements at differing radii in the vortex theory. Then we may *assume*, on the basis of experiment and experience, that the thrust and torque gradings vary along the radius in a standard manner. We can then derive the values of k_T and k_Q from the values of $dk_T/d(r/R)$ and $dk_Q/d(r/R)$ respectively at the standard radius by multiplication by suitable constants. Evidently this procedure is vastly more expeditious than a complete calculation based on the vortex theory and, in the hands of experienced designers, yields very satisfactory results. It is found that the thrust grading for many propellers gives an approximately elliptic plot on a base of $(r/R)^2$. The mid-ordinate of the ellipse occurs where $(r/R)^2 = 0{\cdot}5$ or $r = 0{\cdot}707R$. As an approximation this is usually modified to $r = 0{\cdot}7R$. Let the value of $dk_T/d(r/R)^2$ at the standard radius be k_{TS}. Then

$$k_T = \frac{\pi}{4} k_{TS} \tag{13.10,6}$$

Let us put

$$\xi = \left(\frac{r}{R}\right)^2 \tag{13.10,7}$$

and use the formula for the thrust grading given in the Example appended to § 13.6 together with the relation

$$(1 - a') \sec \phi = \frac{1}{\pi}\left(\frac{W}{V}\right)\left(\frac{R}{r}\right) J \tag{13.10,8}$$

which can be obtained from Fig. 13.6,1. Then

$$\frac{dk_T}{d\xi} = \left(\frac{R}{2r}\right)\frac{dk_T}{d\left(\frac{r}{R}\right)} = \frac{\pi^3}{8}\left(\frac{r}{R}\right)^2 \sigma\lambda_1(1 - a')^2 \sec^2\phi$$

$$= \frac{\pi}{8} J^2\left(\frac{W}{V}\right)^2 \sigma\lambda_1 \tag{13.10,9}$$

where λ_1 is defined in (13.7,9). Accordingly we now derive from (13.10,6)

$$k_T = \frac{\pi^2}{32} J^2\left(\frac{W}{V}\right)^2 \sigma\lambda_1 \tag{13.10,10}$$

where the expression on the right of the equation is to be evaluated at the standard radius. To obtain the torque coefficient we may estimate the efficiency from (13.6,8) or otherwise and use

$$k_Q = \frac{Jk_T}{2\pi\eta}. \tag{13.10,11}$$

13.11 Ship Propulsion

Ships are usually propelled by one or more screw propellers situated at the stern.† Thus the propeller works in a region where the flow is disturbed by the hull and the 'friction belt' or boundary layer; even when propellers are somewhat forward of the stern, as may happen on twin-screw ships, it is customary to say that the propeller works in the 'wake' of the hull. The flow in the wake is very complex and always turbulent; at any given point of a propeller disk the velocity will fluctuate and the average velocity will, in general, have components in the lateral and vertical directions but the most important component is axial and forwards. For a given propeller, hull and speed V we could assign a value to the axial wake velocity V_w such that if the propeller advanced through an undisturbed fluid with velocity

$$V_A = V - V_w \tag{13.11,1}$$

it would have the same *torque* as the propeller in the wake when rotating at the same speed. Then V_w is the effective wake velocity appropriate to the particular circumstances. In designing propellers for ships it is usual (in the absence of tank tests of hull with propeller) to assume that the *thrust* will also have the value corresponding to the speed of advance V_A as already defined but this is no more than a rough approximation. The efficiency, however, is greater than that of the isolated propeller advancing with speed V_A. We have

$$\eta = \frac{VT}{2\pi nQ} = \left(\frac{V}{V_A}\right)\frac{V_A T}{2\pi nQ} = \left(\frac{V}{V_A}\right)\eta' \tag{13.11,2}$$

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† For special applications (e.g. 'double-ended' ferry boats) propellers may be fitted both forward and aft.

where η' is the efficiency of the isolated propeller. This gain in efficiency may be important and is by no means fictitious. It can be explained on general grounds as follows. The velocity in the slipstream of the screw, *measured relative to the wake current*, has (in accordance with our assumptions) the same magnitude v_1 as for the isolated propeller advancing at speed V_A. However, the rearward velocity in the slipstream *measured relative to the undisturbed fluid* is only $(v_1 - V_w)$ and the kinetic energy wasted in the slipstream is therefore reduced by the presence of the forward moving wake current.†

Various methods have been used for specifying the wake velocity. R. E. Froude put

$$V_w = w_F V_A \tag{13.11,3}$$

whereas D. W. Taylor wrote

$$V_w = w_T V \tag{13.11,4}$$

where we have added the suffixes F and T to distinguish the two definitions of the *wake fraction* w. We then have

$$w_F = \frac{V - V_A}{V_A} \tag{13.11,5}$$

or

$$V_A = \frac{V}{1 + w_F} \tag{13.11,6}$$

and

$$w_T = \frac{V - V_A}{V} \tag{13.11,7}$$

or

$$V_A = V(1 - w_T). \tag{13.11,8}$$

Also

$$w_T = \frac{w_F}{1 + w_F} \tag{13.11,9}$$

$$w_F = \frac{w_T}{1 - w_T}. \tag{13.11,10}$$

On account of the existence of the wake current the apparent slip of the propeller differs from the true slip. Let s_a and s_t be the apparent and true slip ratios respectively. Then

$$= 1 - \frac{V}{nP_i} \quad \text{and} \quad s_t = 1 - \frac{V_A}{nP_i}$$

Hence

$$- s_a = \frac{V - V_A}{nP_i} = \frac{w_T V}{nP_i} = w_T(1 - s_a)$$

and

$$s_t = s_a + w_T - s_a w_T. \tag{13.11,11}$$

It is possible for the apparent slip to be negative when the true slip is positive.

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† It should be noted, however, that the isolated propeller when advancing with speed V (and with n and Q unchanged) would probably have a higher efficiency than when advancing at the lower speed V_A.

So far as the argument has hitherto been taken it would appear that there is a distinct advantage as regards efficiency in placing the propeller at the stern. However, there is a second effect which largely offsets the advantage. The action of the propeller increases the velocity of flow over the hull near the stern and thus reduces the fluid pressure on the hull in this region. Consequently there is effectively an addition to the resistance of the hull and this may be augmented by frictional effects. The added resistance of the hull is equivalent to a reduction of the propeller thrust and it is customary to speak of the *thrust deduction.* Let the resistance of the ship without propellers at speed V be R; this is the tension required in a tow rope to maintain the constant speed V and is often called the tow rope resistance. When the ship is self-propelled at speed V the propeller thrust is T and the *thrust deduction coefficient* is defined by

$$t = \frac{T-R}{T} \tag{13.11,12}$$

and then

$$R = T(1-t). \tag{13.11,13}$$

The useful power is RV while the work done by the propeller relative to the wake in unit time is TV_A. The ratio RV/TV_A is known as the *hull efficiency* η_H and we get from (13.11,6) and (13.11,13)

$$\left.\begin{aligned}\eta_H &= (1-t)(1+w_F)\\ &= \frac{1-t}{1-w_T}.\end{aligned}\right\} \tag{13.11,14}$$

The overall hydrodynamic efficiency or *propulsive efficiency* η_P is the ratio of the tow-rope power RV to the power supplied to the propeller, i.e. the shaft power measured aft of the stern bearing. Hence

$$\begin{aligned}\eta_P = \frac{RV}{2\pi nQ} &= (1-t)(1+w_F)\left(\frac{TV_A}{2\pi nQ}\right)\\ &= \eta_H\eta' \end{aligned} \tag{13.11,15}$$

where η' is, as before, the efficiency of the isolated propeller when advancing at speed V_A.

Marine propellers are usually designed on the basis of tests made on isolated model screws and with the additional assumption that the wake is a uniform current.† The non-dimensional coefficients explained in §§ 13.4 and 13.10 are not used in practice because the density of water is very nearly constant and it is accordingly omitted from the formulae which are therefore *incomplete* in Buckingham's sense. The density assumed is an average value for ocean water and is about 3 per cent higher than for fresh water (a density ratio of 36/35 may be assumed). Further, inconsistent units are usually adopted and the practice varies from country to country. We shall here use the following symbols:

D = diameter of propeller (ft.)
V = speed of ship (knots†)
V_A = speed of advance of propeller through the wake (knots)
R = revolutions per minute
H = power supplied to propeller, shaft horse power.

Any power of the speed-power coefficient C_s (see equation 13.4,10) can be used in place of C_s itself and when ρ is omitted we are led to consider powers of $V_A{}^5/R^2H$. R. E. Froude[1], who was the pioneer in the application of model tests to propeller design and was greatly influenced by the ideas of his father, William Froude, adopted a coefficient which is essentially the same as the reciprocal of the foregoing expression while D. W. Taylor[2] used

$$\rho = \sqrt{\left(\frac{R^2H}{V_A{}^5}\right)}, \tag{13.11,16}$$

where ρ is not to be confused with the density of the fluid, as the abscissa of his charts for propeller design. The ordinate of a design chart must depend on D, but J (see equation 13.3,3) has not been adopted for marine purposes. Froude took as ordinate a coefficient which is essentially equivalent to $H/D^2V_A{}^3$. Taylor used

$$\delta = \frac{DR^{2/3}}{(HV_A)^{1/6}} \tag{13.11,17}$$

while W. J. Duncan[3] adopted

$$\theta = \frac{DR^{3/5}}{H^{1/5}}, \tag{13.11,18}$$

which is clearly related to the coefficient C_P (see equation 13.4,5) as ordinate with ρ as abscissa. It will be found that δ and θ become non-dimensional when multiplied by the sixth root and fifth root, respectively, of the density.

It is important for a designer of marine propellers to accumulate information about the wake velocity and this can be done by the systematic analysis of the results of ship trials on the measured mile. The information obtained from the trial will be: shaft power† H, revolutions per minute R and speed of ship V while all particulars of the propeller will be known. From D, H and R the value of θ can be calculated (see equation 13.11,18). Then since the pitch ratio and other geometric parameters of the propeller are known, a point on the $\rho\theta$ chart can be located and ρ read off. The speed of advance through the wake is then given by

$$V_A = \sqrt[5]{\left(\frac{R^2H}{\rho^2}\right)} \tag{13.11,19}$$

and the wake fraction, according to the definition of Froude or Taylor (see above), can be calculated.

Throughout the foregoing discussion we have assumed that cavitation (see § 3.12) does not occur. It is beyond our scope to treat this in detail and we shall content ourselves with pointing out that cavitation is most likely to occur when the tip speed of the propeller is high and the depth of immersion of the blade tip at its highest point is small. Proneness to cavitation is also influenced by blade thickness, profile shape, roughness and waviness of the blade surface.

Example. Estimation of rate of rotation with given torque

The data are the diameter and other geometrical particulars of the propeller, torque Q, rate of advance V_A (and density of the fluid). First let us consider the use of non-dimensional coefficients. Since the rate of rotation (n or R) is unknown we must use a non-dimensional quantity which is independent of this and Q_C (see equation 13.4,3) meets the requirements. The problem may be solved by use of a chart in which the coordinates are Q_C and J, for Q_C can be calculated from the data and then n can be found from J. For marine propellers we could use charts with the coordinates $Q/V_A{}^2D^3$ and true slip ratio, the contours corresponding to various pitch ratios.

13.12 Airscrews

When screw propulsion is adopted for aircraft the propeller or propellers are usually placed forward and are then called *tractor screws*; when placed aft, they are called *pusher screws*. The usual arrangement is thus the opposite of that on ships and there are several reasons for this:

(a) The potential gain in efficiency resulting from working in the wake (see § 13.11) is trifling since the average wake velocity over the propeller disk is very small. This is related to the fact that the ratio of propeller diameter to body diameter is, typically, much greater for an aircraft than for a ship and this, in turn, is related to the very much greater power/volume ratio of the aircraft.

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† If only I. H. P is known, this must be multiplied by an estimated mechanical efficiency.

(b) Adequate ground clearance is more easily obtained with the propellers forward.

(c) Airscrews situated in the wake are liable to suffer vibration troubles on account of the variation of wake velocity around a circle concentric with the axis of rotation; this causes periodic changes of the aerodynamic forces on the blades.

Another typical difference is that airscrews have much narrower blades (i.e. are of lower solidity) than marine screws. The value of the speed-power coefficient (see equation 13.4,10) is, typically, considerably larger for an airscrew than for a marine screw and consequently the value of the solidity at maximum efficiency is smaller. On account of their relatively small solidity airscrews are more amenable to theoretical treatment than marine propellers.

The hub or boss of a tractor airscrew is usually provided with a fairing called a *spinner* so shaped that the spinner and nacelle or body behind it are part of one fair surface while the gap between the rotating spinner and the body is made as small as practicable. Spinners are also fitted to pusher screws. A pair of opposite-handed propellers, mounted on coaxial shafts and with their disks close together, is called a *contra-rotating pair*. It is possible to design the pair so that the rotary component of velocity in the slipstream is absent. This can lead to a small gain in efficiency but the main advantage of the contra-rotating pair is that objectionable effects of the rotating slipstream on the aircraft, particularly at take-off, are avoided; this was of particular value in the case of piston-engined fighter aircraft. With piston engines it is usual (on the larger aircraft) to adopt *constant speed propellers*. Here the blades are arranged so that they can rotate about their root sockets in the propeller boss and the angular setting of the blades is so controlled by an automatic mechanism that the rate of rotation is kept nearly constant. In this manner the engine is permitted to develop its full power when the velocity of advance is small (take-off and climb). With this arrangement the pitch can only be the same at all radii for one blade setting.

Since airscrews in service encounter air whose density varies widely according to altitude etc. it is essential that the *complete* non-dimensional coefficients be employed (see § 13.4). Also the speed of the blade elements relative to the air is usually so high, especially near the blade tips, that compressibility effects (see § 9.1) are not negligible. These effects are usually related to the tip Mach number which is the ratio of the resultant speed at the blade tips relative to the air (but with the inflow velocities omitted) to the velocity of sound in the undisturbed ambient air. The noise generated by an airscrew increases very rapidly when the tip Mach number approaches and exceeds unity.

Lastly, we shall mention that a rotating screw behaves like a fin surface when the relative velocity is not axial. The aerodynamic force lies in the plane containing the axis of the propeller and the relative velocity vector and it is proportional to the angle between this vector and the axis (subject to the angle being small).†

13.13 Fans

We shall suppose that the fan is fitted in a cylindrical duct of circular section; in practice the clearance at the blade tips will be the minimum required to avoid fouling at top speed. Very often the fan is mounted on a spinner of a diameter which may, in extreme cases, be larger than half the diameter of the duct and the spinner will be fair with a fixed nacelle containing the driving motor. We shall assume here that the blades are situated in a duct bounded externally and internally by concentric cylinders so that, if the blades were absent, the flow at their site would be uniform. We shall discuss below the provision of *straighteners* which are fixed vanes intended to eliminate the rotary motion in the region downstream from the fan.

A fan working in a duct differs from a propeller in the open in as much as the walls prevent the contraction of the stream in passing through the propeller. Hence there is no induced axial velocity of inflow and, by continuity, the axial component of the velocity is the same in front of and behind the fan. Thus the fan creates a pressure difference across its disk without changing the axial velocity. However, the rotational component is altered just as for a propeller in the open and likewise the rotational inflow is the same. Hence the axial inflow factor a (see §§ 13.5 and 13.6) is zero for a fan with or without straighteners while the rotational inflow factor a' has the same value for a fan without straighteners *upstream* as for a propeller in the open.

For some applications, in particular to return flow wind tunnels (see § 5.5), it is most desirable that the fluid should leave the fan assembly without swirl, i.e. with the circumferential component of velocity zero at all radii. Now, by the principle of angular momentum (see § 3.2) the fan inevitably communicates angular momentum to the fluid at any radius of amount corresponding to the torque grading (see § 13.6). However, the swirl can be eliminated by providing a set of evenly spaced fixed vanes called straighteners so arranged that the torque which they exert on the fluid at any radius is equal and opposite to that of the fan at that radius. The straighteners can be variously arranged:

(a) All upstream of the fan.
(b) All downstream of the fan.
(c) Some upstream and some downstream, say equally divided.

In all cases the final rotational component of velocity is zero while the change in this component through the fan corresponds to the torque grading;

also the effective inflow velocity is the mean of the rotational components just before and just aft of the fan. Hence we see that the effective value of the inflow factor a' is as follows in the above cases:

(a) a' is numerically equal to the value for a propeller in the open but with reversed sign.

(b) a' is the same as for a propeller in the open.

(c) a' is zero when the contributions to the torque are the same upstream and downstream.

Whatever may be the arrangement, the thrust and torque gradings may be calculated by the method given in §§ 13.6 and 13.7 with the inflow factor a taken as zero and a' determined in accordance with the preceding remarks.

As a rule the aim of the designer is to obtain a uniform stream from the fan. If the approaching stream is uniform the emergent stream will be uniform and remain so if the pressure difference across the fan is uniform. It is usual for the velocity of the approaching stream to be reduced near the blade tips by the boundary layer from the duct wall and it may then be advantageous to increase the pressure difference near the tips. Let the pressure difference at the radius r be p. Then the thrust grading is

$$\frac{dT}{dr} = 2\pi rp \tag{13.13,1}$$

while (see Example I of § 13.6)

$$\frac{dT}{dr} = \pi r\sigma\rho W^2(C_L \cos\phi - C_D \sin\phi) \tag{13.13,2}$$

where W is the resultant velocity of the blade element at radius r measured relative to the fluid. It follows from these equations that

$$\begin{aligned} p &= \tfrac{1}{2}\sigma\rho W^2(C_L \cos\phi - C_D \sin\phi) \\ &= \frac{1}{4\pi r}\rho BcW^2(C_L \cos\phi - C_D \sin\phi). \end{aligned}$$

But $$W = r\Omega(1 - a')\sec\phi \tag{13.13,3}$$

and $$p = \frac{1}{4\pi}\rho Bc(1 - a')^2 r\Omega^2(C_L \cos\phi - C_D \sin\phi)\sec^2\phi. \tag{13.13,4}$$

The blade angle of fans is usually small and ϕ is thus also small under normal working conditions. As an approximation we may take $\cos\phi$ as unity and in preliminary design work we may neglect a' and C_D. We then derive the simple approximate relation

$$p = \frac{1}{4\pi}\rho Bcr\Omega^2 C_L. \tag{13.13,5}$$

If p is to be constant the blades must be such that crC_L is independent of r. It is quite usual to make the chord c constant and then the condition reduces to constancy of rC_L.

In the design of fans for wind tunnels use is made of equation (5.5,1) which gives the power absorbed by the fan as

$$P = \tfrac{1}{2}\lambda\rho A_0 V_0^3 \tag{13.13,6}$$

where A_0 is the cross-sectional area of the working section of the tunnel, V_0 is the velocity at the working section and λ is the power factor, whose value should be estimated from the known performance of wind tunnels of similar design and size. The diameter of the fan will be fixed by the general layout of the tunnel and, in particular, by the value of the 'contraction ratio' in the contraction immediately upstream of the working section. The speed of rotation may be fixed by the characteristics of the motor available but the tip speed must not be excessive or the noise will be intolerable. The required value of the pressure difference p is obtained from

$$\begin{aligned} P &= \frac{TV}{\eta} \\ &= \frac{pAV}{\eta} \end{aligned} \tag{13.13,7}$$

where A is the cross-sectional area of the duct at the site of the blades (with due allowance for the spinner), V is the axial velocity there and η is the efficiency of the fan. The design can then be completed by the application of equation (13.13,5). According to Collar[1] it is advantageous to adopt the value 0·6 for C_L at the blade tips.

13.14 Windmills

Windmills of very varied design have been conceived and some of these have been made and used. However, nearly all windmills now in service are essentially screw propellers worked in reverse, i.e. with a negative torque, and the thrust is also negative. The thrust of a windmill (or rather, the drag which is the thrust reversed in direction) is not important, except in relation to the loads on bearings and the stresses in the supporting structure.

Let P be the power and Q the torque developed *by* the windmill, while R is the resistance or drag (negative thrust), where these are all measured in consistent units. We shall suppose for simplicity that the windmill is placed in a uniform and unbounded stream of incompressible fluid of density ρ, the velocity being V and parallel to the axis of the windmill; the centre of the hub of the rotor is stationary. In accordance with the principle of relativity we may impose a constant velocity $-V$ on the whole system without altering any of the forces. Hence the windmill is exactly equivalent to a propeller advancing through a stationary fluid with speed V in a direction opposite to that of the wind; indeed, small windmills are sometimes carried by aircraft or other vehicles and are then in exactly the condition postulated. All the theory of propellers developed in the earlier parts of this chapter can now be applied to the windmill but it must be kept in mind that the sign of Q is now reversed, while the thrust T is equal to $-R$. The inflow velocities aV and $a'r\Omega$ are also reversed in sign. For a given windmill of fixed pitch the data will be ρ, D, V and Q which is equal to the torque reaction of the load. Hence it is convenient to use the non-dimensional torque coefficient

$$Q_c = \frac{Q}{\rho V^2 D^3}. \tag{13.14,1}$$

For similar rotors (and in the absence of scale effect) Q_c is a function of J alone. Hence J can be found when Q_c is known and the rate of rotation n can be calculated from J. Then the power developed is

$$P = 2\pi n Q = 2\pi n \rho V^2 D^3 Q_c. \tag{13.14,2}$$

For a given rotor and with given values of ρ and V there will be a particular value of Q_c such that nQ_c (and P) is a maximum. If the torque differs from that corresponding to this value of Q_c the windmill will not develop the maximum power for the given wind.

The ordinary definition of the efficiency of a propeller is inapplicable to a windmill since the thrust does no work. However, we can frame a reasonable definition of the efficiency η as the ratio of the useful work done in unit time to the kinetic energy of the fluid passing in unit time through an area A equal to that of the rotor disk. Accordingly

$$\eta = \frac{2\pi n Q}{\frac{1}{2}\rho V^3 A}. \tag{13.14,3}$$

When we substitute for A in terms of D and use (13.14,1) we find that

$$\eta = \frac{16 Q_c}{J} \tag{13.14,4}$$

Thus the efficiency can be found from the known relation between Q_c and J.

A value for the efficiency of a windmill can be deduced from the theory of the actuator (see § 13.5) and it is convenient for the purpose of the argument to suppose that the windmill is advancing with speed V into fluid at rest, as explained above. Since the axial velocity of inflow is actually negative it is convenient to put $v' = -v$ and then v' is positive.† Hence the velocity of the fluid at the site of the actuator is v' in the same sense as V and the work done on the fluid in unit time is $v'R$. However, the rate at which energy is supplied to the actuator is VR and the balance is the useful power P. Thus

$$P = R(V - v'). \tag{13.14,5}$$

Also

$$\begin{aligned} R &= -T = -2\rho v(V + v)A \\ &= 2\rho v'(V - v')A \end{aligned} \tag{13.14,6}$$

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† There is consequently an increase in the diameter of the stream as it passes through the disk whereas the diameter decreases for a propeller with positive thrust.

by equation (13.5,7). Therefore (13.14,5) becomes

$$P = 2A\rho v'(V - v')^2. \tag{13.14,7}$$

Now let us revert to the windmill. The efficiency is given by

$$\eta = \frac{P}{\frac{1}{2}\rho A V^3} = 4\left(\frac{v'}{V}\right)\left(1 - \frac{v'}{V}\right)^2. \tag{13.14,8}$$

We see that η is a unique function of the velocity ratio v'/V and it has its maximum value 16/27 or 59·3 per cent when v'/V has the value 1/3. Since frictional and rotational losses have been ignored in this argument the greatest practically attainable efficiency must be less than the foregoing maximum.

EXERCISES. CHAPTER 13.

1. Show that the diameter of a propeller is given by

$$D = \frac{C_s}{J}\sqrt[5]{\left(\frac{P}{\rho n^3}\right)}.$$

Discuss the dependence of the diameter on power input, rate of rotation and density when J is constant. (Assume that changes in Reynolds and Mach numbers may be neglected.)

2. An actuator disk 0·70 m diameter advances at 6 m/s through water (density 1 Mg/m^3) and develops a thrust of 4·5 kN. Calculate the axial inflow factor a and the efficiency η. (Answer $a = 0{\cdot}141$, $\eta = 0{\cdot}876$)

3. Graph the relationship $\dfrac{P}{\rho V^3 D^2} = \dfrac{\pi}{2}\dfrac{1 - \eta}{\eta^3}$ and estimate the efficiency of an actuator disk from the following data: diameter 0·5 m, density of fluid 1 Mg/m^3, velocity of advance 9 m/s, power absorbed 150 kW. (Answer $\eta = 0{\cdot}765$)

4. When J lies within the range 0·2 to 0·7 the speed-power coefficient of a certain propeller is given with sufficient accuracy by the linear relation $C_s = 1{\cdot}9J - 0{\cdot}05$. Find the diameter of a geometrically similar propeller which will absorb 1,500 kW when $V = 60$ m/s, $\eta = 20$ revs./sec. and $\rho = 1{\cdot}226$ kg/m^3 (Answer $D = 4{\cdot}97$ m.)

5. An airscrew is 4·2 m diam. and has 4 blades; at 1·5 m radius the blade chord is 0·3 m and the blade angle θ is 23°. Find the value of J and the efficiency of the blade elements at this radius when the true local angle of incidence is 4° (use the vortex theory), given that $C_L = 0{\cdot}6$ and $C_D = 0{\cdot}03$ at this incidence for infinite aspect ratio. Find also the total local thrust and torque gradings when $V = 60$ m/s and $\rho = 1{\cdot}226$ kg/m^3. What is the value of n? (Answer $J = 0{\cdot}628$, $\eta = 0{\cdot}698$, thrust grading 20·2 kN/m, torque grading 12·1 kNm/m, $n = 22{\cdot}8$ revs./sec.)

6. Repeat the calculation of the last Exercise when the true local incidence is 2°, given that for infinite aspect ratio $C_L = 0{\cdot}43$ and $C_D = 0{\cdot}024$. (Answer $J = 0{\cdot}765$, $\eta = 0{\cdot}758$, thrust grading 9·95 kN/m, torque grading 6·70 kNm/m, $n = 18{\cdot}7$ revs./sec.)

7. Prove the following relations which may be used for checking calculations:

   (a) $\lambda_1^2 + \lambda_2^2 = C_L^2 + C_D^2$.

   (b) $(a\lambda_2 \sin\phi - a'\lambda_1 \cos\phi) = aa'C_L$.

   (c) $\sigma\lambda_1\lambda_2(a + a') = 4aa'C_L \sin\phi$ or $\dfrac{1}{a} + \dfrac{1}{a'} = \dfrac{4C_L \sin\phi}{\sigma\lambda_1\lambda_2}$.

   Note that (b) and (c) are linear equations in the reciprocals of a and a' and may be solved for these quantities.

8. A ship is making 33·4 km/h relative to the undisturbed sea. The propeller is of 5·5 m pitch and makes 120 revs. per minute. Find the apparent slip (as a fraction) and the true slip, given that the Taylor wake fraction is 0·2. Find also the value of the Froude wake fraction. (Answer $S_a = 0{\cdot}157$, $s_t = 0{\cdot}325$, $w_F = 0{\cdot}25$.)

9. A four-bladed propeller is of 1·2 pitch ratio and 0·51 surface ratio (ratio of developed area of driving faces to disk area) and propels a ship at 14·5 knots through sea water when making 80 revs./minute. The wake fraction $w_T = 0{\cdot}24$ and the shaft horse power is 3,600. Calculate the diameter and pitch of the propeller given that, for a propeller of the given shape, the relation between ρ and θ (see § 13.11) is approximately

   $$\theta = 47{\cdot}8 - 0{\cdot}21\rho$$

   for the range 10–16 of ρ (units:—foot, knot, revolution per minute and H.P. Fluid:—sea water). (Answer Diameter 16·8 ft. Pitch 20·2 ft.)

10. The data are the same as for Exercise 9 except that the pitch ratio ϖ is unknown and the diameter is given as 16 ft. (which may be taken as the maximum permitted). Estimate the pitch in feet (for the same speed, r.p.m. and power) on the basis of the following approximate empirical formula

    $$\theta = 30{\cdot}7 + \frac{20{\cdot}5}{\varpi} - 0{\cdot}21\rho$$

    (valid only for limited ranges of ϖ and ρ and for the kind of propeller described in Exercise 9). (Answer Pitch 22·2 ft.)

11. The propeller of an aircraft absorbs 750 kW at a forward speed of 88·8 m/s at sea level. The diameter of the propeller is 3·35 m. With the assumption that the propeller can be treated as a simple actuator disk find the velocity in the slipstream far downstream and the efficiency when $\rho = 1{\cdot}226$ kg/m^3. (Answer 8·3 m/s, 0·956)

12. For the propeller of Exercise 11 find the value of $\Delta p/(p_0 - p)$, where Δp is the pressure rise across the disc of the propeller, p_0 is the static pressure of the undisturbed air, and p is the static pressure just in front of the propeller disk. (Answer 2·06)

13. For a propeller of 3·35 m diameter on an aircraft flying at 80·6 m/s the slipstream diameter far downstream is found to be 3·2 m at sea level. On the assumption that the propeller can be treated as a simple actuator disk find the aircraft, between that of the air in the slipstream and that of the air outside the slipstream. $\rho = 1{\cdot}226$ kg/m^3. (Answer 1·858 kN/m^2.)

14. A propeller is designed for an aircraft to fly at 164 m/s at 12,400 m (ρ/ρ_0 = 0·246, temperature = −56·5 °C), and at this speed the propeller is to absorb 1,500 kW at 920 r.p.m. A model of the propeller is required for testing in a wind tunnel in air at sea level density and temperature 1·226 kg/m^3, 15 °C). Find the rotational speed of the model propeller and the power to be absorbed by it to ensure dynamic similarity with the full scale propeller. Take the coefficient of viscosity μ as proportional to $T^{3/4}$, where T is the absolute

temperature in degrees Kelvin, and neglect compressibility effects. The aircraft propeller is 3·8 times larger than the model propeller.

(Answer 4,050 r.p.m., 658 kW.)

15. It is found that over a range of aircraft Mach number M_1, where $M_1 < 1{\cdot}0$, the thrust and torque coefficients of a propeller are of the form

$$k_T = \text{const.}(1 + \alpha M_1), \qquad k_Q = \text{const.}(1 + \beta M_1)$$

where α and β are constants that are small compared with unity. If the overall efficiency of the propeller is to remain constant over the range of M_1 show that the following conditions must hold:

$$\alpha = \beta, \qquad \pi n D/a_1 = M_1,$$

where a_1 is the speed of sound in the undisturbed air, n is the number of revolutions per second of the propeller, and D is the diameter.

16. By means of blade element theory, with the assumption that inflow factors can be neglected, show that with the usual maximum values of the sectional lift drag ratio the local efficiency is a maximum when the helical angle is approximately 45°.

APPENDIX

SUMMARY OF BASIC APPLICATIONS OF VECTOR ANALYSIS TO FLUID MECHANICS

A.1 Introductory Remarks

The following as the above title implies provides little more than the bare fundamentals of vector analysis and its applications to the main topics of fluid mechanics as covered in this book. Detailed proofs are not offered but cross-references to relevant places in the main text are quoted to enable the reader to appreciate both the aid to physical understanding and the conciseness of presentation that are the special advantages of vector analysis. Readers unfamiliar with the essentials of vector analysis are advised as a preliminary to study a standard text book on the subject.

A.2 Notation (Vectors are characterised by heavy type)

x, y, z	Cartesian axes.
t	Time.
$\mathbf{r}$	Position vector of a point (x, y, z) relative to the origin.
r	$(x^2 + y^2 + z^2)^{1/2} = \lvert\mathbf{r}\rvert$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors parallel to the x, y, z axes, respectively.

Vector $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} = (A_x, A_y, A_z)$.

$\mathbf{A} . \mathbf{B} =$ scalar product of $\mathbf{A}$ and $\mathbf{B}$

$= A_xB_x + A_yB_y + A_zB_z = AB\cos\theta$

where $A = \lvert\mathbf{A}\rvert = (A_x^2 + A_y^2 + A_z^2)^{1/2}$, similarly $B = \mathbf{B}$

and $\theta =$ angle between $\mathbf{A}$ and $\mathbf{B}$.

$\mathbf{A} \wedge \mathbf{B} =$ vector product of $\mathbf{A}$ and $\mathbf{B}$

$= (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$

and is a vector at right angles to $\mathbf{A}$ and $\mathbf{B}$ such that a rotation from $\mathbf{A}$ to $\mathbf{B}$ is clockwise about it, its magnitude is $AB\sin\theta$.

The operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right),$$

it is sometimes called *del* or *nabla*.

$$\nabla . \mathbf{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial \mathbf{n}}\right),$$

called divergence of $\mathbf{A}$ or div $\mathbf{A}$.

$$\nabla \wedge \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}$$

called curl $\mathbf{A}$.

$\mathbf{n}$ Unit vector taken normal to the surface of a volume and outwards from the volume,

δS an element of the surface.

$\delta \mathbf{S} = \delta S \mathbf{n}$.

$\delta \tau$ an element of the volume.

$\delta \mathbf{s}$ a line element.

$$\nabla P = \frac{\partial P}{\partial x} . \mathbf{i} + \frac{\partial P}{\partial y} \mathbf{j} + \frac{\partial P}{\partial z} \mathbf{k}, \text{ called } \mathbf{grad}\, P,\ P \text{ is a scalar quantity.}$$

$$\nabla^2 P = \text{div } \mathbf{grad}\, P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} .$$

$$\nabla^2 \mathbf{A} = \frac{\partial^2 \mathbf{A}}{\partial x^2} + \frac{\partial^2 \mathbf{A}}{\partial y^2} + \frac{\partial^2 \mathbf{A}}{\partial z^2} .$$

∇^2 is generally referred to as Laplace's operator and is sometimes called del squared.

A.3 Some Important Formulae and Theorems

$$\text{Div } (P\mathbf{B}) = \mathbf{grad}\, P . \mathbf{B} + P \text{ div } \mathbf{B} = \nabla P . \mathbf{B} + P\nabla . \mathbf{B}. \tag{A.3,1}$$

$$\mathbf{Curl}\ (P\mathbf{B}) = \mathbf{grad}\, P \wedge \mathbf{B} + P\, \mathbf{curl}\, \mathbf{B} = \nabla P \wedge \mathbf{B} + P\nabla \wedge \mathbf{B}. \tag{A.3,2}$$

$$\text{Div } (\mathbf{A} \wedge \mathbf{B}) = \mathbf{B} . \mathbf{curl}\, \mathbf{A} - \mathbf{A} . \mathbf{curl}\, \mathbf{B} = \mathbf{B} . (\nabla \wedge \mathbf{A}) - \mathbf{A} . (\nabla \wedge \mathbf{B}). \tag{A.3,3}$$

$$\begin{aligned} \mathbf{Curl}\ (\mathbf{A} \wedge \mathbf{B}) &= \mathbf{B} . \nabla \mathbf{A} - \mathbf{A} . \nabla \mathbf{B} + \mathbf{A} \text{ div } \mathbf{B} - \mathbf{B} \text{ div } \mathbf{A} \\ &= (\mathbf{B} . \nabla)\mathbf{A} - (\mathbf{A} . \nabla)\mathbf{B} + \mathbf{A}(\nabla . \mathbf{B}) - \mathbf{B}(\nabla . \mathbf{A}). \end{aligned} \tag{A.3,4}$$

$$\begin{aligned} \mathbf{Grad}\ (\mathbf{A} . \mathbf{B}) &= \mathbf{B} . \nabla \mathbf{A} + \mathbf{A} . \nabla \mathbf{B} + \mathbf{A} \wedge \mathbf{curl}\, \mathbf{B} + \mathbf{B} \wedge \mathbf{curl}\, \mathbf{A} \\ &= (\mathbf{B} . \nabla)\mathbf{A} + (\mathbf{A} . \nabla)\mathbf{B} + \mathbf{A} \wedge (\nabla \wedge \mathbf{B}) + \mathbf{B} \wedge (\nabla \wedge \mathbf{A}) \end{aligned} \tag{A.3,5}$$

$$\mathbf{Curl}\ \mathbf{grad}\, P = \nabla \wedge \nabla P = 0 \tag{A.3,6}$$

$$\text{Div } \mathbf{curl}\, \mathbf{A} = \nabla . \nabla \wedge \mathbf{A} = 0 \tag{A.3,7}$$

$$\begin{aligned} \mathbf{Curl}\ \mathbf{curl}\, \mathbf{A} = \nabla \wedge \nabla \wedge \mathbf{A} &= \mathbf{grad} \text{ div } \mathbf{A} - \nabla^2 \mathbf{A} \\ &= \nabla(\nabla . \mathbf{A}) - \nabla^2 \mathbf{A}. \end{aligned} \tag{A.3,8}$$

Gauss's Divergence Theorem:—

$$\iint \mathbf{A} . \mathbf{n}\, dS = \iint \mathbf{A} . d\mathbf{S} = \iiint \nabla . \mathbf{A}\, d\tau. \tag{A.3,9}$$

The surface integral is taken over the bounding surfaces of the volume in which the volume integral is taken.

Stokes' Theorem:

$$\oint_C \mathbf{A} . d\mathbf{s} = \iint_S \mathbf{curl}\, \mathbf{A} . \mathbf{n}\, dS = \iint_S (\nabla \wedge \mathbf{A}) . d\mathbf{S} \tag{A.3,10}$$

Here C denotes a closed circuit and S any area based on it and with no other edge. See § 2.10 and 2.11.

A.4 Application to Fluid Mechanics

Equation of Continuity (see § 2.3)

For any fixed volume not enclosing sources continuity requires that

$$\iint \rho \mathbf{V} \cdot \mathbf{dS} + \iiint \frac{\partial \rho}{\partial t} \cdot d\tau = 0$$

where $\mathbf{V}$ is the velocity vector.

But from Gauss's Divergence Theorem (Equation A.3,9)

$$\iint \rho \mathbf{V} \cdot d\mathbf{S} = \iiint \nabla \cdot (\rho \mathbf{V})\, d\tau$$

and making the volume vanishingly small to tend to a point P we have that in the limit at that point

$$\left.\begin{aligned} &\nabla \cdot (\rho \mathbf{V}) + \frac{\partial \rho}{\partial t} = 0 \quad \text{(see equation 2.3,7)} \\ \text{or} \quad &\nabla \cdot \mathbf{V} + \frac{1}{\rho}\frac{D\rho}{Dt} = 0. \end{aligned}\right\} \qquad (\text{A.4,1})$$

If the fluid is incompressible

$$\nabla \cdot \mathbf{V} = 0 \text{ (see equation 2.3,9).}$$

A vector such that its divergence is zero is referred to as *solenoidal.*

Boundary Conditions (see § 2.6).

At any point of a bounding surface we must have

$$\mathbf{n} \cdot \mathbf{V} = \mathbf{n} \cdot \mathbf{V}_B \qquad (\text{A.4,2})$$

where $\mathbf{V}_B$ is the velocity of the boundary.

Velocity fields of sources and doublets (see § 2.7)

For a point source at the origin of strength σ the velocity at point $\mathbf{r}$ is

$$\mathbf{V} = \sigma \mathbf{r}/4\pi r^3 = -\frac{\sigma}{4\pi} \nabla\left(\frac{1}{r}\right). \qquad (\text{A.4,3})$$

For a uniform line source, i.e. a two dimensional source of strength m at the origin

$$\mathbf{V} = m\mathbf{r}/2\pi r^2 = \frac{m}{2\pi} \nabla(\ln r). \qquad (\text{A.4,4})$$

For a doublet of strength $\boldsymbol{\mu}$ at the origin, for which the direction of $\boldsymbol{\mu}$ is defined by the axis of the doublet

$$\left.\begin{aligned} \mathbf{V} &= \frac{\nabla}{4\pi}(-\boldsymbol{\mu}\,.\,\mathbf{r}/r^3) \\ &= \frac{1}{4\pi}\left[-\frac{\boldsymbol{\mu}}{r^3}+\frac{3(\boldsymbol{\mu}\,.\,\mathbf{r})}{r^5}\,.\,\mathbf{r}\right]. \end{aligned}\right\} \qquad \text{(A.4,5)}$$

For a two dimensional doublet of strength $\boldsymbol{\mu}'$ at the origin

$$\left.\begin{aligned} \mathbf{V} &= \frac{\nabla}{2\pi}(-\boldsymbol{\mu}'\,.\,\mathbf{r}/r^2) \\ &= \frac{1}{2\pi}\left[\frac{-\boldsymbol{\mu}'}{r^2}+\frac{2(\boldsymbol{\mu}'\,.\,\mathbf{r})}{r^4}\,.\,\mathbf{r}\right]. \end{aligned}\right\} \qquad \text{(A.4,6)}$$

Rate of change with time of a physical property of a fluid particle (see § 2.9)

If α is the property in question, which can be a scalar or vector, then

$$\frac{D\alpha}{Dt}=\frac{\partial\alpha}{\partial t}+\mathbf{V}\,.\,\nabla\alpha.$$

Put $\alpha = \mathbf{V}$, then the acceleration of the fluid particle is

$$\mathbf{a}=\frac{D\mathbf{V}}{Dt}=\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\,.\,\nabla\mathbf{V}. \qquad \text{(A.4,7)}$$

Circulation (see § 2.10)

The flow along a curve from point A to point B is $\int_A^B \mathbf{V}\,.\,d\mathbf{s}$.
The circulation for a circuit C is then

$$\Gamma = \oint_c \mathbf{V}\,.\,d\mathbf{s}. \qquad \text{(A.4,8)}$$

Vorticity (see § 2.11)

The vorticity vector is

$$\boldsymbol{\omega} = \text{curl}\,\mathbf{V} = \nabla \wedge \mathbf{V}. \qquad \text{(A.4,9)}$$

From (A 3,7)

$$\text{div}\,\boldsymbol{\omega} = 0 \qquad \text{(A.4,10)}$$

i.e. $\boldsymbol{\omega}$ is solenoidal.

From Stokes' Theorem (A.3,10) we see that for any circuit C

$$\Gamma = \oint_C \mathbf{V}\,.\,d\mathbf{s} = \iint_S \text{curl}\,\mathbf{V}\,.\,d\mathbf{S} = \iint \boldsymbol{\omega}\,.\,\mathbf{n}\,dS \qquad \text{(A.4,11)}$$

where S is any finite surface lying within the fluid based on C as its only edge (cf. equation (2.11,7)).

Velocity Potential (see § 2.12)

The velocity potential ϕ is a scalar function such that for irrotational flow ($\boldsymbol{\omega} = 0$)

$$\mathbf{V} = -\mathbf{grad}\,\phi = -\nabla\phi. \tag{A.4,12}$$

For a point source of strength σ at the origin

$$\phi = \frac{\sigma}{4\pi r}. \tag{A.4,13}$$

For a uniform line source (i.e. two dimensional source) of strength m at the origin

$$\phi = -\frac{m}{2\pi}\ln r. \tag{A.4,14}$$

For a doublet $\boldsymbol{\mu}$ at the origin

$$\phi = \boldsymbol{\mu}\,.\,\mathbf{r}/4\pi r^3. \tag{A.4,15}$$

For a two dimensional doublet $\boldsymbol{\mu}'$ at the origin

$$\phi = \boldsymbol{\mu}'\,.\,\mathbf{r}/2\pi r^2. \tag{A.4,16}$$

Straight Line Vortex of Strength $\boldsymbol{\Gamma}$ *Passing Through Origin* (*see* § 2.13)

$$\mathbf{V} = \boldsymbol{\Gamma} \wedge \mathbf{r}/2\pi r^2. \tag{A.4,17}$$

Curved Vortex Line (see § 2.13)

$$\mathbf{V} = \oint (\boldsymbol{\Gamma} \wedge \mathbf{r}/4\pi r^3)\,ds = -\Gamma \oint \mathbf{r} \wedge d\mathbf{s}/4\pi r^3 \tag{A.4,18}$$

where $\mathbf{r}$ is now the position vector of the point to which $\mathbf{V}$ refers relative to the element ds. We see that we can interpret the contribution of the element $\Gamma\mathbf{ds}$ of the vortex to the velocity as

$$d\mathbf{V} = -\Gamma\mathbf{r} \wedge \mathbf{ds}/4\pi r^3 \qquad \text{(cf. equation (2.13,14)).}$$

The Dynamical Equations of an Inviscid Fluid (see § 3.5)

Equating as in § 3.5 the acceleration of a fluid particle to the applied forces (body plus pressure) we get

$$\frac{D\mathbf{V}}{Dt} = -\nabla\chi - \nabla\varpi \tag{A.4,19}$$

where χ is the potential of the body force $\mathbf{F}_B$, i.e. $\mathbf{F}_B = -\nabla\chi$, and $\varpi = \int dp/\rho$. But

$$\frac{D\mathbf{V}}{Dt} = \frac{\partial\mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}.$$

Put $\mathbf{A} = \mathbf{B} = \mathbf{V}$ in equation (A.3,5) and we get

$$\mathbf{grad}\,(V^2) = 2\mathbf{V}\,.\,\nabla\mathbf{V} + 2\mathbf{V} \wedge \operatorname{curl} \mathbf{V}$$

where $$V^2 = V_x^2 + V_y^2 + V_z^2$$

and hence $$\nabla\,.\,\nabla\mathbf{V} = \mathbf{grad}\,(V^2/2) - \mathbf{V} \wedge \boldsymbol{\omega}.$$

Therefore

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{grad}\left(\frac{V^2}{2}\right) - \mathbf{V} \wedge \boldsymbol{\omega} = -\nabla(\chi + \varpi). \qquad \text{(A.4,20)}$$

Take the **curl** of both sides of this equation, then from (A.3,6) it follows that

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \mathbf{curl}\,(\mathbf{V} \wedge \boldsymbol{\omega}). \qquad \text{(A.4,21)}$$

If the flow is irrotational, and hence $\mathbf{V} = -\nabla\phi$ and $\boldsymbol{\omega} = 0$, it follows from (A.4,20) that

$$\Delta\left[\chi + \varpi + \frac{V^2}{2} - \frac{\partial\phi}{\partial t}\right] = 0$$

Hence, for irrotational flow

$$\chi + \varpi + \frac{V^2}{2} - \frac{\partial\phi}{\partial t} = F(t) \qquad \text{(A.4,22)}$$

where $F(t)$ is a function of time determined by the boundary conditions (see equation 3.5,13). If the motion is steady then

$$\chi + \varpi + \frac{V^2}{2} = \text{const.} \qquad \text{(A.4,23)}$$

Reverting to equation (A.4,20) we see that for steady motion

$$\mathbf{V} \wedge \boldsymbol{\omega} = \nabla\left(\chi + \varpi + \frac{V^2}{2}\right)$$

and if we multiply both sides by $\mathbf{V}$ it then follows that

$$\mathbf{V}\,.\,\nabla\left(\chi + \varpi + \frac{V^2}{2}\right) = 0$$

and hence $\chi + \varpi + V^2/2 = \text{const.}$ along a streamline. (A.4,24)

This last relation is Bernoulli's Theorem (see § 3.4).

The Equations of Momentum (see § 3.6)

Equating, as in § 3.6, the rate of change of momentum of the fluid (assumed inviscid) enclosed by a surface S to the forces acting on that fluid we obtain

$$\iiint \frac{\partial}{\partial t}(\rho \mathbf{V})\, d\tau + \iint \rho \mathbf{V}(\mathbf{n}\,.\,\mathbf{V})\, dS = \iiint \rho \mathbf{F}_B\, d\tau - \iint p\mathbf{n}\, dS - \mathbf{F}$$

where $\mathbf{F}_B$ is the body force per unit mass and $\mathbf{F}$ is the force exerted by the fluid on an enclosed body.

Rearranging we get (cf. equation (3.6,6)

$$\iiint \left[\rho \mathbf{F}_B - \frac{\partial}{\partial t}(\rho \mathbf{V})\right] d\tau = \iint \rho \mathbf{V}(\mathbf{n}\,.\,\mathbf{V})\, dS + \iint p\mathbf{n}\, dS + \mathbf{F}. \quad (\text{A.4,25})$$

For steady motion and in the absence of body forces we get

$$\mathbf{F} = -\iint \rho \mathbf{V}(\mathbf{n}\,.\,\mathbf{V})\, dS - \iint p\mathbf{n}\, dS. \quad (\text{A.4,26})$$

Similarly from the Principle of Angular Momentum we obtain

$$\iiint \left[\rho \mathbf{r} \wedge \mathbf{F}_B - \frac{\partial}{\partial t}(\rho \mathbf{r} \wedge \mathbf{V})\right] d\tau = \iint [p\mathbf{r} \wedge \mathbf{n} + \rho \mathbf{r} \wedge \mathbf{V}(\mathbf{n}\,.\,\mathbf{V})]\, dS + \mathbf{M} \quad (\text{A.4,27})$$

where $\mathbf{M}$ is the moment exerted by the fluid on the enclosed body. Here $\mathbf{r}$ is the position vector of a point in the fluid referred to an origin on the axis about which the angular momentum is determined.

Again for steady motion and in the absence of body forces we have

$$\mathbf{M} = -\iint [p\mathbf{r} \wedge \mathbf{n} + \rho \mathbf{r} \wedge \mathbf{V}(\mathbf{n}\,.\,\mathbf{V})]\, dS. \quad (\text{A.4,28})$$

Kelvin's Circulation Theorem (see § 3.7)

We have

$$\Gamma = \oint \mathbf{V}\,.\,d\mathbf{s}$$

and hence

$$\frac{D\Gamma}{Dt} = \oint \frac{D\mathbf{V}}{Dt}\,.\,d\mathbf{s} + \oint \mathbf{V}\,.\,\frac{D}{Dt}(d\mathbf{s}).$$

For a circuit moving with the fluid

$$\frac{D}{Dt}(d\mathbf{s}) = \frac{\partial \mathbf{V}}{\partial s}\, ds$$

and

$$\oint \mathbf{V}\,.\,\frac{D}{Dt}(d\mathbf{s}) = \oint \frac{\partial}{\partial s}\left(\frac{V^2}{2}\right) ds = 0$$

Also for an inviscid fluid

$$\oint \frac{D\mathbf{V}}{Dt}\,.\,d\mathbf{s} = -\oint \nabla(\chi + \varpi)\,.\,d\mathbf{s} = -\oint \frac{\partial}{\partial s}(\chi + \varpi)\, ds$$

and hence is zero if χ and ϖ are single valued functions of position. The latter requires that the fluid is barotropic. With these conditions satisfied it follows that for an inviscid fluid

$$\frac{D\Gamma}{Dt} = 0. \qquad \text{(A.4,29)}$$

Flow in a Rotating Channel (see § 3.11)

If a vector **A** is referred to a frame of axes of origin 0 and this frame is rotating with constant angular velocity $\mathbf{\Omega}$ about 0 relative to a fixed frame of reference then referred to the latter

$$\frac{d\mathbf{A}}{dt} = \left(\frac{d\mathbf{A}}{dt}\right)_1 + \mathbf{\Omega} \wedge \mathbf{A}$$

where suffix 1 denotes quantities as measured in the rotating frame. Thus, if $\mathbf{r}$ is the position vector of a fluid particle relative to 0, then for that particle

$$\mathbf{V} = \left(\frac{d\mathbf{r}}{dt}\right) = \left(\frac{d\mathbf{r}}{dt}\right)_1 + \mathbf{\Omega} \wedge \mathbf{r} = \mathbf{V}_1 + \mathbf{\Omega} \wedge \mathbf{r}$$

and its acceleration is

$$\begin{aligned}\mathbf{a} &= \left(\frac{d\mathbf{V}}{dt}\right)_1 + \mathbf{\Omega} \wedge \mathbf{V}\\ &= \mathbf{a}_1 + 2\mathbf{\Omega} \wedge \mathbf{V}_1 + \mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge \mathbf{r}).\end{aligned}$$

But

$$\mathbf{a}_1 = D\mathbf{V}_1/Dt = \partial\mathbf{V}_1/\partial t + (\mathbf{V}_1 \cdot \nabla)\mathbf{V}_1$$

and so

$$\mathbf{a} = D\mathbf{V}_1/Dt + 2\mathbf{\Omega} \wedge \mathbf{V}_1 + \mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge \mathbf{r}). \qquad \text{(A.4,30)}$$

The 2nd term on the R.H.S. is minus the so-called Coriolis force and the 3rd term is minus the centrifugal force. The boundary condition at the rotating boundary, if solid, is

$$\mathbf{V}_1 \cdot \mathbf{n} = 0.$$

Continuity requires that

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{V}_1) = 0, \quad \text{with} \quad \mathbf{V}_1 = \mathbf{V} - \mathbf{\Omega} \wedge \mathbf{r}.$$

The dynamical equation for an inviscid fluid is

$$\mathbf{a} = -\nabla(\chi + \varpi)$$

and so

$$\left.\begin{aligned}&\frac{D\mathbf{V}_1}{Dt} + 2\mathbf{\Omega} \wedge \mathbf{V}_1 + \mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge \mathbf{r}) = -\nabla(\chi + \varpi)\\ \text{or}\quad &\frac{D\mathbf{V}_1}{Dt} + 2\mathbf{\Omega} \wedge \mathbf{V}_1 - \Omega^2\mathbf{r} + (\mathbf{\Omega} \cdot \mathbf{r})\mathbf{\Omega} = -\nabla(\chi + \varpi)\\ \text{or}\quad &\frac{D\mathbf{V}_1}{Dt_1} + 2\mathbf{\Omega} \wedge \mathbf{V}_1 - \tfrac{1}{2}\nabla[(\mathbf{\Omega} \wedge \mathbf{r})^2] = -\nabla(\chi + \varpi).\end{aligned}\right\} \qquad \text{(A.4,31)}$$

If the motion relative to the rotating axis is steady this last equation can be rearranged to give

$$\nabla\left[\chi + \varpi + \frac{V_1^2}{2} - \tfrac{1}{2}(\boldsymbol{\Omega} \wedge \mathbf{r})^2\right] = \mathbf{V}_1 \wedge (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \qquad (A.4,32)$$

Multiplying both sides by $\mathbf{V}_1$ we then deduce that

$$\chi + \varpi + \frac{V_1^2}{2} - \tfrac{1}{2}(\boldsymbol{\Omega} \wedge \mathbf{r})^2 = \text{const. along a stream line.} \quad (A.4,33)$$

If $\boldsymbol{\Omega}$ and $\mathbf{r}$ are orthogonal then $(\boldsymbol{\Omega} \wedge \mathbf{r})^2 = \Omega^2 r^2$ and then

$$\chi + \varpi + \frac{V_1^2}{2} - \frac{\Omega^2 r^2}{2} = \text{const. along a stream line.}$$

The Equations of Motion for a Viscous, Incompressible Fluid
The Navier-Stokes Equations (see § 11.1 and Examples 1–6 of Ch. 11)

In tensor notation the stress components in a viscous fluid can be written

$$p_{\alpha\beta} = -p \,.\, \delta_{\alpha\beta} + 2\mu e_{\alpha\beta}, \qquad (A.4,34)$$

where

$$\left.\begin{aligned} \delta_{\alpha\beta} &= 0, \quad \text{if} \quad \alpha \neq \beta \\ &= 1, \quad \text{if} \quad \alpha = \beta \\ e_{\alpha\beta} &= \frac{1}{2}\left[\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right]. \end{aligned}\right\} \qquad (A.4,35)$$

and

The equations of motion are

$$\begin{aligned} \rho \frac{Dv_\alpha}{Dt} &= \rho F_\alpha + \partial p_{\alpha\beta}/\partial x_\beta \\ &= \rho F_\alpha - \partial p/\partial x_\alpha + \mu[\partial^2 v_\alpha/\partial x_\beta^2 + \partial^2 v_\beta/\partial x_\alpha \,\partial x_\beta] \\ &= \rho F_\alpha - \frac{\partial p}{\partial x_\alpha} + \mu \nabla^2 v_\alpha, \end{aligned} \qquad (A.4,36)$$

since

$$\frac{\partial^2 v_\beta}{\partial x_\alpha \,\partial x_\beta} = \frac{\partial}{\partial x_\alpha}\left(\frac{\partial v_\beta}{\partial x_\beta}\right) = \frac{\partial}{\partial x_\alpha}(\text{div } \mathbf{V}) = 0.\dagger$$

In vector notation (A.4,36) becomes

$$\begin{aligned} \rho\left(\frac{\partial \mathbf{V}}{\partial t} - \mathbf{V} \wedge \boldsymbol{\omega} + \nabla \frac{V^2}{2}\right) &= \rho \mathbf{F} - \mathbf{grad}\, p + \mu \nabla^2 \mathbf{V} \\ &= \rho \mathbf{F} - \mathbf{grad}\, p - \mu\, \mathbf{curl}\, \boldsymbol{\omega}. \end{aligned} \qquad (A.4,37)$$

The reader is warned that the equation is less simple for a compressible fluid since then div $\mathbf{V} \neq 0$ and μ as well as ρ are not constant.

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† The convention in tensor notation is that if a suffix is repeated in any single expression then summation is implied over all the possible values that suffix can take.

INDEX

Abbott, I. W., 580
Accelerated motion of cylinder, 148, 153, 154
Acceleration along streamline, 67, 128, 129
—, law of, 124
— of fluid particle, 66, 708
— potential, 171
Ackeret, J., 507
Ackeret's Law, 507
Active, H.M.S., 213
Actuator disk, 674
— —, efficiency of, 676
Adams, MacC., 610
Addison, H., 637, 660
Addition of motions, 137
Adiabatic process, 467
Advance ratio of propeller, 671
Aeolipile, 651
Aerofoil section, aerodynamic centre, 583, 584
— —, camber, 580, 583, 597
— —, centre of pressure, 583, 584
— —, drag, 582
— —, equivalent source distribution, 597 et seq.
— —, geometric and aerodynamic characteristics, 579 et seq.
— —, ideal angle of attack, 590
— —, lift, 581, 582, 584 et seq., 596
— —, maximum lift coefficient, 582, 597
— —, pitching moment, 582, 583, 584 et seq., 594, 596, 597
— —, pressure distribution, 581
— —, stall, 581, 582
— —, thickness, 579, 580
Aerofoils, boundary layer, control, 369 et seq.
—, — — or profile drag, 353 et seq.
—, — — pressure drag or form drag, 350 et seq.
—, lift, 368, 379 et seq.
Air, density, 28
—, gas constant for, 31
—, ratio of specific heats, 31
— speed, equivalent, 224
—, standard density of, 28
— vessel, 561
—, viscosity of, 30
Aircraft axes, 570
— stability, 616
— — derivatives, 615 et seq.
Airscrews, 696
Alphabet, Greek, vi
Amplitude of wave, 528
Anemometer, hot-wire, 235
—, Robinson cup, 223
—, vane, 223
Angle of incidence defined, 207
Archimedes, principle of, 24
Area coefficient of runner, 636
— rule, 521
Aspect ratio, 382, 385
Atmosphere, standard, 13
Atwell, N. P., 349
Average depth of channel, 439

Babister, A. W., 452
Bar, 27
Barker's mill, 651
Barotropic fluid, 12, 14, 133
Barr, G., 398, 403, 452
Barye, 27
Batchelor, G. K., 294
Beats 565
Bends, flow in, 410
Bernoulli's theorem, 128, 472, 475, 710
— —, extension to unsteady motion, 129
— — for barotropic fluid, 137
— —, generalization for rotating channel, 168
Betz, A., 364, 688
Bickley, W., 270
Binnie, A. M., 215, 424
Binormal to streamline, 135
Black, J., 215
Blade element theory of propeller, 677
— tip effects, 687
Blasius, H., 260, 266, 312, 314, 407
Blasius theorems, 592 et seq.
Blockage effect, 220
Bodies of revolution, boundary layer or profile drag, 356, 358
— —, boundary layer pressure drag or form drag, 356
Body force 4, 14, 15, 128
Bohlen, T., 273, 280
Borda mouthpiece, 453
Bound vorticity, 602 et seq.
Boundary conditions, 49, 707
— layer, 237 et seq., 572 et seq.
— — control, 369 et seq.
— — displacement thickness, 251, 257 et seq., 262, 272, 311, 320, 350 et seq., 368, 574

Boundary layer, drag, 243, 350 et seq., 572, 623
— —, energy integral equation, 281
— —, energy thickness, 252, 282, 388
— —, general properties, 239 et seq.
— —, laminar, 239, 259 et seq.
— — —, Blasius solution, 259 et seq., 373
— —, momentum integral equation, 254 et seq., 270, 271, 274, 278, 315, 330, 333, 355 et seq., 374
— —, momentum thickness, 252, 257 et seq., 262, 272, 311, 315, 318, 320, 330 et seq., 355 et seq., 388
— — pressure drag, 242, 349 et seq., 572
— — separation, 241, 266, 267, 273, 378 et seq., 385, 480
— —, similar solutions, 259, 263 et seq., 373
— — stability, 239, 244, 286 et seq., 374
— — thickness, 240, 249, 257 et seq., 271 et seq., 299, 308 et seq., 318, 319
— —, transition, 239, 244 et seq., 283, 289, 290, 291 et seq., 330 et seq.
— —, transition point, 244, 330
— —, turbulence in, 291 et seq.
— —, turbulent, 239, 294 et seq.
— — velocity profile, laminar, 240, 271, 277, 293
— — — —, turbulent, 240, 293, 302, 304, 305, 307 et seq.
Bourdon pressure gauge, 233
Boussinesq, J., 398
Branched pipes, 415
Bradshaw, P., 349
Braslow, A. L., 375
Bray, R. S., 380
Broad-bladed propellers, 688
Bruynes, H., 386
Bryant, C. N., 317, 404
Bucket pumps, 631
Bulb turbine, 634
Bulk modulus, 5, 459
— —, effective, of fluid in pipe, 560
Buoyancy, 24
—, centre of, 25
—, horizontal, in tunnel, 221
—, surface of, 34
Buri, A., 344
Burrows, D. L., 375
Burstall, F. H., 235
Busemann, A., 525
Bussmann, K., 374

Calculation of lift of a wing section, 368
Capillary waves, 549
Carter, A. D. S., 622
Cartesian diver, 25
Cascades, 618 et seq.
Cast-off vortex, 573, 574
Cavitation, 168
— in hydraulic machines, 659
— number, critical, 170
— tunnels, 219
Centipoise, 30
Centistokes, 30
Centre of pressure, 19
Centrifugal potential, 168
— pressure, 644
— pumps, 635, 657
C.G.S. system, 27
Channels, flow in, 424
— of arbitrary section, flow of variable depth in, 438
Characteristic equation, 5
Chézy coefficient for channel, 426
— formula for pipe resistance, 408
"China clay" method of detecting transition, 235
Choked flow, 479
Churo, F., 381
Circle theorem, 112
Circular cylinder in uniform stream, 65, 99, 147
— —, virtual mass of, 153
— notch, 452
Circulation, 69, 74, 368, 384, 571 et seq., 608, 613, 620, 708
— theorem, Kelvin's, 142
Clauser, 322
Clifford, W. K., 89
Cloud chamber, 168
Coefficient of contraction, 453
— — discharge, 227
Coherent units, 189
Coles, D., 300, 309, 315, 317, 322, 323, 325, 341, 342, 348
Collar, A. R., 622, 700
Collie, B., 10
Collins, G. A., 386
Combination of motions, 48, 80, 84
Compatibility, conditions of, 137
Complex potential, 94
— variable, 94
Compressible flow in stream-tube, 462, 463, 473, 478 et seq.
— fluid, 459 et seq.
Compressibility of a fluid, 5
Confocal coordinates, 106
Conformal property, 100
— transformation, 100, 594 et seq., 621
Conical flow, 613
Conjugate functions, 94
Conn, J. F. C., 214
Conservative body force, 14
— field, 12, 14
Constant speed propellers, 697

Continuity, 41
—, equation of, 43, 44, 707
— of flow in rotating channel, 166
Contra-rotating propellers, 697
Control surface, 141
Convection, 201
Convex function, 172
Conveyance factor of channel, 427
Corner, flow in, 97
Couette flow, 158
Courant, R., 513
Critical condition in branched pipe, 419
— depth of channel, 432, 440
Cross, Hardy, 419
Curl of a vector, 80, 705
Curvature of streamline, 67, 69
Curved surface, thrust on, 24
Cusec, definition of, 29
Cyclic interchange of symbols, 75

d'Alembert, 95
Danel, P., 634
Dannenberg, R. E., 380, 384
Denny, M. E., 213
Density of air, 28
— of fluid, 2
Design of ship's propellers, 694
Developable surface, 668
Devereaux, A. N., 380
Dhawan, S., 317
Diehl, Z. W., 315
Diffusers, 413
Diffusion, equation of, 159, 162
Diffusivity of temperature, 201
Dimensional analysis, 191
— —, application to flow in pipes, 393
— homogeneity, principle of, 192
Dipole, 54
Discharge, measurement of, 226
— number, 637
Disk, actuator, 674
— friction, 648
Dispersive wave propagation, 529, 542
Divergence of a vector, 80, 171, 705
— of velocity, 44
Dividing streamline, 60
Doublet, 54, 58, 707
Douglas, O., 381
Downwash, 603
Drag, 570, 571 et seq., 621
—, bluff bodies, 242
—, boundary layer or profile, 243, 350 et seq.
—, boundary layer pressure or form, 243, 350 et seq., 356, 358
— coefficient, 208, 509, 518
— defined, 207
— due to roughness, 245, 326 et seq.
Drag, flat plate at zero incidence, 243, 262, 314 et seq.
—, induced, 350
— of body in perfect fluid, 152
—, pressure, 242, 350 et seq., 356, 358
—, skin friction, 242, 257 et seq., 262, 311 et seq., 328, 329, 330 et seq.
Draught tube, 635
Duncan, W. J., 179, 191, 203, 221, 379, 403, 582, 618, 695, 698
Durand, W. F., 285, 314
Dynamic pressure, 148, 224
Dynamical equations, 142, 709
— — for fluid, 133
— principles, 124
— — of rotodynamic machines, 642
— similarity, 180

Earn, Loch, 530
Eckhardt, M. K., 688, 694
Eddy shear stress, 239, 285, 296, 300, 301 et seq.
— stresses, 284 et seq., 294, 296, 300, 301 et seq.
Efficiency of actuator, 676
— of hydraulic machine, measurement of, 660
— of propeller, 672
— of propeller blade element, 678
— of windmill, 701
Efficiency, thermodynamic 661
—, propulsive, 694
Ehlers, F., 383
Elliptic coordinates, 106
— cylinder, circulation about, 109
— — in uniform stream, 123
Ellis, D. L., 219
Emmons, H. W., 291
Energy balance in boundary layer, 285, 310
—, principle of, 125
—, rate of dissipation of, in channel, 430
Enthalpy or total heat, 468, 661
Entropy, 469 et seq., 480, 661
Entry to pipe, phenomena near, 397
Equation of continuity, Lagrangean, 118 707
— of state, 5, 463, 464
Equilibrium of fluids, 10
Equipotentials, 82
Equivalent air speed, 224
Erk viscometer, 421
Error function, 159
Euler's dynamical equations, 133
— relations for rotodynamic machines, 643
Eulerian method, 37
Expansion, 44
— waves, 480

Fage, A., 316, 405
Falkner, V. M., 264, 316
Fans, 698
—, design of, 699
Ferri, A., 513
Ferris, D. H., 349
Filament line, 38
Fin effect of propeller, 697
First Law of Thermodynamics, 464, 465 473
Flaps, 379, 384 et seq.
Flat plate in normal stream, 106
Flotation, surface of, 34
Flow along a curve, 69, 74, 82
— in a channel, 262, 263, 296
— in nozzle, 479, 480
— in pipe, 262, 263, 296, 305, 306, 308, 310, 393 et seq.
— in pipe, dimensional analysis applied to, 393
— separation, 241, 581, 582, 614, 615, 623
Fluid, definition of, 1
—, types of, 1
Fluids, similar, 203
Flumes, 215
Flux, 42
— across a curve, 44
Force de cheval, 31
— on body, 140, 568 et seq.
Francis formula for rectangular notch, 449
— turbine, 633, 655
Free molecule flow, 523
— turbulence, 365 et seq.
Frequency parameter, 204
Friction factor, 409
— velocity, 297, 302 et seq.
— —, defined, 395
Friedrichs, K., 513
Frössling, N., 266
Froude, R. E., 693, 695
Froude, William, 213, 454
Froude number, 182, 195
Froude's law of comparison, 185, 555
Fully developed pipe flow, 398
Function of state, 464

Gallon, Imperial 28
—, U.S., 28
Garner, H. C., 345, 346
Garrick, I. E., 622
Gas constant, 6, 30, 464
Gaster, M., 289
Gault, D. E., 383
Gauss's Divergence Theorem, 706
Gear pumps, 631
Geneva, Lake of, 530
Gibson, N. R., 230
Giffard injector, 173
Ginzel, G. I., 688
Glauert, H., 223, 503, 588, 589, 596, 609, 685
Glauert, M. B., 381
Glide angle of propeller blade profile, 678
Gliders, towing of, 214
Goethert, B. H., 222
Goldstein, A. W., 622
Goldstein, S., 249, 260, 269, 300, 364, 372, 376, 394, 502, 597, 599
Goldstein's propeller theory, 688
Görtler, H., 290, 376
Gradient of a scalar, 706
Gradings of thrust and torque, non-dimensional, 679
Gradual slope of channel, 442
Graphical solution of branched pipe problem, 419
Grashof number, 202
Gray, W. E., 235, 377
Greek alphabet, vi
Gregory, N., 380
Greyhound, H.M.S., 213
Griffith, A. A., 372, 376
Group velocity of wave, 529, 543

Hagen, G., 396, 402
Hagen-Poiseuille formula, 402
Haines, A. B., 521
Half body, 61, 65
Hanlon, 454
Harmonic function, 83, 94
Hartree, D. R., 264
Hawthorne, W. R., 625
Hayes, W. D., 523
Head, M. R., 282, 346, 347, 349, 374, 375
Head, 12
—, defined, 129
—, measurement of, 230
— number, 638
Heart, animal, 631
Helicoid, 666, 669
Helicoidal surface, area of, 670
— —, general, 670
Helix, 667, 669
Helmbold, H. B., 694
Helmholtz, H., 90
Helmholtz's theorem, 144
Hero of Alexandria, 651
Hiemenz, K., 266
Hilton, W. F., 364
Holder, D. W., 216, 218, 219, 222, 234, 235, 236, 292, 513
Holstein, H., 273, 280
Hook gauge, 233
Hopkin, H. R., 618
Horse power, definition of, 31
Horse-shoe vortex, 91
Hot-wire anemometer, 235

Howarth, L., 201, 219, 234, 239, 260, 266, 267, 513
Howell, A. R., 622
Hughes, H. P., 622
Hull efficiency, 694
Hurley, D. G., 386
Hydraulic analogy of supersonic flow, 215
— jump, 434
— —, analogy with plane shock wave, 437
— mean depth of channel, 425
— — — of pipe bore, 399
Hydraulically smooth pipe, 407
Hypersonic flow, 521 et seq.

Iglisch, R., 373
Images, method of, 109
Impeller pump, 635
Impulse turbines, 632
Indicated H.P., 696
Indicial equations, 193
Induced downwash, 603, 604 et seq.
— drag, 604, 606 et seq., 614
— velocity of vortex, 85, 92
Inertia-pressure method, 230
Inflow factor, axial, 676, 677, 685
— —, rotational, 677, 685
Inner velocity region of pipe, 407
— — law, 302
Innis, R. C., 380
Internal energy, 465
Inverse points with respect to circle, 111
Irreducible circuit, 83
Irrotational flow, 81, 472, 475
Isentropic process, 470
Isobar, 174
Isobaric surface, 174
Isolated propeller, 666

Jaeger, C., 562
Jeans, J. H., 415
Jerison, M., 622
Jet impact, 127
— reaction, 126
Jets, 252 et seq., 365 et seq.
Jones, B. M., 360, 375
Jones, R. T., 610
Joukowsky, N., 368, 574, 577, 584, 589 596
Joukowsky profile, 104
— transformation, 102, 596
Joule (unit), 31
Joule's equivalent, 31

Kaplan turbine, 634, 656
Kármán, see von Kármán
Kay, J. M., 373
Kelvin, Lord, 6, 91, 109
Kelvin's circulation theorem, 142, 571, 711
— theorem on impulsive generation of motion, 152
Kempf, G., 317, 322
Kilowatt, 31
Kinematic theory, status of, 113
— viscosity, 8, 248
— waves, 541
Kinematics, 36
Kinetic energy of fluid, 151
— theory of gases, 464
King, L. V., 236
Klebanoff, P. S., 291, 292, 293, 297, 298, 315, 337, 347
Knapp, R. T., 220
Knot, (unit of speed), 29
—, definition of, 29, 695
Knudsen number, 523
Kutta, W. M., 368, 574, 577, 584, 589
Kutta-Joukowsky condition, 368, 574, 584, 585, 589
— — relation, 577
— — theorem, 574, 584

Lachmann, G. V., 376
Lackenby, H., 213
Lagrangean method, 37
Lamb, H., 90, 114, 127, 163, 171, 455, 550, 552
Lamé's theory of thick cylinders, 560
Laminar boundary layer equations, 246 et seq., 259 et seq., approximate methods of solution, 270 et seq.
— — — stability, 283, 286 et seq.
— — layers on bodies of revolution, 249, 250, 273
Laminar flow, in circular tube, 400
— — in pipes, 400
— — in rectangular tube, 403
— sub-layer, 285, 296, 297, 298, 301 et seq. 305, 307, 316, 427
— — — in pipe, 404, 407
— — —, thickness of, 405, 427
Laminar wake, 267
Lamont, P. A., 409
Lanchester, F. W., 604
Lanchester-Prandtl lifting line theory, 604 et seq.
Laplace, P.S., 540
Laplace's equation, 83, 94
— — in polar coordinates, 106
— operator, 706
Lapse rate, 13
Laurmann, J. A., 604, 609, 618
Law of corresponding times, 183
Law of the wall, 302
Lea, F. C., 228
Leading edge slots, 379
— — suction force, 586, 587
— — vortex sheet, 614, 615
Leakage losses, 648
Lees, C. H., 407

Liepmann, H. W., 290, 317
Lift, 570, 571 et seq., 581, 582, 584 et seq., 596, 605, 606, 612, 613, 620
— coefficient, 208, 504, 507, 509, 514 et seq.
— defined, 208
Lighthill, M. J., 237, 381, 541, 597, 622
Lin, C. C., 286
Line source, 53
Linearised perturbation potential function theory for compressible flow, 501 et seq.
Linearization, 172
Loads on bearings, hydrodynamic, 645
Lock, C. N. H., 688
Lock, O. H., 364
Losses in rotodynamic machines, 647
Lucy Ashton, S.S., 213
Ludwieg, H., 317, 337, 341, 342, 343, 344, 349
Luxton, R. E., 270, 280

Maas, J. N., 364
McCullough, G. R., 383
Mach angle, 461, 462
— cone, 461, 462
— number, 7, 187, 200, 459 et seq.
— waves, 462, 482 et seq., 496, 497, 505 et seq., 511, 512
Macmillan, D. H., 530, 553
MacMillan, F. A., 364
Magnus effect, 579
Mangler, W., 273
Manometer, U tube, 231
Manometric efficiency, 659
— head, 657
Marshall, D., 316, 404
Maskell, E. C., 347, 348
Maximum lift coefficient, 380, 384, 385
Maxwell, J. Clerk, 7
Measure formulae, 189
— —, table of, 190
— of physical quantity, 188
Mediratta, O. P., 215
Membrane analogy for laminar flow in tube, 403
Merchant, W., 622
Meredith, W., 372
Metastability, 168
Meters, positive and rotary, 228
Method, Eulerian, 37
—, Lagrangean, 37
— of characteristics, 509 et seq.
— of images, 109
Meyer, T., 489
Milne-Thomson, L. M., 112, 171
Minimum phase velocity of sea waves, 549
— property of kinetic energy in irrotational motion, 152
Minimum property of laminar flow in pipe systems, 415
Mixing length, 299
Model basins, 214
Modulus, bulk, 5, 459, 538, 560, 561
—, isentropic, 5
—, isothermal, 5
Moilliet, J. L., 10
Molecular mean free path, 522, 523
Moment on body, 141
Momentum equations for channel, 428
—, equations of, 138, 710
—, principles, of 124
Morgan, W. B., 688, 694
Morley, A., 560
Morris, D. E., 330
Motzfeld, H., 295
Multiply connected region, 83
Munk, M., 609
Münz, H., 374

Nappe, 446
National Physical Laboratory, 214, 225, 234
Navier-Stokes equations, 626, 713
Newman, B. G., 347
Newton, I., 521, 522, 540,
Newton's laws of motion, 125
Ng, K. H., 349
Nikuradse, J., 310, 312, 326 et seq., 337, 397, 407
Nodes, 528, 529
Non-conservative field, 15
Non-dimensional force and moment coefficients, 207, 617
— — quantity defined, 191
Normal depth of channel, 439
— stress, 3, 568 et seq.
Notch, 446
Nozzles, 414
Nuclei for cavitation etc., 168
Nusselt number, 202

Open channel, influence of change of bed level, 433
Orbits of particles in sea waves, 547, 550
Orifice gauge, 227
Orifices in pipes, 414, 453
—, notches, weirs, 446 et seq.
Orthogonal coordinates, 104
Orthogonality of streamlines and equipotentials, 82
Oscillations in branched tubes, 535
— of liquids in tubes, 530; damping of, 563, 566
Outer velocity law for pipe, 406
Owen, P. R., 377
Ower, E., 223

Pai, S., 366
Pankhurst, R. C., 216, 218, 219, 222, 223, 234, 235, 236, 292, 317
Parabolic notch, 450
Pasamanich, J., 383
Patel, V. C., 349
Path of particle, 37
Péclet number, 203
Pelton wheel, 633, 649
Penny, H. H., 386
Perfect fluid, 2
— fluids, method of solution of problems on motion of, 146
— gas, 464
— —, equation of state for, 5
Periodic phenomena in fluids, 204
Perturbations of velocity, small, 172
Pfenninger, W. P., 376
Phase velocity of wave, 528
Physical dimensions, 191
Pi theorem, 192, 197, 200
Piercy, N. A. V., 260
Pipe flow, effects of roughness, 325 et seq.
— —, friction chart, 409
— —, fully developed, 262, 398
— — near entry, 397
— —, power law velocity distribution, 310
— —, turbulent skin friction, 311 et seq.
Pitch, experimental mean, 688
— of helix and helicoid, 667
— ratio of propeller, 668
Pitching moment, 571, 582, 583, 591, 593, 594, 596
— — coefficient, 208, 504, 508, 518, 519
— — defined, 207
Pitot tube, 499, 500
— —, surface, 404
Pitot-static tube, 224
Pitot-traverse method of measuring drag, 359 et seq.
Plane surface, thrust on, 18
Pohlhausen, K., 270, 271 et seq.
Poise (unit of viscosity), 29
Poiseuille, 402
Poisson-Quinton, 385
Polar coordinates, 105
Poncelet turbine, 655
Pope, A., 216
Poppleton, E. D., 383
Positive displacement hydraulic machines, 631
— — meters, 228
Potential, centrifugal, 168
—, complex, 94
— function, 481, 482, 501 et seq.
— of body forces, 14; multi-valued, 146
—, single valued, 82
—, velocity, 81
Pound weight, standard, 27
Poundal, 27
Power factor of wind tunnel, 218
— number, 638
Prandtl, L., 225, 237, 299, 314, 317, 322, 323, 325, 329, 394, 408, 489, 503, 604
Prandtl number, 201, 203
Prandtl-Glauert Law, 503
Prandtl-Meyer expansion, 489
Pressure, absolute, 11
— at a point, 3, 127; in moving fluid, 127
—, atmospheric, 11
— balance, 232
— equation for irrotational flow, 135
—, gauge, 11
—, measurement of, 230
Preston, J. H., 260, 316, 339, 368, 369, 625
Pretsch, J., 288, 374
Principal normal to streamline, 135
Principle of Archimedes, 24
Probstein, R. F., 523
Product moment of area, 20
Profile drag, 572
Propeller coefficients, 671
— design, 689
— fin effect, 697
— pump, 636
Propellers, 666
—, constant speed, 697
—, contra-rotating, 674, 697
—, screw, 668
Propulsive efficiency, 694
Proudman, J., 530
Pulse, 552
Pumps, 631
Pusher screws, 696

Rafter, 452
Raked helicoid, 669
Ramsay, W. J., 414
Ramsey, A. S., 455
Rankine, W. J. M., 16, 17, 18, 49, 56, 60, 550
Rankine oval, 62
— ovoid, 65
Rarefied gas flow, 522, 523
Raspet, A., 375
Rates of strain, 115, 568, 569
Rayleigh number, 203
Rayleigh's formula for group velocity, 543
Rayner, W. G., 380
Reaction turbines, 632
Rebuffet, P., 385
Receptance, 534
Rectangular notch, 448, 451
Recuperators, 413
Reducible circuit, 83
Reflection of waves in pipes, 559

Refractive index of fluid, 7
Regenscheit, B., 383
Regime of flow, uniform and steady, 398
—, steady and uniform, in channel, 424
Reichardt, H., 307
Relativity, principle of, 125
Relaxation applied to flow in pipe systems, 419
Relf, E. F., 317
Reservoir conditions, 480
Resistance laws for turbulent flow in pipes, 407
Resorber of cavitation tunnel, 220
Reversible and irreversible processes, 465 et seq.
Reynolds, experiment on transition to turbulent flow, 396
— number, 186, 196
— — for pipe, 394; critical 397
— stresses, 284 et seq., 294, 296, 300, 301 et seq.
Rheology, 1
Riabouchinsky, D., 215
Richards, E. J., 235
Riemann surface, 108
Ripples, 549
Robinson, A., 604, 609, 618
Robinson cup anemometer, 223
Rolling moment, 571
Rosenhead, L., 266, 286, 512
Rotary meter, 228
Rotating channel, flow in, 164, 712
Rotation, steady, of a fluid, 15
Rotodynamic machines, types of, 632
Rotta, J. C., 294, 348
Rough channels, resistance of, 427
— pipes, resistance of, 408
Roughness effects, 244, 245, 325 et seq.
— of pipe, 395
— factor, 409, 427
Royal Aeronautical Society Data Sheets, 356, 617, 618
Ruled surface, 668
Runner head, 644
Russell, J. Scott, 552
Russell, R. C. H., 530, 553

Salt dilution method, 229
— velocity method, 230
Schaefer, H., 373
Schiller, L., 398
Schlichting, H., 239, 265, 266, 286, 288, 317, 323, 325, 329, 330, 366, 374, 618, 622, 623, 624
Schlieren method, 234
Scholz, N., 622, 623
Schoenherr, K. E., 322
Schrenk, H., 382
Schrenk, O., 382
Schubauer, G. B., 289, 291, 292, 293, 295, 297, 298, 337, 347, 348
Schuh, H., 345, 346
Schulz-Grunow, F., 317, 322
Scotch turbine, 651
Screw propeller, geometry of, 666
— pump, 635
Sears, W. R., 509, 513, 610
Second Law of Thermodynamics, 466, 469 et seq.
Secondary flows, 624, 625, 649
Seiches, 529
Serby, J. E., 330
Shadow picture of flow, 234
Shape numbers for pumps and turbines, 638
Shear stress, 568 et seq.
— —, laminar, 247 et seq., 261
Shearing stress, 3
Ship propulsion, 692
—, resistance of, 198
— tanks, 214
— waves, 553
Shock stall, 515 et seq., 520
— — Mach number, 520
— wave-boundary layer interaction, 480, 516
— waves, 479, 485, 489 et seq, 514 et seq.
Shock-expansion method, 496 et seq.
Shooting flow, 424
Side force, 571
Similar fluids defined, 203
— physical systems, 179
Similarity of behaviour, 180
— of flow, 178, 185
Simple wave flow, 485 et seq.
Simply connected region, 83
Singular points of transformation, 101
Sink, 52
Skan, S. W., 264
Skin friction, 239 et seq., 261 et seq., 311 et seq.
— — drag, 572
— — for turbulent flow, in rough pipes, 328; in smooth pipes, 311 et seq.; on a flat plate at zero incidence, 314 et seq.
— —, measurement of, 314 et seq.
— — on a cylinder, 336
— — on a rough flat plate, 328 et seq.
— — with partly laminar and partly turbulent boundary layer, 330 et seq.
Skramstad, H. K., 289
Slender wing theory, 609 et seq.
Slip flow, 523
— of propeller, 671
— ratio, 671
Slug, definition of, 27
Smith, D. W., 306, 308, 317, 323, 341, 343

Smoke generator, 234
Smooth entry, 646
Solenoidal vector, 80, 707
Solidity of propeller, 668, 690
Solitary waves, 552
Sound, velocity of, 6, 459, 540
— waves, 538
Source, 52, 707
—, two-dimensional, 53, 707
Source-sink pair, 54, 56
Southwell, R. V., 419
Spalding, D. B., 349
Specific heat at constant pressure, 468 et seq., 661
— — at constant volume, 468 et seq.
— heats of a fluid, 6
— speeds, 639, 642
- - volume, 2
— weight, 12, 128
Speed of sound, 6, 459, 540
Speed-power coefficient, 673, 689
Spence, D. A., 322, 337, 340, 341, 345, 346, 369
Sphere, drag of (Stokes' formula), 164
— in uniform stream, 117
—, virtual mass of, 153, 155
Squire, H. B., 290, 300, 304, 325, 334, 356, 357, 377, 599, 625
Stability of atmosphere, 25
— of stratified fluid, 25
Stagnation point, 60
— pressure, 148
Standard atmosphere, 13
— radius, method of, in propeller design, 691
Standing waves, 528
Stanton, T. E., 317, 404
Stanton number, 201
Starting vortex, 573, 574
Static pressure, 148, 224, 569, 570
— stability, 538
Steady motion defined, 37
Steep slope of channel, 442
Stephens, A. V., 386
Stokes (unit of kinematic viscosity), 30
Stokes' analysis of motion near a point, 114
— formula for resistance of sphere, 164
— stream function, 115
— theorem, 73, 77, 78, 706
Straighteners for fans, 698
Stratosphere, 13
Streak line, 38
Stream function, 44, 481, 482
— —, Stokes', 115
Streamline bodies, 60, 66
—, definition of, 37
—, differential equations of, 38
—, dividing, 60
Streamline, stream function constant on, 46, 116
Streamtube, 41
Stress, 2
Strouhal number, 205
Stuart, J. T., 244
Subsonic flow, 461 et seq., 502 et seq.
Subundal flow, 215, 424
Sudden contraction in pipe, 412
— expansion in pipe, 411
Supersonic flow, 461 et seq., 507 et seq.
Superundal flow, 215, 424
Surface energy, 9
— Pitot-tube, 404
— tension, 8
— —, influence on wave propagation, 547
Surge tanks, 562
Sweepback, 377, 378, 380, 385
—, effect on shock stall, 520, 521
Symmetric body, tunnel interference on, 221
Système International (S.I. Units), 27 et seq.

Tani, I., 283
Tanner, L. W., 386
Taylor, D. W., 693, 695
Taylor, G. I., 163, 222, 290, 291, 299
Taylor, H. D., 386
Temperature, measurement of, 236
—, stagnation, 236
—, total, 236
Temple, G., 513
Temporal boundary, 180
Tetervin, 337, 346, 375
Thermal expansion, coefficient of, 6
Thermodynamic measurement of efficiency, 661
Thermometer, 236, 661
Thermometric conductivity, 201
Thin aerofoil theory, 587 et seq.
Thijsse's formula for resistance of channel, 426
Thom, A. S., 452, 661
Thoma cavitation factor, 659
Thompson, B. G. J., 347, 348
Thomson, James, 451, 655
Thrust deduction, 694
— on curved surface, 24
— on plane surface, 18
Thwaites, B., 270, 273 et seq., 281, 322, 347, 369, 380, 384, 618
Tillmann, W., 317, 337, 341, 342, 343, 344, 349
Tomotika, S., 273
Torda, T. P., 373
Torricelli's theorem, 132
Total energy, 474

Total head, 130
— pressure, 148, 224
— — in compressible flow, 499, 500
Towing tests, 213
Townend, H. C. H., 222, 405
Townsend, A. A., 294, 295
Tractor screws, 696
Tractrix, 455
Tracy, C. M., 386
Trailing vortices of propeller, 681
— vorticity, 600 et seq., 613
Tranquil flow, 424
Transition in boundary layer, detection of, 235
— to turbulent flow in pipe, 396
Transonic flow, 513 et seq.
Trochoidal waves, 550, 551
Troposphere, 13
Truckenbrodt, E., 345, 346
Tufts, depth, 235
—, surface, 235
Tunnel corrections, 220
Turbines, 631
Turbulence energy spectrum, 295
— intermittency factor, 295
Turbulent boundary layer, inner velocity law or law of the wall, 302, 305, 338, 339
— — —, 'log' law, 302, 305, 313, 314, 321 et seq., 332, 333
— — —, mixing length theories, 299 et seq.
— — —, momentum transport theory, 299, 300 et seq.
— — —, outer velocity law, 307 et seq.
— — —, power laws for flow past flat plate, 317 et seq., 330 et seq., 335
— — —, power laws for pipe flow, 310 et seq.
— — —, structure, 294 et seq.
— — —, with non-zero pressure gradient, 333 et seq.
— — —, von Kármán's similarity theory. 303 et seq.
— flow in pipes, 403

U tube, 231
— —, oscillations in, 530
Ulrich, A., 266, 375
Ultramicroscopic technique, 405
Uniform flow defined, 37
Uniqueness of irrotational motion, 114, 152
Units, 26
—, consistent, 189
—, derived, 189
—, fundamental, 189
— of measurement, 188

Valves, resistance of, 413
van der Waals' Law, 31
van Lammeren, W. P. A., 694
van Manen, J. D., 694
Vane anemometer, 223
Vapour pressure of liquid, 6
Vector analysis, 705 et seq.
Vector, solenoidal, 80
Vee notch, 228, 450, 452
Velocity diagrams for runners, 646
—, friction, 297, 302
—, heat, 136
—, inner law, 302
—, measurement of, 222
— of wave, group, 529
— — —, phase, 528
— potential, 81, 708
Vena contracta, 412, 453
Venturi meter, 132, 227
— flume, 215, 229
Virtual mass, 151, 194
— —, general expression for, 154
Visconti, F., 375
Viscosity, 7
—, kinematic, 8
Viscous fluid, 155
— —, flow between parallel planes, 156
— —, flow between rotating coaxial cylinders, 160, 290
von Doenhoff, A. E., 337, 346, 580
von Gerstner, F. J., 550
von Kármán, T., 254, 299, 303, 308, 406, 526
Vortex, 76
—, combined, 18, 86
—, diffusion of, 163
— drag, 604, 606 et seq., 614
—, electromagnetic analogue of, 78, 90
— filament, 76
—, forced, 16, 74, 86
—, —, stream function for, 48
—, free, 17, 86
— generators, 385 et seq.
—, Hill's spherical, 80
—, horse-shoe, 91
— line, 75, 709
— pair, 87
— ring, velocity induced on axis, 93
— sheet, 79
— street, 204
— strength, 76, 78
— — independent of time, 144
— theory of propeller, 679
— tube, 76
—, velocity field of, 85, 709
—, velocity potential of induced motion, 90
Vorticity, 73, 74, 708
—, generation within fluid, 145

Vorticity in axisymmetric flow, 144
— in two-dimensional flow, 144

Wake fraction, 693
— velocity of ship, 692
Wake-adjusted propellers, 694
Wakes, 252 et seq., 267, 355, 356, 365 et seq.
—, laminar, 267 et seq.
—, turbulent, 355, 356, 365 et seq.
Walker, J. W., 306, 308, 317, 323, 341, 343
Walker, P. B., 220
Walker, W. P., 213
Walker, W. S., 317, 380
Wall, law of, 302
Wallis, R. D., 386
Walz, A., 281, 283, 383
Ward, G. N., 501, 504, 509, 610
Water, density of, 28
— hammer, 556
— line, 20
— tunnels, 219
—, viscosity of, 29
Watt, James, 31
Watt (unit), 31
Wave drag, 508
— motion, 527
— number, 528
— propagation, equation of, 95, 539
— — in pipe, 558
— reflection in pipe, 559
Wavelength, 528
Wave-making resistance, 554
Waves, cuspid, 551
—, destructive action of, 553
Waves in sea of uniform depth, 544
— near shelving shore, 551
—, ship, 553
—, solitary, 552
—, trochoidal, 550
Weber number, 206
Weiber, J. A., 380, 384
Wetted cross-section of channel, 425
Wetting of solid by liquid, 10
Whirling arms, 215
Whitcomb, R. T., 521
Whitelaw, J., 652
Whitham, G. B., 541
Wieghardt, K., 282
Williams, J., 381
Wilson, C. T. R., 168
Wind tunnels, high speed, 218
— —, low speed, 216
Windmills, 700
Wing flap, 628
Wings of finite span, 599 et seq.
— — — —, circulation, 608
— — — —, drag, 606 et seq.
— — — —, elliptic loading, 608
— — — —, lift, 605, 606
Winter, K. G., 625
Winternitz, F. A. L., 414
Work, exchange of energy by, 661

Yawing moment, 571
Yawmeter, 225
—, hot-wire, 235
Young, A. D., 270, 278 et seq., 280, 281, 325, 327, 329, 330, 334, 356, 357, 358, 362, 364, 502